Injectivity of 2D Toric Bézier Patches

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***Abstract*—Rational Bézier functions are widely used as mapping functions in surface reparameterization, finite element analysis, image warping and morphing. The injectivity (one-to-one property) of a mapping function is typically necessary for these applications. Toric Bézier patches are generalizations of classical patches (triangular, tensor product) which are defined on the convex hull of a set of integer lattice points. We give a geometric condition on the control points that we show is equivalent to the injectivity of every 2D toric Bézier patch with those control points for all possible choices of weights. This condition refines that of Craciun, et al., which only implied injectivity on the interior of a patch.**

***Keywords*-Bézier patches; toric patches; injectivity; mapping**

I. Introduction

Mapping functions play an important role in computer graphics, computer aided geometric design (CAGD), finite element analysis (FEA) and some related areas. The injectivity of mapping functions, that is, the absence of self-intersection, is crucial in image warping and morphing [11], free form deformation [1], surface reparameterization, and so on. Many authors have investigated conditions which imply injectivity. Goodman and Unsworth [7] proposed a sufficient condition for the injectivity of a 2D Bézier function. For the control points of a $m \times n$ tensor product patch, their condition involves $2m(m+1)+2n(n+1)$ linear inequalities. For image morphing, Choi and Lee [1] presented a sufficient condition for the injectivity of 2D and 3D uniform cubic B-spline functions. Their condition provides a single bound for the displacements of control points that guarantees the injectivity of the cubic B-spline function. Floater [6] studies a sufficient condition for injectivity of convex combination mappings over triangulations.

Fig. 1 displays rational plane cubic Bézier curves with their control polygons (bold lines). The curve in Fig. 1(a) has no points of self-intersection. The curve in Fig. 1(b) has one point of self-intersection, which may be removed by varying the weights as shown in Fig. 1(c). The control polygon of the first curve is in convex position, so there are no positive weights for which the resulting Bézier curve has self-intersection. For the other control polygon there are weights (e.g. Fig. 1(b)) such that the resulting Bézier curve has a point of self-intersection. The cited works provide conditions which imply no self-intersection. Our purpose is different: We give conditions on the control points for 2D patches which are equivalent to there being no self-intersection for any choice of positive weights.

The basic units in the geometric modeling of surfaces are rational Bézier simplices and tensor product patches. Krasauskas [8] introduced toric Bézier patches as a natural extension of classical rational patches and their higher-dimensional generalizations, the Bézier simploids by DeRose, et al. [4]. The theory of toric patches is based upon real toric varieties from algebraic geometry [9], and they provide a general framework in which to pose many questions concerning classical rational patches.

To study dynamical systems arising from chemical reaction networks, Craciun et al. [3] prove an injectivity theorem for certain maps. This was adapted in [2] to give a geometric condition on a set of control points which implies that the resulting toric Bézier patch has no self-intersection, for any choice of positive weights. That result contains a minor flaw in that it only guarantees injectivity in the interior of a patch. We correct that flaw, at least for 2D patches, showing that the condition from [2] plus the mild additional hypothesis that the vertices correspond to distinct control points is equivalent to injectivity for every choice of positive weights.

In Section 2, we introduce toric Bézier patches as generalizations of the classical rational patches. In Section 3 we explain our condition and sketch its equivalence to the injectivity of every 2D patch with a given set of control points, for all possible weights. More details, including examples of the geometric arguments of Lemma 3.5 and Corollaries 3.6 and 3.7 will be added in the complete version of this paper. We conclude some remarks on how to check this condition, argue that it is in fact quite natural, and interpret it in terms of piecewise linear maps.

While our main interest is in establishing a criteria valid in 3D, and in fact in all dimensions, we currently do not know how to add hypotheses to the condition of [2] so that the result will be equivalent to injectivity for any choice of weights in 3D.

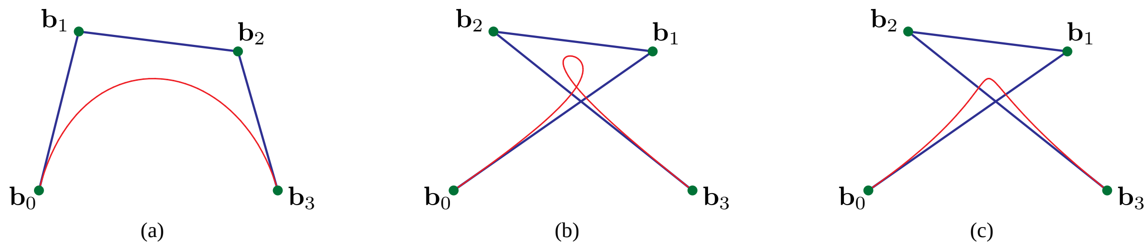


Figure 1. Cubic Bézier curves.

II. Toric Bézier patches

Let $\mathcal{A} \subset \mathbb{Z}^2$ be any finite set of integer *lattice points*. Its *convex hull* $\Delta_{\mathcal{A}}$ is a polygon whose vertices are lattice points. This polygon is also defined by its *edge inequalities*,

$$\Delta_{\mathcal{A}} = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq h_i(x, y)\,, i = 1, \ldots, \ell\},$$

where $h_i(x) = a_i x + b_i y + c_i$ are linear polynomials with integer coefficients and (a_i, b_i) is relatively prime.

For each integer lattice point $\mathbf{a} \in \mathcal{A}$, Krasauskas [8] defined the *toric Bernstein polynomial*

$$\beta_{\mathbf{a}}(x) \;:=\; h_1(x)^{h_1(\mathbf{a})} h_2(x)^{h_2(\mathbf{a})} \cdots h_\ell(x)^{h_\ell(\mathbf{a})}, \qquad (1)$$

These toric Bernstein polynomials are non-negative on $\Delta_{\mathcal{A}}$, and the collection of all $\beta_{\mathbf{a}}$ has no common zeroes in $\Delta_{\mathcal{A}}$.

Let $\mathbb{R}^{\mathcal{A}}_{>}$ be $\mathbb{R}^{|\mathcal{A}|}_{>}$ with coordinates $(w_{\mathbf{a}} \in \mathbb{R}_{>} \mid \mathbf{a} \in \mathcal{A})$ indexed by elements of $\mathcal{A}$.

Definition 2.1: Let $\mathcal{A} \subset \mathbb{Z}^2$ be a finite set. A toric Bézier patch associated with $\mathcal{A}$ requires an assignment $f\colon \mathcal{A} \to \mathbb{R}^d$ $(d = 2, 3)$ of *control points* and a choice of weights $w \in \mathbb{R}^{\mathcal{A}}_{>}$. The *toric Bézier patch* $F_w\colon \Delta_{\mathcal{A}} \to \mathbb{R}^d$ is the function

$$F_w(x) \;=\; F_{\mathcal{A},f,w}(x) \;:=\; \frac{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}} f(\mathbf{a}) \beta_{\mathbf{a}}(x)}{\sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}} \beta_{\mathbf{a}}(x)}\,, \qquad (2)$$

written F_w as $\mathcal{A}$ and f are understood.

The degree of a toric Bézier patch is encoded in its domain, differing from the classical patches as developed in [5]. These two types of patches share many properties, which is explained in [8], [9]. Two properties in particular are important for us.

One is the convex hull property, that the image of $\Delta_{\mathcal{A}}$ under F_w is contained in the convex hull of the control points $f(\mathcal{A})$ with $F_w(\mathbf{b}) = f(\mathbf{b})$ if $\mathbf{b}$ is a vertex of $\Delta_{\mathcal{A}}$, and the other is the boundary property, that the restriction of F_w to an edge δ of $\Delta_{\mathcal{A}}$ is a rational Bézier curve, defined by control points and weights corresponding to lattice points of δ.

The boundary property may be seen directly by considering the restriction to an edge. For the convex hull property, note that as $w_{\mathbf{a}}\beta_{\mathbf{a}}(x)$ is nonnegative, $F_w(x)$ is a convex combination of the control points, and if $\mathbf{b}$ is a vertex, then $\beta_{\mathbf{a}}(\mathbf{b})$ is zero unless $\mathbf{a} = \mathbf{b}$. Since the toric Bernstein polynomials are strictly positive on the interior of Δ (and those corresponding to an edge δ are strictly positive on the interior of δ), we may deduce a little more.

Proposition 2.2: The image of the interior of Δ lies strictly in the interior of the convex hull of the control points $f(\mathcal{A})$, and the image of the interior of an edge δ lies strictly within the interior of the convex hull of $f(\delta \cap \mathcal{A})$.

Toric Bézier patches include the classical Bézier patches and some multi-sided patches such as Warren's polygonal surface [10] which is a reparameterized toric Bézier surface.

Example 2.3 (Tensor product patches): Let m, n be positive integers. Let $\mathcal{A}$ be the integer points in the $m \times n$ rectangle $\mathcal{A} := \{(i, j) : 0 \leq i \leq m,\; 0 \leq j \leq n\}$. Then the corresponding toric Bernstein polynomials (1) are

$$\beta_{(i,j)}(x, y) \;:=\; x^i (m - x)^{m-i} y^j (n - y)^{n-j}\,, \qquad (3)$$

and the toric Bézier patch (2) (with weights $w_{i,j} = \binom{m}{i}\binom{n}{j}$) is the rational tensor product Bézier patch of bidegree (m, n) after the simple reparameterization $s = x/m$, $t = y/n$.

Example 2.4 (Triangular Bézier patches): Let m be a positive integer and $\mathcal{A}$ be the integer points in the triangle with vertices $(0, 0)$, $(m, 0)$, and $(0, m)$, $\mathcal{A} := \{(i, j) \mid 0 \leq i, j, \quad 0 \leq m - i, j\}$. The corresponding Bernstein polynomials (1) are

$$\beta_{i,j}(x, y) \;=\; x^i y^j (m - x - y)^{m-i-j}.$$

Then the toric Bézier patch (2) (with weights $w_{i,j} = \frac{m!}{i!j!(m-i-j)!}$) is the rational Bézier triangle of degree m after the simple reparameterization $s = x/m$, $t = y/m$.

III. Injectivity of 2D toric Bézier patches

Given a finite set $\mathcal{A} \subset \mathbb{Z}^2$ and a choice $f\colon \mathcal{A} \to \mathbb{R}^2$ of control points, we consider the injectivity of toric Bézier patches as mapping functions $F_w\colon \Delta_{\mathcal{A}} \mapsto \mathbb{R}^2$ (2), for all choices $w \in \mathbb{R}^{\mathcal{A}}_{>}$ of positive weights.

Affinely independent points $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ determine an orientation via the ordered basis $\mathbf{a}_1 - \mathbf{a}_0, \mathbf{a}_2 - \mathbf{a}_0$ of $\mathbb{R}^2$.

Definition 3.1: A choice $f\colon \mathcal{A} \to \mathbb{R}^2$ of control points is *weakly compatible* if

1) There are affinely independent points $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ of $\mathcal{A}$ such that $f(\mathbf{a}_0), f(\mathbf{a}_1), f(\mathbf{a}_2)$ is also affinely independent, and
2) For any affinely independent points $\mathbf{a}'_0, \mathbf{a}'_1, \mathbf{a}'_2$ of $\mathcal{A}$ with the same orientation as $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$, if

$f(\mathbf{a}'_0), f(\mathbf{a}'_1), f(\mathbf{a}'_2)$ is also affinely independent, then it has the same orientation as $f(\mathbf{a}_0), f(\mathbf{a}_1), f(\mathbf{a}_2)$.

Fig. 2 shows three sets of labeled points, indicating assignments between them. The assignment between the first

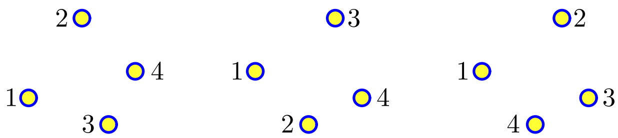


Figure 2. Weak compatibility.

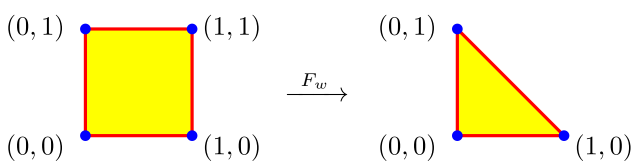
two sets is weakly compatible, but neither assignment to the third set is weakly compatible.

We state Theorem 3.5 of [2] for $\mathbb{R}^2$, which is their main result on injectivity of toric Bézier functions (it holds in any dimension). Write $\Delta^\circ_{\mathcal{A}}$ for the interior of $\Delta_{\mathcal{A}}$.

Theorem 3.2: The map $F_w : \Delta^\circ_{\mathcal{A}} \mapsto \mathbb{R}^2$ is injective for all $w \in \mathbb{R}^{\mathcal{A}}_{>}$ if and only if the assignment $f \colon \mathcal{A} \to \mathbb{R}^2$ is weakly compatible.

In [2], the authors incorrectly stated this result as F_w is injective on all of $\Delta_{\mathcal{A}}$, even though their proof was only valid for the interior of the convex hull. Their proof showed that F_w has no critical points in the interior, which shows that it is an open map on $\Delta^\circ_{\mathcal{A}}$.

This is the best possible result with these hypotheses: Consider a bilinear patch where two control points coincide. Specifically, let $\mathcal{A} = \{(0,0),(0,1),(1,0),(1,1)\}$ and suppose that the control points are $\{(0,0),(0,1),(1,0)\}$, where $f(\mathbf{a}) = \mathbf{a}$, except that $f(1,1) = (1,0)$. This assignment of control points is weakly compatible, but F_w collapses the edge between $(1,0)$ and $(1,1)$ to the point $(1,0)$.



This example shows that more hypotheses are needed to ensure that F_w is injective on $\Delta_{\mathcal{A}}$, and those hypotheses should imply that faces of $\Delta_{\mathcal{A}}$ are not collapsed. In fact, this is the only additional hypothesis needed.

Definition 3.3: A choice $f \colon \mathcal{A} \to \mathbb{R}^2$ of control points is *compatible* if it is weakly compatible, and no two vertices have the same image under f.

We state our main result.

Theorem 3.4: The map $F_w : \Delta_{\mathcal{A}} \mapsto \mathbb{R}^2$ is injective for all $w \in \mathbb{R}^{\mathcal{A}}_{>}$ if and only if the assignment $f \colon \mathcal{A} \to \mathbb{R}^2$ is compatible.

If $\mathbf{a} \in \mathcal{A}$ is a vertex of $\Delta_{\mathcal{A}}$, then $F_w(\mathbf{a}) = f(\mathbf{a})$. Theorem 3.2, together with this observation, shows that if F_w is injective for all $w \in \mathbb{R}^{\mathcal{A}}_{>}$, then $f \colon \mathcal{A} \to \mathbb{R}^2$ is compatible.

For the other implication, suppose that $f \colon \mathcal{A} \to \mathbb{R}^2$ is compatible. We show that the assumption that F_w is not injective leads to a contradiction.

We first make several observations about the relative positions of the points $f(\mathbf{a})$ for $\mathbf{a} \in \mathcal{A}$ which are implied by compatibility. Composing with a reflection of $\mathbb{R}^2$ if necessary, we may assume that if $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ and $f(\mathbf{a}_0), f(\mathbf{a}_1), f(\mathbf{a}_2)$ are both affinely independent, then they induce the same orientation on $\mathbb{R}^2$.

Let δ be an edge of $\Delta_{\mathcal{A}}$. There is some triple of points $\mathbf{d}, \mathbf{d}', \mathbf{a}$ of $\mathcal{A}$ with $f(\mathbf{d}), f(\mathbf{d}'), f(\mathbf{a})$ affinely independent where $\mathbf{d}, \mathbf{d}' \in \delta$ and $\mathbf{a} \notin \delta$. Indeed, if there are no such triples, then every point of $f(\mathcal{A})$ lies on every line segment between two distinct points of $f(\delta \cap \mathcal{A})$, which implies that the points of $f(\delta \cap \mathcal{A})$ are collinear and the line they span contains $f(\mathcal{A})$, which contradicts the first condition for weak compatibility of Definition 3.1. This argument requires that there be at least two distinct points of $f(\delta \cap \mathcal{A})$, which follows as the endpoints of δ (which are vertices of $\Delta_{\mathcal{A}}$) are mapped to different points under f.

Suppose that we list the points $\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_m$ of $\delta \cap \mathcal{A}$ so that if $\mathbf{a} \in \mathcal{A} \setminus \delta$, and $i < j$, then $\mathbf{d}_i, \mathbf{d}_j, \mathbf{a}$ are positively oriented. Then either $f(\mathbf{d}_i), f(\mathbf{d}_j), f(\mathbf{a})$ are collinear or positively oriented. Since there must be at least one such triple with $f(\mathbf{d}_i), f(\mathbf{d}_j), f(\mathbf{a})$ affinely independent, we deduce the following.

Lemma 3.5: Every control point $f(\mathcal{A} \setminus \delta)$ lies in the intersection of closed halfspaces

$$\overline{\{x \in \mathbb{R}^2 \mid f(\mathbf{d}_i), f(\mathbf{d}_j), x \text{ are positively oriented}\}}$$

for $i < j$ with $f(\mathbf{d}_i) \neq f(\mathbf{d}_j)$, and this intersection has a nonempty relative interior.

Corollary 3.6: For every edge δ of $\Delta_{\mathcal{A}}$ and every $\mathbf{b} \in \mathcal{A} \setminus \delta$, the control point $f(\mathbf{b})$ does not lie in the relative interior of the convex hull of $f(\delta \cap \mathcal{A})$.

To see this, note that the intersection of halfspaces of Lemma 3.5 is either interior or exterior to the convex hull of $f(\delta \cap \mathcal{A})$, and if it is exterior, then it is separated from the relative interior of the convex hull by a line. If there is an edge δ so that this intersection lies in the interior of the convex hull of $f(\delta \cap \mathcal{A})$, let δ' be a different edge. Then the positions of the points of $\delta \cap \mathcal{A}$ relative to the intersection of halfspaces for δ' leads to a contradiction.

Corollary 3.7: If $f \colon \mathcal{A} \to \mathbb{R}^2$ is compatible, then the restriction of F_w to any edge δ of $\Delta_{\mathcal{A}}$ is injective.

To see this, fix an edge δ and consider the intersection of halfspaces of Lemma 3.5. This intersection is exterior to the convex hull of $f(\delta \cap \mathcal{A})$ and so consists of an unbounded polyhedron, P. Consider the orthogonal projection $\pi \colon \mathbb{R}^2 \to \mathbb{R}$ along an unbounded direction of P. Then the map $\pi \circ f \colon \delta \cap \mathcal{A} \to \mathbb{R}$ is a weakly compatible choice of control points for $\delta \cap \mathcal{A}$, and so the map $\pi \circ F_w$ restricted to the edge δ is injective, by Theorem 3.2. But this implies that the restriction of F_w to δ is injective.

Proof of Theorem 3.4: We suppose that $f\colon \mathcal{A} \to \mathbb{R}^2$ is compatible and that F_w is not injective. Let $x, y \in \Delta_{\mathcal{A}}$ be distinct points with $F_w(x) = F_w(y)$.

First, neither x nor y can be a point of $\Delta^\circ_{\mathcal{A}}$. To see this, suppose that $x \in \Delta^\circ_{\mathcal{A}}$ and let V be a neighborhood of x in $\Delta_{\mathcal{A}}$ whose closure does not contain y. Then $F_w(V)$ is an open set containing $F_w(x) = F_w(y)$, so $F_w^{-1}(V) \setminus V$ contains an open subset U of y in $\Delta_{\mathcal{A}}$. But then points of $U \cap \Delta^\circ_{\mathcal{A}}$ are mapped by F_w to points of $F_w(V)$, and so F_w is not injective on the interior of Δ, which contradicts Theorem 3.2, as the choice f of control points is weakly compatible.

Thus x and y are points of some edges of $\Delta_{\mathcal{A}}$. They cannot be points of the same edge δ, for then the restriction of F_w to δ is not injective, contradicting Corollary 3.7. Thus they are points of different edges, $x \in \delta$ and $y \in \delta'$ with $\delta \neq \delta'$. We cannot have one of them be an interior point of its edge, for then the relative interiors of the convex hulls of $f(\delta \cap \mathcal{A})$ and $f(\delta' \cap \mathcal{A})$ meet, contradicting Corollary 3.6.

The only possibility left is that x and y are vertices of Δ, but then $F_w(x) = f(x)$ and $F_w(y) = f(y)$, which are different, as the choice f was compatible. ■

Remark 3.8: By definition, to check weak compatibility for 2D patches, it suffices to check determinants for each triple of points of $\mathcal{A}$ and the corresponding control points, giving a simple $(\#(\mathcal{A}))^3$ algorithm. The complexity may be reduced if we start from a triangulation of $\Delta_{\mathcal{A}}$, or with careful bookkeeping. Such triangulations can be obtained from control nets for tensor product patches or Bézier triagles. We will treat the complexity of checking weak compatibility in the complete version of this extended abstract.

Mapping functions that are weakly compatible exist; for example the identity assignment of control points is weakly compatible. A designer may choose weakly compatible control points for aesthetic or other reasons. For example, if only a few control points are moved such as in image warping, morphing, or reparameterization, then the control points may be weakly compatible by design, or else only a few determinants need to be computed.

For any triagulation of $\mathcal{A}$, the assignment of control points induces a piecewise linear map to the image. This piecewise linear map is injective (except possibly collapsing an interior simplex) for every such triangulation if and only if the assignment of control points is weakly compatible.

IV. Conclusions

In this paper, we study the injectivity of toric Bézier patch geometrically. We present a simple condition on a set of control points which implies that the resulting 2D toric Bézier patch is injective, for any choice of positive weights. For higher dimension, the best result remains Theorem 3.2 by Craciun et al. in [2] (Theorem 3.5 in [2]). We plan to continue this investigation of injectivity for 3D and higher dimensions in a future publication.

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