Rational Curves on Grassmannians:
systems theory, reality, and transversality

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ABSTRACT. We discuss a particular problem of enumerating rational curves on a Grassmannian from several perspectives, including systems theory, real enumerative geometry, and symbolic computation. We also present a new transversality result, showing this problem is enumerative in all characteristics.

While it is well-known how this enumerative problem arose in mathematical physics and also its importance to the development of quantum cohomology, it is less known how it arose independently in mathematical systems theory. We describe this second story.

1. Introduction

The enumerative geometry of curves on algebraic varieties has become an important theme in algebraic geometry. One motivation for this development was to understand (and prove) remarkable formulae from theoretical physics, including a formula of Vafa and Intriligator [**30, 62**] involving curves on Grassmannians. The story of this direct influence of theoretical physics on algebraic geometry is well-known. What is less known is how the problem of enumerating rational curves on Grassmannians also arose and was solved in systems theory. Our purpose is to make that story better known and to relate the different solutions, from physics and from systems theory, of this enumerative problem. We also discuss some related work in algebraic geometry inspired by systems theory.

We describe this enumerative problem. Let $m, p \geq 1$ be integers. The space $\mathcal{M}_{m,p}^q$ of maps M of degree q from $\mathbb{P}^1$ to $Grass(p, \mathbb{C}^{m+p})$, the Grassmannian of p-planes in $\mathbb{C}^{m+p}$, has dimension $N := q(m + p) + mp$ [**14, 58**]. Given a point

$s \in \mathbb{P}^1$ and an m-plane L in $\mathbb{C}^{m+p}$, the set of maps M which satisfy $M(s) \cap L \neq \{0\}$ (the p-plane $M(s)$ meets the m-plane L non-trivially) is a divisor on this space of maps. We consider the following enumerative problem:

Question 1. *Given general points $s_1, s_2, \ldots, s_N \in \mathbb{P}^1$ and general m-planes $L_1, L_2, \ldots, L_N \subset \mathbb{C}^{m+p}$, how many degree q maps $M \colon \mathbb{P}^1 \to Grass(p, \mathbb{C}^{m+p})$ satisfy*

$$(1.1) \qquad\qquad M(s_i) \cap L_i \; \neq \; \{0\} \qquad for \;\; i = 1, 2, \ldots, N \; ?$$

This is a special case of the more general enumerative problem considered by Vafa and Intriligator [**30, 62**] who replaced the Schubert condition $M(s) \cap L \neq \{0\}$ by general Schubert conditions and the map $M \colon (\mathbb{P}^1, s_1, s_2, \ldots, s_N) \to Grass(p, \mathbb{C}^{m+p})$ by a map of a general pointed curve. There, a formula was proposed involving residues. This formula was justified by Siebert and Tian [**51**] by computing the (small) quantum cohomology ring of the Grassmannian, whose structure was also conjectured by Vafa and Intriligator. We describe this part of our story in Section 5.

A completely different approach (and motivation) to this enumerative problem came from systems theory. Briefly, conditions of the form $M(s) \cap L \neq \{0\}$ arise in the problem of stabilizing a given linear system using dynamic output compensation [**44**]. In the critical dimension when there are finitely many compensators, the problem of enumeration was solved by Ravi, Rosenthal, and Wang [**40, 41**], who gave the closed formula for the intersection number $d(m, p; q)$ of Question 1:

$$(1.2) \qquad (-1)^{q(p+1)} \, N! \; \cdot \sum_{\nu_1 + \cdots + \nu_p = q} \frac{\prod_{j<k}(k-j+(\nu_k-\nu_j)(m+p))}{\prod_{j=1}^{p}(m+j+\nu_j(m+p)-1)!} \; .$$

One of their motivations was to determine when this number is odd, for then there exists a real compensator stabilizing a given real linear system. We describe how this problem in systems theory is a special case of the general enumerative problem described above, and also how Ravi, Rosenthal, and Wang solved this enumeration in Section 2. We remark that the quantum cohomology of the Grassmannian also has applications to matrix interpolation problems [**3, 38**].

The geometric formulation from systems theory (and ideas from numerical homotopy continuation [**2**]) were exploited to prove the following result in real enumerative geometry: There exist real points $s_1, s_2, \ldots, s_N \in \mathbb{P}^1_{\mathbb{R}}$ and real m-planes $L_1, L_2, \ldots, L_N \subset \mathbb{R}^{m+p}$ such that there are $d(m, p; q)$ rational maps $M \colon \mathbb{P}^1 \to Grass(p, \mathbb{C}^{m+p})$ of degree q satisfying (1.1), and each of these maps is real [**54**]. Thus the enumerative problem of Question 1 is fully real (in the sense of [**52**]). A variant of this argument gives the new result that Question 1 makes enumerative sense in any characteristic: If $\mathbb{K}$ is any algebraically closed field, then for general points $s_1, s_2, \ldots, s_N \in \mathbb{P}^1_{\mathbb{K}}$ and general m-planes $L_1, L_2, \ldots, L_N \subset \mathbb{K}^{m+p}$ there are exactly $d(m, p; q)$ degree q rational maps $M \colon \mathbb{P}^1 \to Grass(p, \mathbb{K}^{m+p})$ satisfying (1.1) [**53**]. The point here is that the corresponding varieties intersect transversally and so the solutions occur without multiplicities. We give a proof of these results in Section 3, where we also solve the enumerative problem of Question 1 without reference to the Chow or quantum Chow rings, the usual tools of enumerative geometry.

Ravi, Rosenthal, and Wang [**40, 41**] also showed that $d(m, p; q)$ equals the number of saturated chains in a certain poset of quantum Plücker coordinates. This is the degree of the singular Uhlenbeck compactification [**47, 7**] of the space of rational curves in the Grassmannian in a natural projective embedding, also called

the quantum Grassmannian. Its degree may be computed from its defining ideal. In [**56**], quantum Plücker relations for this ideal were constructed, giving a different proof that this degree equals the number of chains in the poset of quantum Plücker coordinates. We describe that in Section 4 and give another proof that $d(m, p; q)$ equals the number of chains in that poset.

In the last section, we not only describe some of the classical story motivated by physicists, but also relate the formula (1.2) of Ravi, Rosenthal, and Wang to the formula of Vafa and Intriligator. This involves another, intermediate formula (5.10). We conclude by discussing some further aspects of the quantum cohomology ring of the Grassmannian, including how it arose in representation theory and open problems involving quantum Littlewood-Richardson numbers.

2. Dynamic Control of Linear Systems

In control theory, the greatest interest is to obtain results valid over the real numbers $\mathbb{R}$. As in algebraic geometry, the strongest and most elegant results are true only for the complex numbers $\mathbb{C}$. Also as in algebraic geometry, much of the theory may be developed over any field. To that end, we let $\mathbb{K}$ denote an arbitrary field, keeping in mind the special cases of when $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Suppose we have a time-invariant physical system with m inputs $\mathbf{u} \in \mathbb{K}^m$ and p outputs $\mathbf{y} \in \mathbb{K}^p$ whose evolution is governed by a system of constant coefficient linear differential equations

$$0 \;=\; F(\mathbf{u}, \mathbf{u}', \ldots; \mathbf{y}, \mathbf{y}', \ldots)\,.$$

One important way in which such a linear system arises is from a linear perturbation of a non-linear system.

Introducing auxiliary variables or *internal states* $\mathbf{x} \in \mathbb{K}^n$, we can transform this into a first order system of linear differential equations

$$(2.1) \qquad \begin{aligned} \tfrac{d}{dt}\mathbf{x} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u}\,, \end{aligned}$$

where A, B, C, and D are matrices of the appropriate size. The matrix D represents a direct linear dependence of $\mathbf{y}$ on $\mathbf{u}$. Systems with $D = 0$, where the dependence of $\mathbf{y}$ on $\mathbf{u}$ is purely dynamic, are called *strictly proper*. The representation (2.1) is called a *state space form* or *state space realization* of the original system. There are many ways to realize a given system in state-space form and a fundamental invariant, the *McMillan degree*, is the minimal number n of internal states needed to obtain such a first order linear evolution equation. The McMillan degree measures the complexity of a linear system.

A system is *observable* if the joint kernel of the matrices CA^k for $0 \leq k < n$ is zero, which implies that the internal states ($\mathbf{x}$) may be recovered from knowledge of $\mathbf{y}(t)$ and $\mathbf{u}(t)$. It is *controllable* if the matrices $A^k B$ for $0 \leq k < n$ span $\mathbb{K}^n$, which implies that the system may be driven to any fixed internal state. A state space realization (2.1) of a system is *minimal* (n is its McMillan degree) if and only if it is both observable and controllable [**19**, §13].

2.1. Rational curves on Grassmannians. We give another fundamental representation of a linear system that links systems theory to the (quantum) cohomology of the Grassmannian. Consider the Laplace transform of (2.1)

$$
\begin{aligned}
s \cdot \mathbf{x}(s) &= A \cdot \mathbf{x}(s) + B \cdot \mathbf{u}(s)\,, \\
\mathbf{y}(s) &= C \cdot \mathbf{x}(s) + D \cdot \mathbf{u}(s)\,.
\end{aligned}
$$

We eliminate $\mathbf{x}$ and solve

$$
\mathbf{y}(s) = \big(C(sI_n - A)^{-1}B + D \big)\, \mathbf{u}(s)\,.
$$

This p by m matrix $\Gamma(s) := C(sI_n - A)^{-1}B + D$ of rational functions is called the *transfer function* of the original system. It represents the response of the system in the frequency domain.

The transfer function determines a curve in $Grass(p, \mathbb{K}^{m+p})$ by

$$
\mathbb{P}^1 \ni s \longmapsto \text{ column space} \begin{bmatrix} I_m \\ \Gamma(s) \end{bmatrix}\,,
$$

whenever this is well-defined. This *Hermann-Martin* curve extends to $\mathbb{P}^1$ and its degree is equal to the McMillan degree of the system. Recall that the degree of a curve $M\colon \mathbb{P}^1 \to Grass(p, \mathbb{K}^{m+p})$ has three equivalent descriptions:

(1) The number of points $s \in \mathbb{P}^1$ such that $M(s) \cap L \neq \{0\}$, where L is a general m-plane.
(2) The maximum degree of the (rational-function) minors of any $(m+p)$ by p matrix of rational functions whose column space gives the map M.
(3) The degree of the pullback of the generator $\mathcal{O}(1)$ of the Picard group of $Grass(p, \mathbb{K}^{m+p})$.

One concrete way to see that the transfer function defines a curve in the Grassmannian is via the algebra of polynomial matrices. A matrix $\Gamma(s)$ of rational functions is *proper* if $\lim_{s\to\infty} \Gamma(s)$ exists and *strictly proper* if that limit is zero. The transfer function of the linear system (2.1) is proper, since $\lim_{s\to\infty} \Gamma(s) = D$, and strictly proper linear systems have strictly proper transfer functions. Given a proper matrix of rational functions $\Gamma(s)$ of size p by m, consider factorizations

$$
\Gamma(s) = P(s)Q(s)^{-1}
$$

where $P(s)$ is a p by m matrix of polynomials and $Q(s)$ is a m by m matrix of polynomials with non-zero determinant. There are many ways to do this: One could, for instance, let $Q(s)$ be the diagonal matrix with entries $f(s)$, the least common multiple of the denominators of the entries of $\Gamma(s)$. There is a unique minimal, or (right) coprime factorization.

THEOREM 2.1. *Suppose $\Gamma(s)$ is a proper transfer function of a linear system of McMillan degree n. Then there exist matrices $P(s), Q(s)$ of polynomials such that*

(*i*) *$P(s)$ and $Q(s)$ are coprime in that there exist matrices of polynomials $X(s)$ and $Y(s)$ satisfying*

$$
X(s)Q(s) + Y(s)P(s) = I_m\,.
$$

(*ii*) *Any other factorization $\Gamma(s) = N(s)D(s)^{-1}$ into matrices of polynomials has*

$$
\deg \det D(s) \geq \deg \det Q(s) = n\,.
$$

(*iii*) *$P(s)$ and $Q(s)$ are unique up to multiplication on the right by elements of $GL_m(\mathbb{K}[s])$.*

Theorem 2.1 is proven, for instance in any of [**42**] or [**16**, §22] or [**19**, §4].

By (i) and the factorization, $P(s)Q(s)^{-1} = C(sI_n - A)^{-1}B + D$, the determinants of $Q(s)$ and of $sI_n - A$ have the same roots. We call $P(s)Q(s)^{-1}$ a *right coprime factorization* of $\Gamma(s)$. By (i), the Hermann-Martin curve is also represented by

$$(2.2) \qquad s \longmapsto \text{ column space } \begin{bmatrix} Q(s) \\ P(s) \end{bmatrix},$$

which has dimension m for all $s \in \mathbb{P}^1$, as $\Gamma(s)$ is proper. Since $\lim_{s\to\infty} \Gamma(s) = D$, the value of the curve at infinity is the column space of

$$\lim_{s\to\infty} \begin{bmatrix} Q(s) \\ P(s) \end{bmatrix} = \begin{bmatrix} I_m \\ D \end{bmatrix}.$$

Thus the maximal minors of the matrix

$$\begin{bmatrix} Q(s) \\ P(s) \end{bmatrix}$$

have degree at most the degree of the principal minor $\det Q(s)$, which is n. This shows that the Hermann-Martin curve has degree n. In this way, a linear system (2.1) with m inputs and p outputs of McMillan degree n corresponds to a rational curve $M \colon \mathbb{P}^1 \to Grass(p, \mathbb{K}^{m+p})$ of degree n. In fact, every such rational curve comes from a linear system [**37**].

An informal way to see this is to first observe that the entries of the matrices A, B, C, and D in (2.1) give the set of all possible state-space realizations of m-input p-output linear systems with n internal states the structure of affine space of dimension $n^2 + nm + np + mp$. The conditions of controllability and observability for the system to be minimal are the non-vanishing of certain polynomials in the entries of A, B, C, and so the set of all such systems of McMillan degree n is an open subset of this affine space. Changing coordinates of the internal states $\mathbf{x}$ gives a free $GL_n(\mathbb{K})$-action on these minimal realizations whose orbits are exactly the fibres of the map

$$\{\text{Minimal state-space realizations}\} \quad \longrightarrow \quad \{\text{Proper transfer functions}\}.$$

Thus the space of Hermann-Martin curves of m-input p-output linear systems of McMillan degree n has dimension $nm + np + mp$, which is equal to the dimension of the space $\mathcal{M}^q_{p,m}$ of degree n rational maps to $Grass(m, \mathbb{K}^{m+p})$.

In fact, the Hermann-Martin curves constitute an open subset of this space of rational curves, and there are very natural objects from systems theory that yield the full space of rational curves, as well as various compactifications of this space. The work of Hermann and Martin [**37**] continued work of Clark [**14**], who showed that the space of transfer functions is a smooth manifold. Later, Helmke [**26**] studied topological properties of this space and Hazewinkel [**25**] and Byrnes [**11**] studied compactifications of this space. This work was revived by Rosenthal, who introduced the quantum Grassmannian into systems theory in his 1990 PhD thesis [**43**]. See [**39**] for a discussion and further references.

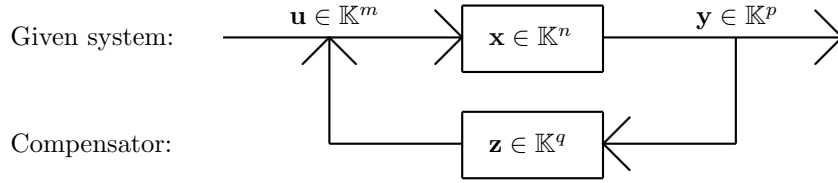**2.2. Feedback control and Schubert calculus.** Given a strictly proper linear system

$$(2.3) \qquad \begin{aligned} \tfrac{d}{dt}\mathbf{x} &= A\mathbf{x} + B\mathbf{u}, \\ \mathbf{y} &= C\mathbf{x}, \end{aligned}$$

we would like to control its behavior using dynamic output feedback. That is, we couple its inputs $\mathbf{u}$ to its outputs $\mathbf{y}$ through a p-input, m-output linear system of McMillan degree q, called a *dynamic compensator*. Consider a minimal state-space realization of this compensator

$$(2.4) \qquad \begin{aligned} \tfrac{d}{dt}\mathbf{z} &= F\mathbf{z} + G\mathbf{y} \\ \mathbf{u} &= H\mathbf{z} + K\mathbf{y}\,, \end{aligned}$$

where $\mathbf{z} \in \mathbb{K}^q$ are the internal states, and F, G, H, and K are matrices of the appropriate size, K representing a constant (residual) linear feedback law.

Schematically we have:

Given system:

$$\mathbf{u} \in \mathbb{K}^m \qquad\qquad \boxed{\mathbf{x} \in \mathbb{K}^n} \qquad\qquad \mathbf{y} \in \mathbb{K}^p$$

Compensator:

$$\boxed{\mathbf{z} \in \mathbb{K}^q}$$

We obtain a closed-loop or autonomous system from (2.1) and (2.4) by eliminating $\mathbf{y}$ and $\mathbf{u}$

$$(2.5) \qquad \frac{d}{dt}\begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} A + BKC & BH \\ GC & F \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}.$$

The behavior of this autonomous system is determined by the $n+q$ eigenvalues of the matrix, that is, by the zeroes of the (monic) characteristic polynomial

$$(2.6) \qquad \varphi(s) := \det\left(sI_{n+q} - \begin{bmatrix} A + BKC & BH \\ GC & F \end{bmatrix} \right).$$

The pole placement problem asks the inverse question:

Pole Placement Problem. Given a strictly proper m-input p-output linear system of McMillan degree n (2.3) and a desired behavior represented by a monic characteristic polynomial $\varphi(s)$ of degree $n + q$, for which dynamic compensators (2.4) does the corresponding autonomous system (2.5) have characteristic polynomial φ?

The reason for the word pole is that the zeroes of the characteristic polynomial are the poles of a transfer function. A linear system of McMillan degree n is *arbitrarily pole-assignable by degree q compensators* (over $\mathbb{K}$) if the pole placement problem may be solved for all monic polynomials φ of degree $n + q$.

REMARK 2.2. Pole placement is a fundamental design problem for linear systems. When $\mathbb{K} = \mathbb{R}$, an important property of an autonomous real linear system is whether or not it is stable, that is, whether or not all of the roots of its characteristic polynomial have negative real parts. In other situations, the control engineer may wish to destabilize a system. For discrete-time systems (which have an identical formalism), stability is achieved by placing the roots of the characteristic polynomial on the unit circle. These questions of placing poles in subsets of the complex plane are strictly weaker than the pole placement problem, yet little is known about them. Here is an important related question concerning stability.

Minimal Stability. Given a strictly proper m-input p-output real linear system of McMillan degree n, what is the minimal McMillan degree q of a real dynamic compensator (2.4) for which the corresponding autonomous system (2.5) is stable?

When $\mathbb{K}$ is algebraically closed, the pole placement problem may be solved for $q \geq n - 1$ [**8**] and $q \geq (n - mp)/(m + p - 1)$ is necessary [**44**] and sufficient [**65**] for generic systems. Thus for q large enough there exist stabilizing dynamic compensators. The minimal stability problem is particularly important when the original system arises as a linear perturbation of a non-linear system. In this case, it asks how cheaply may we damp linear perturbations of McMillan degree n.

We investigate the pole placement problem. Given a strictly proper system (2.3) and a monic characteristic polynomial $\varphi(s)$ with distinct roots $s_1, s_2 \ldots, s_{n+q}$, we seek matrices $F, G, H,$ and K for which

$$\det \left(s_i I_{n+q} - \left[\begin{array}{cc} A + BKC & BH \\ GC & F \end{array} \right] \right) = 0 \qquad \text{for} \quad i = 1, 2, \ldots, n + q.$$

This gives $n + q$ equations in the $q^2 + pq + mq + mp$ entries of $F, G, H,$ and K. Since $GL_q(\mathbb{K})$ acts on these data, giving equivalent systems and fixing φ, we expect that the pole placement problem is solvable over the complex numbers when

$$n + q \ \leq \ pq + mq + mp.$$

This is in fact the case for generic systems, as we shall see.

We reformulate the dynamic pole placement problem geometrically. Each step below involves only row or column operations applied to the matrix involved.

$$\begin{aligned}
\varphi(s) \ &= \ \det \left[\begin{array}{cc} sI_n - A - BKC & -BH \\ -GC & sI_q - F \end{array} \right] \\[2mm]
&= \ \det \left[\begin{array}{cccc} sI_n - A - BKC & -BH & BK & -B \\ -GC & sI_q - F & 0 & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & I_m \end{array} \right] \\[2mm]
&= \ \det \left[\begin{array}{cccc} sI_n - A & 0 & 0 & -B \\ 0 & sI_q - F & G & 0 \\ C & 0 & I_p & 0 \\ 0 & -H & K & I_m \end{array} \right] \\[2mm]
&= \ \begin{array}{c} \det[sI_n - A] \\ \times \det[sI_q - F] \end{array} \times \det \left[\begin{array}{cccc} I_n & 0 & 0 & -(sI_n - A)^{-1}B \\ 0 & I_q & (sI_q - F)^{-1}G & 0 \\ C & 0 & I_p & 0 \\ 0 & -H & K & I_m \end{array} \right]
\end{aligned}$$

This becomes

$$\begin{array}{c} \det[sI_n - A] \\ \times \det[sI_q - F] \end{array} \times \det \left[\begin{array}{cccc} I_n & 0 & 0 & -(sI_n - A)^{-1}B \\ 0 & I_q & (sI_q - F)^{-1}G & 0 \\ 0 & 0 & I_p & C(sI_n - A)^{-1}B \\ 0 & 0 & H(sI_q - F)^{-1}G + K & I_m \end{array} \right]$$

And thus we obtain

$$\varphi(s) \;=\; \det \left[\begin{array}{cc} I_p & C(sI_n - A)^{-1}B \\ H(sI_q - F)^{-1}G + K & I_m \end{array} \right]$$
$$\times \det[sI_n - A] \times \det[sI_q - F]\,.$$

The off-diagonal entries in the first matrix are the transfer functions of the original system (2.3) and of the compensator (2.4). Consider coprime factorizations

$$N(s)D(s)^{-1} \;=\; C(sI_n - A)^{-1}B$$
$$P(s)Q(s)^{-1} \;=\; H(sI_q - F)^{-1}G + K\,.$$

Because n and q are the respective McMillan degrees, we have

$$\det D(s) \;=\; \det[sI_n - A]\,, \;\;\text{and}$$
$$\det Q(s) \;=\; \det[sI_q - F]\,,$$

and so our characteristic polynomial becomes

(2.7) $$\varphi(s) = \det \left[\begin{array}{cc} Q(s) & N(s) \\ P(s) & D(s) \end{array} \right]\,.$$

The first column of this 2 by 2 block matrix represents the Hermann-Martin curve $M \colon \mathbb{P}^1 \to Grass(p, \mathbb{K}^{m+p})$ of the compensator and the second column the Hermann-Martin curve $L \colon \mathbb{P}^1 \to Grass(m, \mathbb{K}^{m+p})$ of the original system.

The determinant (2.7) must vanish at each root of the characteristic polynomial. Since, for every s, the columns giving the Hermann-Martin curves have full rank, we obtain the following version of the pole placement problem, when the characteristic polynomial has distinct roots.

Geometric Version of the pole placement problem. Suppose we have a strictly proper m-input p-output linear system (2.3) of McMillan degree n with Hermann-Martin curve L and a monic polynomial $\varphi(s)$ of degree $n+q$ with distinct roots $s_1, s_2, \ldots, s_{n+q}$. Which rational curves $M \colon \mathbb{P}^1 \to Grass(p, \mathbb{K}^{m+p})$ of degree q satisfy

$$M(s_i) \cap L(s_i) \;\neq\; \{0\} \qquad \text{for} \quad i = 1, 2, \ldots, n + q \,?$$

Thus we are looking for rational curves M which satisfy $n + q$ Schubert conditions of the type in Question 1.

Note that when $q = 0$ (the case of static compensators), $Q(s) = I_p$ and $P(s) = K$, so a static compensator is represented by the matrix

$$\left[\begin{array}{c} I_p \\ K \end{array} \right],$$

whose column space is just a point in the Grassmannian $Grass(p, \mathbb{K}^{m+p})$. This observation of Byrnes [10] was the point of departure for the subsequent application of Schubert calculus to the pole placement problem.

2.3. Number of dynamic compensators in the critical dimension. Let $\mathcal{M}_{m,p}^q$ be the space of degree q maps $M \colon \mathbb{P}^1 \to Grass(p, \mathbb{K}^{m+p})$, which is also the space of Hermann-Martin curves of possible degree q dynamic compensators for m-input, p-output linear systems (2.3). (This includes both proper and improper compensators.) An important geometric perspective on the characteristic equation (2.7) is that a given strictly proper linear system of McMillan degree n

(represented by its Hermann-Martin curve $L \colon \mathbb{P}^1 \to Grass(p, \mathbb{K}^{m+p})$) determines a *pole placement map*

$$\Lambda_L \ : \ \mathcal{M}^q_{m,p} \longrightarrow \ \{\text{Polynomials of degree } n+q\}$$

by

$$\Lambda_L \ : \ M \ \longmapsto \ \det[M(s) : L(s)] \,.$$

This map gets its name from the fact that $\Lambda_L^{-1}(\varphi(s))$ is the set of Hermann-Martin curves of degree q dynamic compensators giving characteristic polynomial $\varphi(s)$. Thus a strictly proper linear system is arbitrarily pole assignable when the corresponding pole placement map is surjective.

Consider expanding this determinant along the columns of $M(s)$:

$$\Lambda_L(M) \ = \ \sum_{\alpha \in \binom{[m+p]}{p}} M_\alpha(s) \cdot L_\alpha(s) \,.$$

Here $\binom{[m+p]}{p}$ is the collection of subsets of $\{1, 2, \ldots, m+p\}$ of size p, $M_\alpha(s)$ is the αth maximal minor of $M(s)$ (given by the rows of $M(s)$ indexed by α), and $L_\alpha(s)$ is the appropriately signed complementary maximal minor of $L(s)$. The point of this exercise is that the pole placement map is a linear function of the coefficients of the polynomials $M_\alpha(s)$.

Thus we are led to consider the Plücker map

$$(2.8) \qquad\qquad \mathcal{M}^q_{m,p} \ \longrightarrow \ \mathbb{P}(\wedge^p \mathbb{K}^{m+p} \otimes \mathbb{K}^{q+1})$$

which associates a $m+p$ by p matrix $M(s)$ of polynomials (representing a degree q compensator or degree q curve) to its $\binom{m+p}{p}$ maximal minors $M_\alpha(s)$, which are polynomials of degree q. A more intrinsic definition of this map is given in Section 3 just before (3.1). This gives a map to projective space as multiplying $M(s)$ by an invertible p by p matrix F multiplies each minor by the factor $\det F$ but does not change the curve. This Plücker map is an embedding, and one compactification of $\mathcal{M}^q_{m,p}$ is the closure $\mathcal{K}^q_{m,p}$ of the image, which we call the *quantum Grassmannian*. This space was introduced to systems theory by Rosenthal [**43**].

In this way, we see that the pole placement map factors

$$\mathcal{M}^q_{m,p} \ \longrightarrow \ \mathcal{K}^q_{m,p} \ \xrightarrow{\ \pi_L\ } \ \mathbb{P}^{n+q} \,,$$

with the last map π_L a linear projection on $\mathbb{P}(\bigwedge^p \mathbb{K}^{m+p} \otimes \mathbb{K}^{q+1})$. Here, $\mathbb{P}^{n+q}$ is the space of polynomials of degree at most $n+q$, modulo scalars. (If the compensator is on the boundary of the compactification, then the polynomial has degree less than $n+q$.)

Thus a necessary condition for arbitrary pole assignability of a strictly proper linear system L is that π_L is surjective. The surjectivity of π_L is sufficient for solving the pole placement problem for L and for generic polynomials $\varphi(s)$. Rosenthal [**44**] shows that if $q(m+p) + mp \leq n+q$ and $\mathbb{K}$ is algebraically closed, then π_L is surjective for generic strictly proper linear systems L. This gives the criterion $n+q \leq q(m+p) + mp$ for a generic m-input, p-output system (2.3) of degree n to be arbitrarily pole assigned with degree q compensators.

For generic systems L in the critical dimension ($q(m+p) + mp = n+q$ so that $\dim \mathcal{K}^q_{m,p} = n+q$) the map π_L is finite and hence surjective, again, when $\mathbb{K}$ is algebraically closed. Thus $\#(\pi_L^{-1}\varphi(s) \cap \mathcal{M}^q_{m,p})$ is the number of compensators solving the pole placement problem for $\varphi(s)$. Since π_L is a linear projection,

this number is bounded by the degree of the quantum Grassmannian $\mathcal{K}_{m,p}^q$ in its Plücker embedding. Suppose $\mathbb{K}$ is algebraically closed. Since $\mathcal{M}_{m,p}^q$ is open in $\mathcal{K}_{m,p}^q$, for generic $\varphi(s)$ the degree of $\mathcal{K}_{m,p}^q$ equals the number of dynamic compensators, possibly counted with multiplicity. This is the main theorem of [**44**].

When $\mathbb{K} = \mathbb{R}$ so that A, B, C, and $\varphi(s)$ are real, $\pi^{-1}\varphi(s) \cap \mathcal{M}_{m,p}^q$ gives the *complex* dynamic compensators which solve the pole placement problem for these data. If $n + q \leq q(m + p) + mp$ and $\mathcal{K}_{m,p}^q$ has odd degree, then the set of dynamic compensators is a projective variety defined over the real numbers of odd degree, and hence contains a real point. We deduce the following result.

THEOREM 2.3. *Suppose $n \leq q(m + p - 1) + mp$, and $\deg \mathcal{K}_{m,p}^q$ is odd. Then a general strictly proper real linear system (2.3) with m inputs, p outputs, and McMillan degree n is arbitrarily pole assignable by real degree q dynamic compensators.*

When the degree of $\mathcal{K}_{m,p}^q$ is even the strongest result is due to Rosenthal and Wang.

THEOREM 2.4 ([**45**]). *A generic strictly proper linear system (2.3) with m inputs, p outputs, and McMillan degree n is arbitrarily pole assignable by real degree q compensators if*

$$n \;<\; q(m + p - 1) + mp - \min\{r_m(p - 1), r_p(m - 1)\} ,$$

where r_p and r_m are the remainders of q upon division by p and m, respectively.

The special case when $q = 0$ of static compensation has an interesting history (see the excellent survey of Byrnes [**12**]). In this case, the Grassmannian $Grass(p, \mathbb{K}^{m+p})$ plays the rôle of $\mathcal{K}_{m,p}^q$ and once it was discovered that the equations for pole placement were linear equations on the Grassmannian in its Plücker embedding, significant progress was made. This included Brockett and Byrnes' calculation of the number of static compensators for a generic m-input p-output linear system of McMillan degree mp as the degree of the Grassmannian [**9**]:

$$(2.9) \qquad (mp)! \frac{\prod_{1 \leq j < k \leq p}(k - j)}{\prod_{j=1}^{p}(m + j - 1)!} \;=\; (mp)! \frac{1! \, 2! \, 3! \cdots (p - 2)! \, (p - 1)!}{m! \, (m + 1)! \cdots (m + p - 1)!}$$
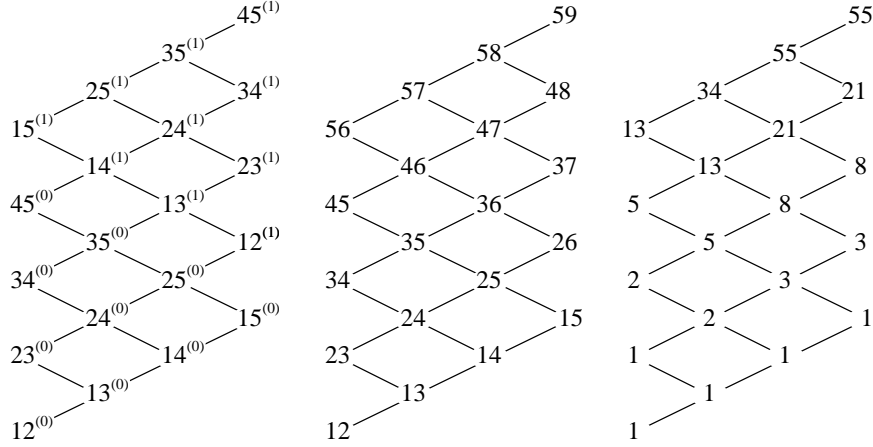
We can deduce the analog of Theorem 2.3 from this; unfortunately, this number is odd only when $\min(m, p) = 1$ (and then it is 1), or else $\min(m, p) = 2$ and $\max(m, p) + 1$ is a power of 2 [**4**]. The analog of Theorem 2.4 is due to Wang [**64**]: $n < mp$ is sufficient to guarantee arbitrary pole assignability over $\mathbb{R}$, for generic systems.

2.4. Formulae for $\deg \mathcal{K}_{m,p}^q$**.** Let $z_{\alpha^{(a)}}$ be the coefficient of s^a in the αth maximal minor $M_\alpha(s)$ of $M(s)$. These coefficients provide *quantum Plücker coordinates* for $\mathbb{P}(\bigwedge^p \mathbb{K}^{m+p} \otimes \mathbb{K}^{q+1})$. Let $\mathcal{C}_{m,p}^q := \{\alpha^{(a)} \mid \alpha \in \binom{[m+p]}{p} \text{ and } 0 \leq a \leq q\}$ be the indices of these quantum Plücker coordinates. This index set has a natural partial order

$$\alpha^{(a)} \;\leq\; \beta^{(b)} \quad \Longleftrightarrow \quad a \leq b \text{ and } \alpha_i \leq \beta_{b-a+i} \text{ for } i = 1, 2, \ldots, p - b + a .$$

The poset $\mathcal{C}_{m,p}^q$ is graded with the rank, $|\alpha^{(a)}|$, of $\alpha^{(a)}$ equal to $a(m+p) + \sum_i(\alpha_i - i)$. It is also a distributive lattice. Figure 1 shows $\mathcal{C}_{3,2}^1$ on the left. Given $\alpha^{(a)} \in \mathcal{C}_{m,p}^q$, define the *quantum Schubert variety*

$$(2.10) \qquad Z_{\alpha^{(a)}} \quad := \quad \{z \in \mathcal{K}_{m,p}^q \mid z_{\beta^{(b)}} = 0 \text{ if } \beta^{(b)} \not\leq \alpha^{(a)}\} .$$

$$45^{(1)} \qquad 59 \qquad 55$$
$$35^{(1)} \qquad 58 \qquad 55$$
$$25^{(1)} \quad 34^{(1)} \qquad 57 \quad 48 \qquad 34 \quad 21$$
$$15^{(1)} \quad 24^{(1)} \qquad 56 \quad 47 \qquad 13 \quad 21$$
$$14^{(1)} \quad 23^{(1)} \qquad 46 \quad 37 \qquad 13 \quad 8$$
$$45^{(0)} \quad 13^{(1)} \qquad 45 \quad 36 \qquad 5 \quad 8$$
$$35^{(0)} \quad 12^{(1)} \qquad 35 \quad 26 \qquad 5 \quad 3$$
$$34^{(0)} \quad 25^{(0)} \qquad 34 \quad 25 \qquad 2 \quad 3$$
$$24^{(0)} \quad 15^{(0)} \qquad 24 \quad 15 \qquad 2 \quad 1$$
$$23^{(0)} \quad 14^{(0)} \qquad 23 \quad 14 \qquad 1 \quad 1$$
$$13^{(0)} \qquad 13 \qquad 1$$
$$12^{(0)} \qquad 12 \qquad 1$$

FIGURE 1. $\mathcal{C}^1_{3,2}$, $J(\mathcal{C}^1_{3,2})$, and $\deg Z_{\alpha^{(a)}}$.

From this definition, we see that $Z_{\alpha^{(a)}} \cap Z_{\beta^{(b)}} = Z_{\gamma^{(c)}}$ (set-theoretically), where $\gamma^{(c)}$ is the greatest lower bound of $\alpha^{(a)}$ and $\beta^{(b)}$.

Let $\mathcal{H}_{\alpha^{(a)}}$ be the hyperplane defined by $z_{\alpha^{(a)}} = 0$. We write $\beta^{(b)} \lessdot \alpha^{(a)}$ to indicate that $\beta^{(b)} < \alpha^{(a)}$ and that there is no other index $\gamma^{(c)}$ with $\beta^{(b)} < \gamma^{(c)} < \alpha^{(a)}$. The main technical lemma of [40, 41] is the following

PROPOSITION 2.5 ([40, 41]). *Let $\alpha^{(a)} \in \mathcal{C}^q_{m,p}$. Then*

(i) *$Z_{\alpha^{(a)}}$ is an irreducible subvariety of $\mathcal{K}^q_{m,p}$ of dimension $|\alpha^{(a)}|$.*

(ii) *The intersection of $Z_{\alpha^{(a)}}$ and $\mathcal{H}_{\alpha^{(a)}}$ is generically transverse and we have*

$$Z_{\alpha^{(a)}} \cap \mathcal{H}_{\alpha^{(a)}} \;=\; \bigcup_{\beta^{(b)} \lessdot \alpha^{(a)}} Z_{\beta^{(b)}} \,.$$

This result is proven essentially by working in local coordinates for $Z_{\alpha^{(a)}}$. Part (ii) is the geometric version of the (codimension-1) Pieri formula. It generalizes the result of Schubert [50], who proved it for the classical Grassmannian ($a = 0$). By Bézout's Theorem (see [20, §8]), we deduce the following fundamental recursion

(2.11) $$\deg Z_{\alpha^{(a)}} \;=\; \sum_{\beta^{(b)} \lessdot \alpha^{(a)}} \deg Z_{\beta^{(b)}} \,.$$

The minimal quantum Schubert variety is a point, so we deduce a formula for $\deg \mathcal{K}^q_{m,p}$.

THEOREM 2.6 ([40, 41]). *The degree $d(m, p; q)$ of $\mathcal{K}^q_{m,p}$ is the number of maximal chains in the poset $\mathcal{C}^q_{m,p}$.*

For example, the degree of $\mathcal{K}^1_{3,2}$ is 55, as shown by the diagram on the right in Figure 1, which recursively computes the degrees of the quantum Schubert varieties $Z_{\alpha^{(a)}}$. In Section 4 we give an alternative proof of Theorem 2.6 using Gröbner bases.

Ravi, Rosenthal, and Wang also solve this recursion to obtain the closed formula (1.2). A first step is to change the indexing of the quantum Plücker coordinates, embedding $\mathcal{C}^q_{m,p}$ into the set of increasing sequences $0 < i_1 < i_2 < \cdots < i_p$

of positive integers of length p. Given $\alpha^{(a)} \in \mathcal{C}^q_{p,m}$, write $a = pl + r$ with $p > r \geq 0$ and define a sequence $J(\alpha^{(a)})$ by

$$(2.12) \qquad J(\alpha^{(a)})_k \; := \; \begin{cases} l(m{+}p) + \alpha_{r+k} & \text{if } 1 \leq k \leq p - r \\ (l{+}1)(m{+}p) + \alpha_{k-p+r} & \text{if } p - r < k \leq p \end{cases} .$$

For instance, when $m = p = 5$, we have $J((2,3,5,6,9)^{(7)}) = (15,16,19,22,23)$. Note that we have $J(\alpha^{(a)})_p < J(\alpha^{(a)})_1 + m + p$. This gives an order isomorphism of the poset $\mathcal{C}^q_{p,m}$ with the poset of sequences $i_1 < i_2 < \cdots < i_p$ of positive integers where $i_p < i_1 + m + p$. This is illustrated in the middle diagram of Figure 1, which shows the image of $\mathcal{C}^1_{3,2}$. This isomorphism (of course) preserves the rank function of the two posets:

$$(2.13) \quad |\alpha^{(a)}| \; := \; a(m+p) + \sum_{j=1}^{p}(\alpha_j - j) \; = \; \sum_{i=1}^{p}\big(J(\alpha^{(a)})_i - i\big) \; =: \; |J(\alpha^{(a)})| \, .$$

Observe that $J(\alpha^{(a)})$ is congruent to α modulo $m + p$.

LEMMA 2.7. *Let* $d(i_1, i_2, \ldots, i_p)$ *be a function defined for all weakly increasing sequences of non-negative integers* $i_1, i_2, \ldots, i_p$ *with* $i_p \leq i_1 + m + p$. *Suppose that for any sequence* $0 < i_1 < \cdots < i_p$ *with* $i_p < i_1 + m + p$, $d(i_1, i_2, \ldots, i_p)$ *satisfies the recursion*

$$(2.14) \qquad d(i_1, i_2, \ldots, i_p) \; = \; \sum_{k=1}^{p} d(i_1, i_2, \ldots, i_k - 1, \ldots, i_p) \, ,$$

is subject to the initial condition

$$(2.15) \qquad\qquad\qquad d(1, 2, \ldots, p) = 1 \, ,$$

and the boundary conditions

$$(2.16) \qquad\qquad d(\ldots, l, l, \ldots) \;\; = \;\; 0 \, ,$$

$$(2.17) \qquad\qquad\qquad d(0, \ldots) \;\; = \;\; 0 \, , \quad \text{and}$$

$$(2.18) \qquad d(i_1, i_2, \ldots, i_p) \;\; = \;\; 0 \quad \text{if } i_p = i_1 + m + p \, .$$

Then $d(J(\alpha^{(a)})) = \deg Z_{\alpha^{(a)}}$.

PROOF. Let $j_1, j_2, \ldots, j_p = J(\alpha^{(a)})$ for $\alpha^{(a)} \in \mathcal{C}^q_{m,p}$. Then the sequence $I := j_1, j_2, \ldots, j_k - 1, \ldots, j_p$ fails to equal $J(\beta^{(b)})$ for some $\beta^{(b)} \in \mathcal{C}^q_{m,p}$ only if I has either two repeated indices (2.16), or if $i_1 = 0$ (2.17), or else if $i_p = i_1 + m + p$ (2.18). In each of these cases $d(I) = 0$, and so the function $d(J(\alpha^{(a)}))$ defined for $\alpha^{(a)} \in \mathcal{C}^q_{m,p}$ satisfies the recursion (2.11) for $\deg Z_{\alpha^{(a)}}$.

Since the index of a minimal quantum Schubert variety (which is a point) is $(1, 2, \ldots, p)^{(0)}$, and $J(\alpha^{(0)}) = \alpha$, the function $d(J(\alpha^{(a)}))$ also satisfies the initial condition for $\deg Z_{\alpha^{(a)}}$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Sequences $I : 0 < i_1 < i_2 < \cdots < i_p \leq l$ index Schubert varieties Ω_I of $Grass(p, \mathbb{K}^l)$. Schubert [50] showed that the degree $g(I)$ of the Schubert variety Ω_I in the Plücker embedding satisfies the recursion, initial condition, and boundary conditions (2.16) and (2.17) of Lemma 2.7. He later [49] gave the following closed formula for this degree (compare with (2.9)):

$$(2.19) \qquad\qquad\qquad |I|! \frac{\prod_{j<k}(i_k - i_j)}{\prod_j (i_j - 1)!} \, ,$$

where $|I| = \sum_j i_j - j$. This formula (2.19) defines $g(I)$ as an alternating function on all sequences of integers if we set $1/l! = 0$ when $l < 0$.

THEOREM 2.8 ([**40**]). *Let* $\alpha^{(a)} \in \mathcal{C}^q_{m,p}$ *and set* $I := J(\alpha^{(a)})$. *Then we have*

$$d(I) \;=\; \sum_{b_1 + \cdots + b_p = 0} g(i_1 + b_1(m+p),\, i_2 + b_2(m+p),\, \ldots,\, i_p + b_p(m+p))\,.$$

Observe that the sum is in fact finite, as only sequences $b_1, b_2, \ldots, b_m$ for which every term $i_j + b_j(m+p)$ is positive contribute to the sum.

PROOF. Let $\delta(I)$ be the function defined by the sum. First observe that if $\max I \le m + p$, then there is only the trivial summand (all $b_i = 0$) and so $\delta(I) = g(I)$. Also, since g is alternating, δ is an alternating function.

We show that the function δ satisfies the conditions of Lemma 2.7, when I is a weakly increasing sequence of non-negative integers with $i_p \le i_1 + m + p$. First, δ satisfies the recursion of Lemma 2.7 because the function g satisfies the recursion. Second, $\delta(1, 2, \ldots, p) = g(1, 2, \ldots, p) = 1$, giving the initial condition. Next, since δ is alternating, it satisfies (2.16). Suppose $i_p = i_1 + m + p$. Then every summand indexed by $b_1, b_2, \ldots, b_p$ with $b_p - 1 = b_1$ vanishes as g is alternating, and every summand with $b_p - 1 \ne b_1$ is paired with another summand indexed by $b_p + 1, b_2, \ldots, b_1 - 1$, which has the same absolute value, but opposite sign, as g is alternating. Thus (2.18) holds for δ. Finally, if $i_1 = 0$, then either $i_p = m + p$ and so $\delta(I) = 0$ or else $i_p < 0$ and so $\delta(I) = g(I) = 0$, giving (2.17) and proving the theorem. $\qquad\square$

We now deduce the formula (1.2) from these results. The quantum Grassmannian $\mathcal{K}^q_{m,p}$ is the maximal quantum Schubert variety $Z_{(m+1,\ldots,m+p)^{(q)}}$. Let $q = pl + r$ with $0 \le r < p$. Set $\alpha := (m+1, \ldots, m+p)$ and $n := m + p$. Then $J(\alpha^{(q)})$ is the sequence

$$(ln + m + r + 1,\, \ldots,\, ln + m + p,\, (l+1)n + m + 1,\, \ldots,\, (l+1)n + m + r)$$

and we have

$$|\alpha^{(q)}| \;=\; |J(\alpha^{(q)})| \;=\; pln + rn + mp \;=\; q(m+p) + mp\,.$$

By Theorem 2.8, the degree $d(J(\alpha^{(q)}))$ of $\mathcal{K}^q_{m,p}$ is the sum over all sequences of integers $b_1, b_2, \ldots, b_p$ satisfying $b_1 + b_2 + \cdots + b_p = 0$ of the terms

$$g((l+b_1)n + m + r + 1,\, \ldots,\, (l+b_{p-r})n + m + p,$$
$$(l+b_{p-r+1}+1)n + m + 1,\, \ldots,\, (l+b_p+1)n + m + r)\,.$$

This term equals

$$(-1)^{r(p-r)}\, g(\nu_1 n + m + 1,\, \nu_2 n + m + 2,\, \ldots,\, \nu_p n + m + p)\,,$$

where

$$(\nu_1, \nu_2, \ldots, \nu_p) \;=\; (l + b_{p-r+1} + 1, \ldots, l + b_p + 1,\, l + b_1, \ldots, l + b_{p-r})\,,$$

and the sign $(-1)^{r(p-r)}$ comes from the resulting permutation of the argument of g. Since $\nu_1 + \cdots + \nu_p = q$, we obtain the formula

$$\deg \mathcal{K}^q_{m,p} \;=\; (-1)^{r(p-r)} \sum_{\nu_1 + \cdots + \nu_p = q} g(\nu_1 n + m + 1, \nu_2 n + m + 2, \ldots, \nu_p n + m + p)\,.$$

Finally, to obtain the formula (1.2), we use Schubert's formula (2.19) for g and the following identity

$$
\begin{aligned}
r(p-r) \;&=\; (q-pl)((l+1)p-q) \;=\; 2pql + pq - q^2 - p^2 l(l+1) \\
&\equiv\; pq + q^2 \mod 2 \;\equiv\; pq + q \mod 2 \,.
\end{aligned}
$$

3. Reality and Transversality

Traditionally, intersection theory and enumerative geometry (both classical and quantum) treat the case of complex solutions to enumerative problems, for it is in this case that the most general and elegant results hold. The real numbers pose special problems as the number of real figures satisfying conditions imposed by general (fixed) real figures depends subtly on the configuration of the fixed real figures. Algebraically closed fields of positive characteristic also pose special problems in enumerative geometry as the number of solutions may depend upon the characteristic of the field. One reason for this is that the solutions may occur with multiplicities; the subvarieties defined by the conditions may not intersect transversally in positive characteristic. In characteristic zero, Kleiman's Theorem on the transversality of a general translate [**32**] may be invoked to show that each solution to many enumerative problems (including that of Question 1) occurs without multiplicities. In positive characteristic, general translates are not necessarily transverse, and other techniques must be employed to determine whether the solutions occur without multiplicity.

For the enumerative problem of Question 1, both these difficulties may be overcome using the same elementary arguments, which are a version of the theory of [**55**] adapted to this particular enumerative problem. These arguments are based upon the Pieri homotopy algorithm of [**28**] and related to a numerical homotopy continuation algorithm for computing numerical solutions to these enumerative problems when $\mathbb{K} = \mathbb{C}$. See [**29**, §5] for details of this algorithm.

THEOREM 3.1 ([**54**, **53**]). *Let $m, p > 1$ and $q \geq 0$ be integers. Set $N :=$ $q(m+p) + mp$. Suppose $\mathbb{K}$ is an infinite field with algebraic closure $\overline{\mathbb{K}}$. Then there exist points $s_1, s_2, \ldots, s_N \in \mathbb{P}^1_{\mathbb{K}}$ and m-planes $L_1, L_2, \ldots, L_N \subset \mathbb{K}^{m+p}$ for which there are exactly $d(m,p;q)$ maps $M \colon \mathbb{P}^1 \to Grass(p, \overline{\mathbb{K}}^{m+p})$ of degree q satisfying*

$$
M(s_i) \cap L_i \;\neq\; \{0\} \quad for\ i = 1, \ldots, N \,.
$$

When $\mathbb{K} = \mathbb{R}$, we may further choose the real points and real m-planes so that all of the resulting maps are real.

Thus the enumerative problem of Question 1 is enumerative in all characteristics and when $\mathbb{K} = \mathbb{R}$, there is some choice of points and m planes for which all of the *a priori* complex solutions are real.

Suppose $\mathbb{K}$ is an infinite field. Let $L \subset \mathbb{K}^{m+p}$ be an m-plane, none of whose Plücker coordinates vanish. That is, if L is the column space of a $m + p$ by m matrix, also written L, then none of the m by m maximal minors of L vanishes. This choice is possible as $\mathbb{K}$ is infinite. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m+p}$ be the distinguished basis of $\mathbb{K}^{m+p}$ corresponding to the rows of such matrices. We equip $\mathbb{K}^{m+p}$ with an action of $\mathbb{K}^\times$. For $s \in \mathbb{K}^\times$, set

$$
(s, \mathbf{e}_i) \;\longmapsto\; s.\mathbf{e}_i \;:=\; s^{m+p-i}\mathbf{e}_i \,.
$$

For $s \in \mathbb{K}^\times$, let $\mathcal{H}(s, L)$ be the hyperplane in Plücker space $\mathbb{P}(\bigwedge^p \mathbb{K}^{m+p} \otimes \mathbb{K}^{q+1})$ whose intersection with the space $\mathcal{M}_{m,p}^q$ of rational curves of degree q consists of those maps $M \colon \mathbb{P} \to Grass(p, \mathbb{K}^{m+p})$ satisfying

$$M(s^{m+p}) \cap s.L \ \neq \ \{0\}.$$

This is just the set of maps satisfying $M(t) \cap K \neq \{0\}$, where $t = s^{m+p}$ and $K = s.L$.

This condition is equivalent to the vanishing of the determinant

$$\det[M(s^{m+p}) : s.L].$$

If we expand this determinant along the columns of $M(s^{m+p})$, we obtain

$$\sum_\alpha M_\alpha(s^{m+p}) \, L_\alpha \, s^{\binom{m}{2}} s^{|\alpha|},$$

where $M_\alpha(s)$ is the αth maximal minor of $M(s)$ and L_α is the appropriately signed complementary maximal minor (Plücker coordinate) of L. If we now expand the polynomials $M_\alpha(s^{m+p})$ in terms of the quantum Plücker coordinates $z_{\alpha^{(a)}}$ of M and divide out the common factor $s^{\binom{m}{2}}$, we obtain the following equation for the hyperplane $\mathcal{H}(s, L)$:

$$\Phi(s, L) \ = \ \sum_{\alpha^{(a)} \in \mathcal{C}_{m,p}^q} z_{\alpha^{(a)}} \, L_\alpha \, s^{|\alpha^{(a)}|},$$

since $a(m + p) + |\alpha| = |\alpha^{(a)}|$.

We prove Theorem 3.1 by first showing that given any m-plane $L \subset \mathbb{K}^{m+p}$ with no vanishing Plücker coordinates, there exist points $s_1, s_2, \ldots, s_N \in \mathbb{K}$ such that there are exactly $d(m, p; q)$ points in $\mathcal{K}_{m,p}^q$ common to the hyperplanes $\mathcal{H}(s_i, L)$, and then argue that none of these $d(m, p; q)$ points lie in the boundary $\mathcal{K}_{m,p}^q - \mathcal{M}_{m,p}^q$.

We first study the boundary of $\mathcal{K}_{m,p}^q$, using results of Bertram [5]. While Bertram works over the complex numbers $\mathbb{C}$, his results we invoke remain valid over any field. A smooth compactification of $\mathcal{M}_{m,p}^q$ is provided by a quot scheme $\mathcal{Q}_{m,p}^q$ [58]. By definition, there is a universal exact sequence

$$0 \ \to \ \mathcal{S} \ \to \ \mathbb{K}^{m+p} \otimes \mathcal{O}_{\mathbb{P}^1 \times \mathcal{Q}_{m,p}^q} \ \to \ \mathcal{T} \ \to 0$$

of sheaves on $\mathbb{P}^1 \times \mathcal{Q}_{m,p}^q$ where $\mathcal{S}$ is a vector bundle of degree $-q$ and rank p. Twisting the determinant of $\mathcal{S}$ by $\mathcal{O}_{\mathbb{P}^1}(q)$ and pushing forward to $\mathcal{Q}_{m,p}^q$ induces a Plücker map

$$(3.1) \qquad\qquad \mathcal{Q}_{m,p}^q \ \to \ \mathbb{P}\left(\textstyle\bigwedge^p \mathbb{K}^{m+p} \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(q))^*\right).$$

The restriction to $\mathcal{M}_{m,p}^q$ is the Plücker map (2.8) and the image is the quantum Grassmannian $\mathcal{K}_{m,p}^q$.

The Plücker map fails to be injective on the boundary $\mathcal{Q}_{m,p}^q - \mathcal{M}_{m,p}^q$ of $\mathcal{Q}_{m,p}^q$. Indeed, Bertram constructs a $\mathbb{P}^{p-1}$ bundle over $\mathbb{P}^1 \times \mathcal{Q}_{m,p}^{q-1}$ that maps onto the boundary of $\mathcal{Q}_{m,p}^q$, with its restriction over $\mathbb{P}^1 \times \mathcal{M}_{m,p}^{q-1}$ an embedding. On this projective bundle, the Plücker map factors through the base $\mathbb{P}^1 \times \mathcal{Q}_{m,p}^{q-1}$ and the image of a point in the base is $s \cdot M$, where s is the section of $\mathcal{O}_{\mathbb{P}^1}(1)$ vanishing at $s \in \mathbb{P}^1$ and M is the image of a point in $\mathcal{Q}_{m,p}^{q-1}$ under its Plücker map.

This identifies the image of the exceptional locus of the Plücker map, which is the boundary of $\mathcal{K}_{m,p}^q$, with the image of $\mathbb{P}^1 \times \mathcal{K}_{m,p}^{q-1}$ in $\mathcal{K}_{m,p}^q$ under a map π which

we now describe. Let $x_{\alpha^{(a)}}$ be the quantum Plücker coordinates for $\mathcal{K}_{m,p}^{q-1}$. Then the boundary of $\mathcal{K}_{m,p}^{q}$ is the image of the map $\pi : \mathbb{P}^1 \times \mathcal{K}_{m,p}^{q-1} \to \mathcal{K}_{m,p}^{q}$ defined by

$$(3.2) \qquad \left([A, B], (x_{\beta^{(b)}} : \beta^{(b)} \in \mathcal{C}_{m,p}^{q-1}) \right) \longmapsto \left(Ax_{\alpha^{(a)}} - Bx_{\alpha^{(a-1)}} : \alpha^{(a)} \in \mathcal{C}_{m,p}^{q} \right),$$

where $x_{\alpha^{(q)}} = x_{\alpha^{(-1)}} = 0$.

For a variety X defined over $\mathbb{K}$, let $X(\overline{\mathbb{K}})$ be the $\overline{\mathbb{K}}$-points of X. Theorem 3.1 is a consequence of the following two theorems.

THEOREM 3.2. *Suppose $L \subset \mathbb{K}^{m+p}$ is an m-plane with no vanishing Plücker coordinates. Then there exist $s_1, s_2, \ldots, s_N \in \mathbb{K}$ so that the intersection*

$$(3.3) \qquad\qquad Z_{\alpha^{(a)}}(\overline{\mathbb{K}}) \cap \mathcal{H}(s_1, L) \cap \mathcal{H}(s_2, L) \cap \cdots \cap \mathcal{H}(s_{|\alpha^{(a)}|}, L)$$

is transverse for any $\alpha^{(a)} \in \mathcal{C}_{m,p}^{q}$.

If $\mathbb{K} = \mathbb{R}$, then we may further choose these numbers $s_1, s_2, \ldots, s_N$ so that for any $\alpha^{(a)} \in \mathcal{C}_{m,p}^{q}$, all points in the intersection (3.3) are real.

THEOREM 3.3. *Suppose $L \subset \mathbb{K}^{m+p}$ is an m-plane with no vanishing Plücker coordinates. If $s_1, s_2, \ldots, s_k \in \mathbb{K}$ are distinct, then for any $\alpha^{(a)} \in \mathcal{C}_{m,p}^{q}$ the intersection*

$$(3.4) \qquad\qquad Z_{\alpha^{(a)}} \cap \mathcal{H}(s_1, L) \cap \mathcal{H}(s_2, L) \cap \cdots \cap \mathcal{H}(s_k, L)$$

is proper in that it has dimension $|\alpha^{(a)}| - k$.

PROOF OF THEOREM 3.1. By Theorem 3.2, there exist $s_1, s_2, \ldots, s_N \in \mathbb{K}$ so that the intersection

$$(3.5) \qquad\qquad \mathcal{K}_{m,p}^{q}(\overline{\mathbb{K}}) \cap \mathcal{H}(s_1, L) \cap \mathcal{H}(s_2, L) \cdots \cap \mathcal{H}(s_N, L)$$

is transverse and consists of exactly $d(m, p; q)$ points, and when $\mathbb{K} = \mathbb{R}$, these points of intersection are real. Furthermore, we may choose these numbers s_i so that their $(m+p)$th powers are distinct. To prove Theorem 3.1, we show that these points all lie in $\mathcal{M}_{m,p}^{q}$. Thus each point in (3.5) represents a map $M : \mathbb{P}^1 \to Grass(p, \mathbb{K}^{m+p})$ of degree q satisfying $M(s_i^{m+p}) \cap s_i.L \neq \{0\}$ for $i = 1, 2, \ldots, N$.

Let $\pi : \mathbb{P}^1 \times \mathcal{K}_{m,p}^{q-1} \to \mathcal{K}_{m,p}^{q}$ be the map (3.2) whose image is the complement of $\mathcal{M}_q$ in $\mathcal{K}_q$. Then

$$\begin{aligned}
\pi^* \Phi(s, L) &= \sum_{\alpha^{(a)} \in \mathcal{C}_{m,p}^{q}} \left(Ax_{\alpha^{(a)}} - Bx_{\alpha^{(a-1)}} \right) L_\alpha\, s^{|\alpha^{(a)}|} \\
&= (A - Bs^{m+p}) \sum_{\beta^{(b)} \in \mathcal{C}_{m,p}^{q-1}} x_{\beta^{(b)}} L_\beta\, s^{|\beta^{(a)}|} \\
&= (A - Bs^{m+p})\, \Phi'(s, L),
\end{aligned}$$

where $\Phi'(s, L)$ is the linear form for $\mathcal{K}_{m,p}^{q-1}$ analogous to $\Phi(s, L)$. Let $\mathcal{H}'(s, L)$ be the hyperplane given by the linear form $\Phi'(s, L)$.

Any point in the intersection (3.5) but not in $\mathcal{M}_{m,p}^{q}$ is the image of a point $([A, B], x)$ in $\mathbb{P}^1 \times \mathcal{K}_{m,p}^{q-1}$ satisfying $\pi^* \Phi(s_i, L) = (A - Bs_i^{m+p}) \Phi'(s_i, L)$ for each $i = 1, 2, \ldots, N$. As the $(m+p)$th powers of the s_i are distinct, such a point can satisfy $A - Bs_i^{m+p} = 0$ for at most one i. Thus $x \in \mathcal{K}_{m,p}^{q-1}$ lies in at least $N-1$ of the hyperplanes $\mathcal{H}'(s_i, L)$. Since $N-1$ exceeds the dimension $N-(m+p)$ of $\mathcal{K}_{m,p}^{q-1}$, there are no such points $x \in \mathcal{K}_{m,p}^{q-1}$, by Theorem 3.3 applied to maps of degree $q-1$. $\quad\square$

PROOF OF THEOREM 3.3. For any $s_1, s_2, \ldots, s_k$, the intersection (3.4) has dimension at least $|\alpha^{(a)}| - k$. We show it has at most this dimension, if $s_1, s_2, \ldots, s_k$ are distinct.

Suppose $k = |\alpha^{(a)}| + 1$ and let $z \in Z_{\alpha^{(a)}}$. Then $z_{\beta^{(b)}} = 0$ if $\beta^{(b)} \not\le \alpha^{(a)}$ and so the form $\Phi(s, L)$ defining $\mathcal{H}(s, L)$ evaluated at z is

$$\sum_{\beta^{(b)} \le \alpha^{(a)}} z_{\beta^{(b)}} \, L_\beta \, s^{|\beta^{(b)}|} \, .$$

This is a non-zero polynomial in s of degree at most $|\alpha^{(a)}|$ and thus it vanishes for at most $|\alpha^{(a)}|$ distinct values of s. It follows that (3.4) is empty for $k > |\alpha^{(a)}|$.

If $k \le |\alpha^{(a)}|$ and $s_1, s_2, \ldots, s_k$ are distinct, but (3.4) has dimension exceeding $|\alpha^{(a)}| - k$, then we may complete $s_1, s_2, \ldots, s_k$ to a set of distinct numbers $s_1, s_2, \ldots, s_{|\alpha^{(a)}|+1}$ which give a non-empty intersection in (3.4), a contradiction. $\square$

PROOF OF THEOREM 3.2. We prove both parts of the theorem simultaneously, making note of the differences when $\mathbb{K} = \mathbb{R}$.

We construct the sequence s_i inductively. The unique element of rank 1 in $\mathcal{C}^q_{m,p}$ is $\alpha^{(0)}$, where α is the sequence $1 < 2 < \cdots < p-1 < p+1$. The quantum Schubert variety $Z_{\alpha^{(0)}}$ is a line in Plücker space. Indeed, it is isomorphic to the set of p-planes containing a fixed $(p-1)$-plane and lying in a fixed $(p+1)$-plane. By Theorem 3.3 or direct observation, $Z_{\alpha^{(0)}} \cap \mathcal{H}(s, L)$ is then a single point, for any non-zero s. When $\mathbb{K} = \mathbb{R}$, this point is real. Let $s_1 \in \mathbb{K}^\times$ be arbitrary.

Suppose $s_1, s_2, \ldots, s_k \in \mathbb{K}$ are distinct points with the property that for any $\beta^{(b)}$ with $|\beta^{(b)}| = k$,

$$Z_{\beta^{(b)}} \cap \mathcal{H}(s_1, L) \cap \mathcal{H}(s_2, L) \cap \cdots \cap \mathcal{H}(s_{|\beta^{(b)}|}, L)$$

is transverse. When $\mathbb{K} = \mathbb{R}$, we suppose further that all points of intersection are real.

Let $\alpha^{(a)}$ be an index with $|\alpha^{(a)}| = k + 1$ and consider the 1-parameter family $\mathcal{Z}(s)$ of schemes defined by $Z_{\alpha^{(a)}} \cap \mathcal{H}(s, L)$, for $s \in \mathbb{K}^\times$. If we restrict the form $\Phi(s, L)$ to $z \in Z_{\alpha^{(a)}}$, then we obtain

$$\sum_{\beta^{(b)} \le \alpha^{(a)}} z_{\beta^{(b)}} \, L_\beta \, s^{|\beta^{(b)}|} \, ,$$

a polynomial in s with leading term $z_{\alpha^{(a)}} \, L_\alpha \, s^{|\alpha^{(a)}|}$. Since the Plücker coordinate L_α is non-zero, $\mathcal{Z}(\infty) \subset Z_{\alpha^{(a)}}$ is defined by $z_{\alpha^{(a)}} = 0$, and so $\mathcal{Z}(\infty)$ equals

$$Z_{\alpha^{(a)}} \cap \mathcal{H}_{\alpha^{(a)}} \;=\; \bigcup_{\beta^{(b)} \lessdot \alpha^{(a)}} Z_{\beta^{(b)}} \, ,$$

by Proposition 2.5 (ii).

Claim: The cycle

$$\mathcal{Z}(\infty) \cap \mathcal{H}(s_1, L) \cap \mathcal{H}(s_2, L) \cap \cdots \cap \mathcal{H}(s_k, L)$$

is free of multiplicities.

If not, then there are two components $Z_{\beta^{(b)}}$ and $Z_{\gamma^{(c)}}$ of $\mathcal{Z}(\infty)$ such that

$$Z_{\beta^{(b)}} \cap Z_{\gamma^{(c)}} \cap \mathcal{H}(s_1, L) \cap \mathcal{H}(s_2, L) \cap \cdots \cap \mathcal{H}(s_k, L)$$

is non-empty. But this contradicts Theorem 3.3, as $Z_{\beta^{(b)}} \cap Z_{\gamma^{(c)}} = Z_{\delta^{(d)}}$, where $\delta^{(d)}$ is the greatest lower bound of $\beta^{(b)}$ and $\gamma^{(c)}$ in $\mathcal{C}_{m,p}^q$, and so $\dim Z_{\delta^{(d)}} < \dim Z_{\beta^{(b)}} = k$.

Because the intersection of $\mathcal{Z}(\infty)$, the fibre of $\mathcal{Z}$ at infinity, with the cycle $\mathcal{H}(s_1, L) \cap \mathcal{H}(s_2, L) \cap \cdots \cap \mathcal{H}(s_k, L)$ is zero dimensional and free of multiplicities, it is transverse, and so the general fibre of $\mathcal{Z}$ meets $\mathcal{H}(s_1, L) \cap \mathcal{H}(s_2, L) \cap \cdots \cap \mathcal{H}(s_k, L)$ transversally. Thus there is a non-empty Zariski open subset $O_{\alpha^{(a)}}$ of $\mathbb{A}_{\mathbb{K}}^1$ consisting of points s for which

$$Z_{\alpha^{(a)}} \cap \mathcal{H}(s_1, L) \cap \mathcal{H}(s_2, L) \cap \cdots \cap \mathcal{H}(s_k, L) \cap \mathcal{H}(s, L)$$

is transverse. Choose s_{k+1} to be any point common to all $O_{\alpha^{(a)}}$ for $|\alpha^{(a)}| = k + 1$.

When $\mathbb{K} = \mathbb{R}$, the claim implies there is a real number $\mathcal{N}_{\alpha^{(a)}} > 0$ such that if $s > \mathcal{N}_{\alpha^{(a)}}$, then

$$\mathcal{Z}(s) \cap \mathcal{H}(s_1, L) \cap \cdots \cap \mathcal{H}(s_k, L)$$

is transverse with all points of intersection real. Set

$$\mathcal{N}_{k+1} := \max\{\mathcal{N}_{\alpha^{(a)}} : |\alpha^{(a)}| = k + 1\}$$

and let s_{k+1} be any real number satisfying $s_{k+1} > \mathcal{N}_{k+1}$.   □

REMARK 3.4. While these results rely upon work from systems theory, the result when $\mathbb{K} = \mathbb{R}$ unfortunately does not give any insight into the dynamic pole placement problem: In the dynamic pole placement problem, the planes $L(s)$ lie on a rational curve $L(s)$ of degree $mp + q(m + p) - q$ while the planes $s_i.L$ of Theorem 3.2 lie on the rational curve $s.L$, which has degree mp. Thus there is overlap only when $q = 0$, which is the static pole placement problem.

REMARK 3.5. Theorem 3.2 proves reality and transversality for the enumerative problem of Question 1. There are more general enumerative problems involving rational curves on a Grassmannian obtained by replacing the Schubert condition $M(s) \cap L \neq \{0\}$ with more general Schubert conditions. It is not known, but is expected, that the transversality and reality properties established in Theorem 3.2 for the enumerative problem of Question 1 hold also for these more general enumerative problems.

REMARK 3.6. The argument given at the end of the Proof of Theorem 3.1, that the intersection (3.5) contains no points in the boundary and hence lies in $\mathcal{M}_{m,p}^q$, may be generalized to show that when $a = q$, the intersection (3.3) similarly lies in $\mathcal{M}_{m,p}^q$.

4. Equations for the Quantum Grassmannian

In Section 3, we solved the enumerative problem of Question 1 by arguing directly from the equations describing the conditions (1.1). This is an unusual feature of that enumerative problem: despite the fact that algebraic geometry is ostensibly concerned with solutions to polynomial equations, enumerative geometric problems are not typically solved in this manner. What is more unusual is that this enumerative problems admits a second solution also based upon equations, in this case equations for the quantum Grassmannian.

We first argue that the number of solutions to the enumerative problem is the degree $\deg \mathcal{K}_{m,p}^q$ of the quantum Grassmannian, and then use the form of a Gröbner basis for the ideal $\mathcal{I}_{m,p}^q = \mathcal{I}(\mathcal{K}_{m,p}^q)$ of the quantum Grassmannian to give another

proof of Theorem 2.6, that $\deg \mathcal{K}^q_{m,p}$ is the number of maximal chains in the poset $\mathcal{C}^q_{m,p}$ of quantum Plücker coordinates.

4.1. The enumerative problem of Question 1. Given an m-plane $L \subset \mathbb{K}^{m+p}$ and a point $s \in \mathbb{P}^1$, the set of degree q maps $M \in \mathcal{M}^q_{m,p}$ which satisfy

(4.1) $$M(s) \cap L \neq \{0\}$$

is a hyperplane section of $\mathcal{M}^q_{m,p}$ in its Plücker embedding. This was shown both in Section 2.3 and, for special versions of (4.1), in Section 3. Thus the enumerative problem of Question 1 asks for the number of points common to $\mathcal{M}^q_{m,p}$ and to $N := q(m+p) + mp (= \dim \mathcal{M}^q_{m,p})$ hyperplanes. Hence if there are finitely many solutions, their number is bounded by $\deg \mathcal{K}^q_{m,p}$ and it equals this degree if there are no points in the boundary $\mathcal{K}^q_{m,p} - \mathcal{M}^q_{m,p}$ common to all the hyperplanes.

Given a point $s \in \mathbb{P}^1$, the *evaluation map* $\mathrm{ev}_s \colon \mathcal{M}^q_{m,p} \to Grass(p, \mathbb{K}^{m+p})$ associates a map M to the p-plane $M(s)$. The evaluation map extends to the quantum Grassmannian. One way to see this is that the evaluation map is defined on the Quot scheme $\mathcal{Q}^q_{m,p}$ and it factors through the Plücker map $\mathcal{Q}^q_{m,p} \to \mathcal{K}^q_{m,p}$ [**5**]. Concretely, points M of $\mathcal{K}^q_{m,p}$ are represented (possibly non-uniquely) by matrices of homogeneous forms whose minors are forms of degree q, and a point is on the boundary if the minors are not relatively prime. The (classical) Plücker coordinates of $\mathrm{ev}_s(M) \in Grass(p, \mathbb{K}^{m+p})$ are given by first dividing each minor by the common polynomial factor, and then evaluating at the point $s \in \mathbb{P}^1$.

Since $Grass(p, \mathbb{K}^{m+p})$ is a homogeneous space, Kleiman's Properness Theorem [**32**, Theorem 2(i)] shows that for distinct points $s_1, s_2, \ldots, s_N \in \mathbb{P}^1$ and general m-planes $L_1, L_2, \ldots, L_N \in \mathbb{K}^{m+p}$, the collection of hyperplanes given by

$$M(s_i) \cap L_i \neq \{0\}$$

meet the boundary properly. Since the boundary has dimension $N - m - p + 1$, there are no points common to the boundary and these hyperplanes.

Again by Kleiman's Properness Theorem, there are finitely many points M in $\mathcal{K}^q_{m,p}$ (and hence in $\mathcal{M}^q_{m,p}$) common to these hyperplanes, as N is the dimension of $\mathcal{K}^q_{m,p}$. Since this set is a particular complementary linear section of $\mathcal{K}^q_{m,p}$, its number of points (possibly counted with multiplicity) is $\deg \mathcal{K}^q_{m,p}$. When $\mathbb{K}$ is algebraically closed of characteristic zero, Kleiman's Transversality Theorem [**32**, Theorem 2(ii)] implies that the solutions appear without multiplicity, and so $\deg \mathcal{K}^q_{m,p}$ solves the enumerative problem of Question 1.

4.2. The degree of $\mathcal{K}^q_{m,p}$ via Gröbner bases. For basics on Gröbner bases, we recommend either [**15**] or [**60**], whose Chapter 11 has a description of the classical $q = 0$ version of the results discussed here. Let $\mathbb{K}[\mathcal{C}^q_{m,p}]$ be the ring generated by the quantum Plücker coordinates $z_{\alpha^{(a)}}$ for $\alpha^{(a)} \in \mathcal{C}^q_{m,p}$, the coordinate ring of the Plücker space $\mathbb{P}(\bigwedge^p \mathbb{K}^{m+p} \otimes \mathbb{K}^{q+1})$. Let $\prec$ be the degree reverse lexicographic term order on ring $\mathbb{K}[\mathcal{C}^q_{m,p}]$ induced by an ordering of the variables $z_{\alpha^{(a)}}$ corresponding to any (fixed) linear extension of the poset $\mathcal{C}^q_{m,p}$. The poset $\mathcal{C}^q_{m,p}$ is in fact a distributive lattice, with $\alpha^{(a)} \wedge \beta^{(b)}$ the meet (greatest lower bound) and $\alpha^{(a)} \vee \beta^{(b)}$ the join (least upper bound) of the indices $\alpha^{(a)}$ and $\beta^{(b)}$.

THEOREM 4.1 ([**56**]). *The reduced Gröbner basis of the Plücker ideal $\mathcal{I}^q_{m,p}$ of the quantum Grassmannian $\mathcal{K}^q_{m,p}$ consists of quadratic polynomials in $\mathbb{K}[\mathcal{C}^q_{p,m}]$*

which are indexed by pairs of incomparable variables $\gamma^{(c)}, \delta^{(d)}$ *in the poset* $\mathcal{C}^q_{p,m}$,

$$S(\gamma^{(c)}, \delta^{(d)}) \quad = \quad z_{\gamma^{(c)}} \cdot z_{\delta^{(d)}} \;-\; z_{\gamma^{(c)} \vee \delta^{(d)}} \cdot z_{\gamma^{(c)} \wedge \delta^{(d)}} \;+\; \textit{lower terms in} \prec,$$

and all lower terms $\lambda z_{\alpha^{(a)}} z_{\beta^{(b)}}$ *of* $S(\gamma^{(c)}, \delta^{(d)})$ *satisfy* $\alpha^{(a)} < \gamma^{(c)} \wedge \delta^{(d)}$ *and* $\gamma^{(c)} \vee \delta^{(d)} < \beta^{(b)}$.

By Theorem 4.1, the initial ideal $\mathrm{in}_\prec(\mathcal{I}^q_{m,p})$ of the Plücker ideal is generated by all monomials $z_{\alpha^{(a)}} z_{\beta^{(b)}}$ with $\alpha^{(a)}, \beta^{(b)} \in \mathcal{C}^q_{m,p}$ incomparable. We write this initial ideal as an intersection of prime ideals. For this, let $Ч^q_{m,p}$ be the set of (saturated) chains in the poset $\mathcal{C}^q_{m,p}$.

LEMMA 4.2. *Let* $\mathcal{I}^q_{m,p}$ *be the Plücker ideal. Then*

$$\mathrm{in}_\prec(\mathcal{I}^q_{m,p}) \;=\; \bigcap_{ч \in Ч^q_{m,p}} \langle z_{\delta^{(d)}} \mid \delta^{(d)} \notin ч \rangle.$$

PROOF. If $z_{\alpha^{(a)}} z_{\beta^{(b)}}$ is a generator of $\mathrm{in}_\prec(\mathcal{I}^q_{m,p})$, then $\alpha^{(a)}$ and $\beta^{(b)}$ are incomparable in the poset $\mathcal{C}^q_{m,p}$. Thus if $ч \in Ч^q_{m,p}$ is a saturated chain, at most one of $\alpha^{(a)}$ or $\beta^{(b)}$ lies in the chain $ч$, and so $z_{\alpha^{(a)}} z_{\beta^{(b)}}$ lies in the ideal $\langle z_{\delta^{(d)}} \mid \delta^{(d)} \notin ч \rangle$.

Suppose now that z is a monomial not in the initial ideal $\mathrm{in}_\prec(\mathcal{I}^q_{m,p})$. Then the variables appearing in z have indices which are comparable in the poset $\mathcal{C}^q_{m,p}$. Thus we may write $z = z_{\alpha^{(a)}} \cdot z_{\beta^{(b)}} \cdots z_{\gamma^{(c)}}$ with $\alpha^{(a)} \leq \beta^{(b)} \leq \cdots \leq \gamma^{(c)}$. There is some chain $ч \in Ч^q_{m,p}$ containing the indices $\alpha^{(a)}, \beta^{(b)}, \ldots, \gamma^{(c)}$ and so the monomial z does not lie in the ideal $\langle z_{\delta^{(d)}} \mid \delta^{(d)} \notin ч \rangle$. This proves the equality of the two monomial ideals. □

Each ideal $\langle z_{\delta^{(d)}} \mid \delta^{(d)} \notin ч \rangle$ defines the coordinate subspace of Plücker space spanned by the coordinates $z_{\alpha^{(a)}}$ with $\alpha^{(a)} \in ч$, which is isomorphic to $\mathbb{P}^{q(m+p)+mp}$, as every maximal chain $ч$ of $\mathcal{C}^q_{m,p}$ has length $q(m+p) + mp + 1$. Thus the zero scheme of $\mathrm{in}_\prec(\mathcal{I}^q_{m,p})$ is the union of these coordinate subspaces, and so it has degree equal to their number.

ALTERNATIVE PROOF OF THEOREM 2.6. The degree of the quantum Grassmannian is the degree of its ideal $\mathcal{I}^q_{m,p}$. By Macaulay's Theorem [**35**] (see also [**18**, §1.10]), this is the degree of the initial ideal, $\deg(\mathrm{in}_\prec(\mathcal{I}^q_{m,p}))$, which is equal to the number of chains in $Ч^q_{m,p}$, by Lemma 4.2. □

REMARKS 4.3.
(1) In [**56**], reduced Gröbner bases for the quantum Schubert varieties $Z_{\alpha^{(a)}}$ which are restrictions of the Gröbner basis of Theorem 4.1 are also constructed, and a consequence is that the definition (2.10) is in fact ideal-theoretic:

$$\mathcal{I}(Z_{\alpha^{(a)}}) \;=\; \mathcal{I}^q_{m,p} + \langle z_{\beta^{(b)}} \mid \beta^{(b)} \not\leq \alpha^{(a)} \rangle.$$

The form of these Gröbner bases also significantly strengthens Proposition 2.5 (ii) to the level of homogeneous ideals.
(2) The reduced Gröbner basis for the Plücker ideal of the classical Grassmannian ($q = 0$) may be constructed as follows [**27**, **59**]: First a Gröbner basis consisting of linearly independent quadratic polynomials, one for each incomparable pair, is constructed using invariant theory. Then this

basis is reduced to obtain the desired reduced Gröbner basis. In contrast to that approach, the reduced Gröbner basis of Theorem 4.1 was constructed explicitly using a double induction on the poset $\mathcal{C}_{m,p}^q$.

For $\alpha^{(a)} \geq \beta^{(b)}$ in $\mathcal{C}_{m,p}^q$, define the skew quantum Schubert variety

$$Z_{\alpha^{(a)}/\beta^{(b)}} \quad := \quad \{(z_{\gamma^{(c)}}) \in \mathcal{K}_{m,p}^q \colon z_{\gamma^{(c)}} = 0 \ \text{ unless } \ \alpha^{(a)} \geq \gamma^{(c)} \geq \beta^{(b)}\}\,.$$

A first step is to show that if $\alpha^{(a)} = \gamma^{(c)} \vee \delta^{(d)}$, then

$$z_{\gamma^{(c)}} \cdot z_{\delta^{(d)}} \ - \ z_{\gamma^{(c)} \vee \delta^{(d)}} \cdot z_{\gamma^{(c)} \wedge \delta^{(d)}} \ \in \ \mathcal{I}(Z_{\alpha^{(a)}/\beta^{(b)}})\,,$$

by downward induction on $\beta^{(b)}$. When $\beta^{(b)} \in \mathcal{C}_{m,p}^q$ is minimal, $Z_{\alpha^{(a)}/\beta^{(b)}} = Z_{\alpha^{(a)}}$. Then the forms $S(\gamma^{(c)}, \delta^{(d)})$ of Theorem 4.1 are constructed by increasing induction on $\alpha^{(a)}$.

(3) An important part of [**56**] was to study the rational parameterization of $\mathcal{K}_{m,p}^{pn}$ given by $m+p$ by p matrices whose entries are generic polynomials of degree n, and also by an intermediate variety, the Grassmannian of p-planes in $\mathbb{K}^{(pn+1)(m+p)}$. This 'long Grassmannian' was used by Byrnes [**11**] to obtain a different compactification of $\mathcal{M}_{m,p}^q$ than $\mathcal{K}_{m,p}^q$. It was also used to prove Proposition 2.5(ii) [**40, 41**], and the indices of its Schubert varieties appeared implicitly in the indexing scheme of Section 2.5. Lastly, the classical (ideal-theoretic) version of Proposition 2.5(ii) for Schubert varieties in the long Grassmannian was used in the inductive steps of item (2) above.

(4) We expect this approach and these results to generalize to other flag manifolds, giving an analog of standard monomial theory [**34**] for spaces of rational curves in all flag manifolds.

5. Quantum Cohomology and the Formula of Vafa and Intriligator

We describe some of the standard story of the enumerative problem of Question 1.We first briefly review some history of the formula of Vafa and Intriligator. Next, we visit the classical cohomology ring of the Grassmannian and its quantum deformation, and then give the formula of Vafa and Intriligator. We then show how this formula of Vafa and Intriligator agrees with the formula (1.2) of Ravi, Rosenthal, and Wang. We next give an alternative way to view the quantum cohomology ring of the Grassmannian, and discusses how this same ring arose in two different contexts in representation theory. This survey concludes with some open problems concerning quantum Littlewood-Richardson coefficients.

Inspired by Donaldson's invariants of 4-manifold [**17**], Gromov [**24**] proposed that topological invariants of moduli spaces of pseudo-holomorphic curves in a symplectic manifold X would give invariants of the symplectic structure of X. Following ideas of Witten [**66**], Vafa [**62**] proposed so-called quantum multiplications in the cohomology rings of symplectic manifolds with structure constants certain correlation functions, and conjectured remarkable residue formulae for these correlation functions when X is a Grassmannian. This was made more precise by Intriligator [**30**]. Ruan (see [**46**]) was perhaps the first to link this work in theoretical physics to the work of Gromov, realizing that Witten's correlation functions were in fact Gromov's invariants, and hence the formula of Vafa and Intriligator computes intersection numbers of curves of all genera on Grassmannians. Siebert

and Tian [**51**] generalized the program of Vafa and Intriligator from the Grass-
mannian to certain Fano manifolds—in particular, they proved the formula of Vafa
and Intriligator and constructed the (small) quantum cohomology rings of these
manifolds. Previously (and with different methods), Bertram, Daskalopoulos, and
Wentworth [**7**] had proven this formula for genus 1 invariants of high degree curves
in Grassmannians of 2-planes, and Bertram [**5**] later developed a quantum Schubert
calculus which enabled the computation of intersection numbers involving arbitrary
Schubert conditions.

5.1. The cohomology ring of the Grassmannian. The cohomology ring
of the complex Grassmannian $Grass(p, \mathbb{C}^{m+p})$ has a standard presentation

$$(5.1) \qquad H^*(Grass(p, \mathbb{C}^{m+p})) \xrightarrow{\ \sim\ } \mathbb{C}[c_1, c_2, \ldots, c_p]/\langle h_{m+1}, h_{m+2}, \ldots, h_{m+p}\rangle \,,$$

where $\deg c_i = 2i$ and $h_1, h_2, \ldots, h_{m+p}$ are defined recursively in terms of the c_i as
follows

$$(5.2) \qquad\qquad h_j - c_1 h_{j-1} + \cdots + (-1)^{j-1} c_{j-1} h_1 + (-1)^j c_j \ = \ 0 \,,$$

with $c_i = 0$ for $i > p$. The isomorphism is given by associating c_i to the ith Chern
class of the dual S^* of the tautological rank p subbundle S over the Grassmannian.
Then h_i is the ith Chern class of the rank m quotient bundle Q, and these classes
vanish for $i > m$. The relation (5.2) between these classes c_i and h_i is succinctly
expressed via the splitting principle,

$$1 \ = \ c(\mathbb{C}^{m+p}) \ = \ c(S^*)c(Q^*) \,,$$

where $\mathbb{C}^{m+p}$ is the trivial bundle and $c(\cdot)$ is the total Chern class. (Here, $c(Q^*) =
1 - h_1 + h_2 - \cdots$.)

Because the cohomology ring is a complete intersection and the h_j are homoge-
neous of degree $2j$, it is Gorenstein with socle in dimension $2mp = \sum_j (\deg h_{m+j} -
\deg c_j)$. A generator of the socle is the image of c_p^m, and the degree map (used
to compute intersection numbers) is simply the coefficient of c_p^m in an element of
this quotient ring. Thus, given some classes $\xi_1, \xi_2, \ldots, \xi_l$ in cohomology which are
Poincaré dual to cycles $X_1, X_2, \ldots, X_l$ in general position, the coefficient of c_p^m in
the product $\xi_1 \cdot \xi_2 \cdots \xi_l$ is the number of points in the intersection

$$X_1 \cap X_2 \cap \cdots \cap X_l \,,$$

when there are finitely many such points.

What is less known is that the degree map may be computed using the local
residue associated to the map $\mathbf{H} := ((-1)^{p-1} h_{m+1}, (-1)^{p-2} h_{m+2}, \ldots, h_{m+p}) \colon \mathbb{C}^p \to
\mathbb{C}^p$. (See [**51**, §4] for details.) This residue is

$$\mathrm{Res}_{\mathbf{H}}(F) \ = \ \frac{1}{(2\pi i)^p} \int_{\Gamma^\epsilon} \frac{F}{(-1)^{p-1} h_{m+1} (-1)^{p-2} h_{m+2} \cdots h_{m+p}} dc_1 \cdot dc_2 \cdots dc_p \,,$$

for $F \in \mathbb{C}[c_1, c_2, \ldots, c_p]$. Here Γ^ϵ is a smooth canonically oriented cycle in the region
where no component h_{m+i} of $\mathbf{H}$ vanishes. Standard properties of residues [**23**, §5]
imply that the residue vanishes on the ideal of (5.1), and so gives a well-defined
map on the cohomology ring. Furthermore, when F is homogeneous, the residue
vanishes unless $\deg F = 2mp$, for otherwise the form is exact. Thus the residue is
proportional to the degree map, and the calculations we do below show the constant
of proportionality is $(-1)^{\binom{p}{2}}$.

The presentation (5.1) has another form. Let $W = \frac{1}{m+p+1}P_{m+p+1}$, where P_{m+p+1} is the $(m+p+1)$th Newton power sum symmetric polynomial in the variables $x_1, \ldots, x_p$. If we express P_{m+p+1} as a polynomial in the elementary symmetric polynomials $c_1, c_2, \ldots, c_p$, then we have (see below)

$$(5.3) \qquad (-1)^{1+j}h_{m+p+1-j} = \frac{\partial W}{\partial c_j},$$

where h_i is the ith complete homogeneous symmetric polynomial in the variables x_j (these satisfy (5.2) when the c_i are elementary symmetric polynomials). Thus the presentation becomes

$$H^*(Grass(p, \mathbb{C}^{m+p})) \xrightarrow{\sim} \mathbb{C}[c_1, c_2, \ldots, c_p]/\langle dW = 0\rangle.$$

We derive (5.3), working in the ring Λ of symmetric functions in the indeterminates $x_1, x_2, \ldots$ [**57**, Sect. 7] [**36**, I.2] [**48**]. To obtain this formula for polynomials, specialize x_i to 0 for $i > p$. First, we have the fundamental relations

$$(5.4) \qquad 1 = \left(\sum_{r\geq 0} h_r(-t)^r\right) \cdot \left(\sum_{r\geq 0} e_r t^r\right),$$

$$(5.5) \qquad \sum_{r > 0} p_r(-t)^{r-1} = \frac{d}{dt}\log\left(\sum_{r\geq 0} e_r t^r\right),$$

where e_i, h_i, and p_i are, respectively, the elementary, complete homogeneous, and power sum symmetric functions of degree i. (Note that (5.4) gives (5.2).) Differentiating (5.5) with respect to e_j gives

$$\sum_{r > 0}\frac{\partial p_r}{\partial e_j}(-t)^{r-1} = \frac{d}{dt}\frac{t^j}{\sum_{r\geq 0}e_r t^r} = \frac{d}{dt}t^j\sum_{r\geq 0}h_r(-t)^r.$$

Equating coefficients of t^{m+p} gives

$$(-1)^{m+p}\frac{\partial p_{m+p+1}}{\partial e_j} = (-1)^{m+p+1-j}(m+p+1)h_{m+p+1-j},$$

from which (5.3) follows.

5.2. Quantum cohomology and the formula of Vafa and Intriligator. The quantum cohomology ring is a perturbation (depending on a Kähler form) of the classical cohomology ring whose structure encodes the genus zero Gromov-Witten invariants. For the Grassmannian, Vafa and Intriligator began with the perturbation of W

$$QW := W + (-1)^p\beta c_1,$$

where β is a complex number associated to the perturbing Kähler form. For our enumerative problem, $\beta = 1$. They then proposed the following presentation for the quantum cohomology ring

$$QH^*(Grass(p, \mathbb{C}^{m+p})) = \mathbb{C}[c_1, c_2, \ldots, c_p]/\langle dQW = 0\rangle$$

which is

$$(5.6) \qquad \mathbb{C}[c_1, c_2, \ldots, c_p]/\langle(-1)^{p-1}h_{m+1}, (-1)^{p-2}h_{m+2}, \ldots, h_{m+p} + (-1)^p\beta\rangle.$$

They also proposed the following formula. Let $X_1, X_2, \ldots, X_l$ be special Schubert cycles in the Grassmannian which are in general position. Suppose $\beta = 1$. For a genus $g \geq 0$, set $\langle X_1, X_2, \ldots, X_l \rangle_g := 0$ unless the sum of the codimensions of the X_j is equal to $d(m + p) + mp(1 - g)$, for some non-negative integer d. When there is such an integer d, let the Gromov-Witten invariant $\langle X_1, X_2, \ldots, X_l \rangle_g$ be the number of maps

$$f : (\Sigma, s_1, s_2, \ldots, s_l) \longrightarrow Grass(p, \mathbb{C}^{m+p})$$

satisfying $f(s_i) \in X_i$. Here Σ is a fixed genus g curve, $s_1, s_2, \ldots, s_l$ are fixed, but general, points of Σ, and $f_*[\Sigma] = d \cdot c_1$. Determining when this definition is well-founded and providing a satisfactory alternative when it is not is an important and subtle story which we do not relate. When $\beta \neq 1$, the definition involves pseudo-holomorphic curves, and we omit it.

Suppose we have special Schubert classes $c_{i_1}, c_{i_2}, \ldots, c_{i_l}$ with c_{i_j} Poincaré dual to X_j. Then the formula of Vafa and Intriligator for $\langle X_1, X_2, \ldots, X_l \rangle_g$ is

$$(5.7) \qquad (-1)^{\binom{p}{2}(g-1)} \sum_{dQW(c_1, c_2, \ldots, c_p) = 0} \det\left(\frac{\partial^2 QW}{\partial c_i \partial c_j}\right)^{g-1} \cdot c_{i_1} \cdot c_{i_2} \cdots c_{i_l} \, .$$

One remarkable feature of this formula involves the determinant $J = \det\left(\frac{\partial^2 QW}{\partial c_i \partial c_j}\right)$. The formula implies that the genus g Gromov-Witten invariant of a monomial $c^{\mathbf{i}}$ equals the genus $g - 1$ Gromov-Witten invariant of $c^{\mathbf{i}}/J$ (up to a sign).

We relate this to the classical intersection formula when $g = 0$. Since dQW is the vector $((-1)^{p-1}h_{m+1}, \ldots, -h_{m+p-1}, h_{m+p} + (-1)^p \beta)$, the determinant J in the formula (5.7) is also the Jacobian of the map $\mathbf{H}$, and so the summand of (5.7) becomes

$$\sum_{c \in \mathbf{H}^{-1}(y)} \frac{F(c)}{J} \, ,$$

where $y = (0, \ldots, 0, (-1)^{p+1}\beta)$ and $F(c) = c_{i_1} \cdot c_{i_2} \cdots c_{i_l}$. Let $\mathrm{res}_y(F)$ denote this number, which is a trace but also a residue as y is a regular value of the map $\mathbf{H}$. A further property of the residue is that $\mathrm{res}_y(F)$ extends holomorphically to a neighborhood of 0, and

$$\lim_{y \to 0} \mathrm{res}_y(F) \; = \; \mathrm{Res}_{\mathbf{H}}(F) \, .$$

This shows rather explicitly how this formula of Vafa and Intriligator is a deformation of the classical intersection formula.

5.3. Relation between the formulae of Vafa and Intriligator and of Ravi, Rosenthal, and Wang. We relate the formula (5.7) for genus 0 curves to the formula (1.2) of Ravi, Rosenthal, and Wang. For $\alpha \in \binom{[n]}{p}$, let $\Omega_\alpha = Z_{\alpha^{(0)}}$, a Schubert subvariety of $Grass(p, \mathbb{C}^{m+p})$.

THEOREM 5.1 ([**41**]). *Let $\alpha^{(a)} \in \mathcal{C}^q_{m,p}$. Then*

$$\deg Z_{\alpha^{(a)}} \; = \; (-1)^{\binom{p}{2}} \sum_{dQW = 0} \frac{\sigma_{\alpha^\vee} \cdot c_1^{|\alpha^{(a)}|}}{J} \, ,$$

where $\sigma_{\alpha^\vee}$ is the cohomology class Poincaré dual to the fundamental cycle of Ω_α.

Proof. Since the Gromov-Witten invariants of genus zero curves on the Grassmannian may be computed in the quantum cohomology ring of the Grassmannian, the obvious linear extension of the formula of Vafa and Intriligator (5.7) for genus zero curves to arbitrary cycles X_i is valid. Thus the right hand side above computes the Gromov-Witten invariant

$$(5.8) \qquad \langle \Omega_\alpha, \, X_1, \, X_2, \, \ldots, \, X_{|\alpha^{(a)}|} \rangle_0 \, ,$$

where $X_1, X_2, \ldots, X_{|\alpha^{(a)}|}$ are special Schubert varieties in general position, each dual to c_1. Since the cohomological degree of the class $\sigma_{\alpha^\vee} \cdot c_1^{|\alpha^{(a)}|}$ is

$$\begin{aligned} 2(mp - |\alpha|) + 2|\alpha^{(a)}| \quad &= \quad 2mp - 2|\alpha| + 2(a(m+p) + |\alpha|) \\ &= \quad 2a(m+p) + 2mp \, , \end{aligned}$$

this is an invariant of degree a curves.

We first express this Gromov-Witten invariant as the number of points in an intersection. Given a point $s \in \mathbb{P}^1$, the evaluation map $\mathrm{ev}_s \colon \mathcal{M}_{m,p}^a \to Grass(P, \mathbb{C}^{m+p})$ associates a curve M to the p-plane $M(s)$. Each cycle X_i has the form

$$\Omega_{L_i} \;=\; \{ H \in Grass(p, \mathbb{C}^{m+p}) \mid H \cap L_i \neq \{0\} \} \, ,$$

where $L_1, \ldots, L_{|\alpha^{(a)}|}$ are m-planes in $\mathbb{C}^{m+p}$ in general position. Thus (5.8) counts the number of points in the intersection

$$(5.9) \qquad \mathrm{ev}_{s_0}^{-1} \Omega_\alpha \bigcap \mathrm{ev}_{s_1}^{-1} \Omega_{L_1} \bigcap \cdots \bigcap \mathrm{ev}_{s_{|\alpha^{(a)}|}}^{-1} \Omega_{L_{|\alpha^{(a)}|}} \, ,$$

where $s_0, s_1, \ldots, s_{|\alpha^{(a)}|}$ are general points in $\mathbb{P}^1$.

Observe that we may choose s_0 to be the point ∞ at infinity in $\mathbb{P}^1$. Then, in the quantum Plücker coordinates $(z_{\beta^{(b)}} \mid \beta^{(b)} \in \mathcal{C}_{m,p}^a)$ for a point $z \in \mathcal{M}_{m,p}^a$,

$$\mathrm{ev}_\infty \colon \; z \; \longmapsto \; \left(z_{\beta^{(a)}} \mid \beta \in \binom{[m+p]}{p} \right) .$$

Since, in the Plücker coordinates $(y_\beta \mid \beta \in \binom{[m+p]}{p}))$ for $Grass(p, \mathbb{C}^{m+p})$ we have the analog of (2.10) for Ω_α,

$$\Omega_\alpha \; := \; \{ y \in Grass(p, \mathbb{C}^{m+p} \mid y_\beta = 0 \text{ if } \beta \not\leq \alpha \} \, ,$$

we see that $\overline{\mathrm{ev}_\infty^{-1}(\Omega_\alpha)} = Z_{\alpha^{(a)}}$.

Finally, observe that for an m-plane $L \subset \mathbb{C}^{m+p}$ and a point $s \in \mathbb{P}^1$,

$$\mathrm{ev}_s^{-1} \Omega_L \;=\; \{ M \in \mathcal{M}_{m,p}^a \mid M(s) \cap L \neq \{0\} \}$$

which is a hyperplane section of $\mathcal{M}_{m,p}^a$. Thus the number of points in the intersection (5.9) is bounded by the degree of $Z_{\alpha^{(a)}}$ and it equals this degree if all points of intersection lie in $Z_{\alpha^{(a)}} \cap \mathcal{M}_{m,p}^a$. But this occurs for general m-planes L_i and points s_i, by Remark 3.6. $\qquad\qquad\square$

This proof is unsatisfactory in that both sides of the equation have a simple algebraic-combinatorial interpretation, yet we argued using the definition of the Gromov-Witten invariants, rather than something more elementary. We now give a more direct proof, following [**41**]. For a sequence $I : 0 < i_1 < i_2 < \cdots < i_p$ of integers, let S_I be the Schur symmetric polynomial in $x_1, \ldots, x_p$ associated to the partition $(i_p - p, \ldots, i_2 - 2, i_1 - 1)$, which is also a polynomial in the elementary symmetric polynomials.

THEOREM 5.2. *For $\alpha^{(a)} \in \mathcal{C}^q_{m,p}$, define*

$$\delta(\alpha^{(a)}) \ := \ (-1)^{\binom{p}{2}} \sum_{dQW=0} \frac{c_1^{|\alpha^{(a)}|} \cdot S_{\alpha^\vee}}{J} \,.$$

Then the function $\delta(\alpha^{(a)})$ satisfies the recursion (2.11).

This will prove the equality of the two formulae, since under the map (5.1) to cohomology we have

$$S_\alpha \ \longmapsto \ \sigma_\alpha \qquad \text{for } \alpha \in \binom{[m+p]}{p} \,.$$

Proof. We set $n := m+p$ and change coordinates, working in the ring of symmetric polynomials $\mathbb{C}[x_1, x_2, \ldots, x_p]^{\mathcal{S}_p}$ in $x_1, x_2, \ldots, x_p$. If we let each x_i have cohomological degree 2, then this ring is isomorphic to the ring $\mathbb{C}[c_1, c_2, \ldots, c_p]$ with the isomorphism given by $c_i = e_i(x_1, x_2, \ldots, x_p)$. Here $e_i(x_1, x_2, \ldots, x_p)$ is the ith elementary symmetric polynomial in $x_1, x_2, \ldots, x_p$.

This theorem is a consequence of Lemma 2.7 and the following lemma. Let $y_1, y_2, \ldots, y_n$ be the nth roots of $(-1)^{p+1}$.

LEMMA 5.3. *For $K = k_1, k_2, \ldots, k_p$, define the function $D(K)$ to be*

$$(5.10) \qquad \frac{(-1)^{\binom{p}{2}}}{n^p} \sum (x_1 \cdots x_p)(x_1 + \cdots + x_p)^{\sum_j k_j - j} \det(x_j^{n-k_i}) \det(x_i^{p-j}) \,,$$

the sum over all $I \in \binom{[n]}{p}$, where $x_j = y_{i_j}$. Then

  (i) *For sequences of integers $k_1 \leq k_2 \leq \cdots \leq k_p$ with $\sum_j k_j - j \geq 0$ and $k_p < k_1 + n$, the function $D(k_1, \ldots, k_p)$ satisfies the recursion, initial condition, and boundary conditions of Lemma 2.7. In particular, $D(J(\alpha^{(a)})) = \deg Z_{\alpha^{(a)}}$.*
  (ii) *For $\alpha^{(a)} \in \mathcal{C}^q_{m,p}$, $\delta(\alpha^{(a)}) = D(J(\alpha^{(a)}))$.*

Proof of Lemma 5.3(i). For a sequence $K = (k_1, k_2, \ldots, k_p)$ of integers, let $f(K) := \det(x_i^{n-k_j})$. This determinant is the sum of terms $f(\pi; K) := \text{sgn}(\pi) x_{\pi(1)}^{n-k_1} \cdots x_{\pi(p)}^{n-k_p}$ over all permutations π of $\{1, 2, \ldots, p\}$, where $\text{sgn}(\pi)$ is the sign of the permutation π. Multiplying this term by $x_1 + \cdots + x_p$ gives

$$(x_1 + \cdots + x_p) \cdot f(\pi; K) \ = \ \sum_{a=1}^{p} f(\pi; k_1, \ldots, k_a - 1, \ldots, k_p) \,.$$

Summing over all permutations π gives the Pieri formula

$$(x_1 + \cdots + x_p) \cdot f(K) \ = \ \sum_{a=1}^{p} f(k_1, \ldots, k_a - 1, \ldots, k_p) \,,$$

and thus $D(K)$ satisfies the recursion (2.14) of Lemma 2.7. Since $D(K)$ is antisymmetric in its arguments, it satisfies the boundary condition (2.16). If $k_p = k_1 + n$, then the first and last rows of the matrix $(x_j^{n-k_i})$ are the scalar multiples $(-1)^{p+1}$ of each other, and so the function D satisfies the boundary condition (2.18).

To show that D satisfies the initial condition (2.15) and the remaining boundary condition (2.17), consider the values of $D(k_1, \ldots, k_p)$ when $k_1 < \cdots < k_p$, $0 = \sum_j (k_j - j)$, and $k_p < k_1 + n$. For such sequences K, we show that

$$(5.11) \qquad D(K) \ = \ \begin{cases} 1 & \text{if } K = (1, 2, \ldots, p), \\ 0 & \text{otherwise} \,. \end{cases}$$

The first case of this is the initial condition (2.15). We deduce the boundary condition (2.17) from the second case of (5.11).

Let $J = j_1 < j_2 < \cdots < j_p$ be sequence of integers satisfying $\sum_i j_i - i \geq 0$ with $j_1 = 0$ and $j_p < n = j_1 + n$. Applying the recursion (2.14) $\sum_i (j_i - i)$ times to $D(J)$, and the boundary conditions (2.16) and (2.18) shows that $D(J)$ is a sum of terms $D(K)$ for K satisfying the conditions for (5.11), but with $k_1 \leq 0$. Thus every such term vanishes, and so $D(J) = 0$.

We prove (5.11), which will complete the proof of Lemma 5.3 (i). For the sequences K of (5.11), we have

$$
\begin{aligned}
D(K) &= \frac{(-1)^{\binom{p}{2}}}{n^p} \sum (x_1 \cdots x_p) \det(x_j^{n-k_i}) \det(x_i^{p-j}) \\
&= \frac{(-1)^{\binom{p}{2}}}{n^p} \sum \det(x_j^{n-k_i}) \det(x_i^{p+1-j}),
\end{aligned}
$$

where the sum is over all (ordered) p-tuples $(x_1, \ldots, x_p)$ of the nth roots $(y_1, \ldots, y_n)$ of $(-1)^{p+1}$. We apply the Cauchy-Binet formula to this sum of products of determinants to obtain

$$
D(K) = \frac{(-1)^{\binom{p}{2}}}{n^p} \det \left(\begin{bmatrix} y_1^{n-k_1} & y_2^{n-k_1} & \cdots & y_n^{n-k_1} \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{n-k_p} & y_2^{n-k_p} & \cdots & y_n^{n-k_p} \end{bmatrix} \cdot \begin{bmatrix} y_1^p & \cdots & y_1 \\ y_2^p & \cdots & y_2 \\ \vdots & \ddots & \vdots \\ y_n^p & \cdots & y_n \end{bmatrix} \right).
$$

Expanding this product, we obtain

$$
(5.12) \qquad D(K) = \frac{(-1)^{\binom{p}{2}}}{n^p} \det \begin{bmatrix} P(n+p-k_1) & \cdots & P(n+1-k_1) \\ \vdots & \ddots & \vdots \\ P(n+p-k_p) & \cdots & P(n+1-k_p) \end{bmatrix},
$$

where $P(b)$ is the sum of the bth powers of the y_i. Since the y_i are the nth roots of $(-1)^{p+1}$, we have

$$
P(b) = \begin{cases} (-1)^{(p+1)a} \cdot n & \text{if } b = an, \\ 0 & \text{otherwise}. \end{cases}
$$

The ith row of the determinant (5.12) has at most one non-zero entry, in the column j where $j + k_i \equiv p + 1$ modulo n. Suppose that K satisfies the conditions of (5.11) and the determinant of (5.12) does not vanish. Then each component of K is congruent to one of $\{1, 2, \ldots, p\}$ modulo n. Since each congruence must occur for the determinant to be non-zero (if you like, since no two components of K are congruent modulo n), we have that $K \equiv \{1, 2, \ldots, p\}$ modulo n. In particular, no component of K vanishes. Let r be the index such that $k_r < 0 < k_{r+1}$. Since $k_p < k_1 + n$ and $\sum_j (k_j - j)$, we must have $k_1 \leq 1$, and also $-n < k_1 < k_p < n$. In fact the condition that $K \equiv \{1, 2, \ldots, p\}$ modulo n implies that $k_1, \ldots, k_r \leq -m$ and $0 < k_{r+1}, \ldots, k_p \leq p$. This implies $k_1 \leq -m + 1 - r$ and $p - r \leq k_p$, and hence $p - r < k_p < k_1 + n \leq p + 1 - r$, which implies that $K = (-m+1-r, -m+2-r, \ldots, -m, 1, 2, \ldots, p-r)$ and so $\sum_j k_j - j = -nr$. Since this sum must equal 0, we see that K must be $(1, 2, \ldots, p)$.

When $K = (1, 2, \ldots, p)$, the matrix of power sums is antidiagonal with entries $(-1)^{p+1} n$, and so the determinant is $(-1)^{p^2 + p + \binom{p}{2}} n^p = (-1)^{\binom{p}{2}} n^p$, which shows that $D(1, 2, \ldots, p) = 1$, as claimed. This completes the proof of Lemma 5.3(i). $\square$

Proof of Lemma 5.3(ii). We show that for $\alpha^{(a)} \in \mathcal{C}^q_{m,p}$, we have the equality $\delta(\alpha^{(a)}) = D(J(\alpha^{(a)}))$. Since

$$QW(x_1, \ldots, x_p) \;=\; \sum_{j=1}^{p} \frac{x_j^{n+1}}{n+1} + (-1)^p x_j\,,$$

the set of solutions for $dQW = 0$ are just the set of p-tuples of nth roots of $(-1)^{p+1}$.

The Schur polynomial $S_{\alpha^\vee}$ is equal to the quotient of alternants [**13**]

$$\frac{\det(x_j^{n-\alpha_i})}{\det(x_i^{p-j})}\,.$$

The denominator is the Vandermonde determinant $\Delta := \prod_{i<j}(x_i - x_j)$.

The Jacobian J is the determinant of the Hessian of QW with respect to the variables c_i, which we compute using the multivariate chain rule

$$\left(\frac{\partial^2 QW}{\partial x_i \partial x_j}\right) \;=\; \left(\frac{\partial^2 QW}{\partial c_i \partial c_j}\right) \cdot \left(\frac{\partial c_i}{\partial x_j}\right)^2 + \left(\sum_k \frac{\partial QW}{\partial c_k}\frac{\partial^2 c_k}{\partial x_i \partial x_j}\right)\,.$$

Since we evaluate this where $dQW = 0$, we obtain

$$\det\left(\frac{\partial^2 QW}{\partial x_i \partial x_j}\right)\bigg|_{dQW=0} \;=\; \det\left(\frac{\partial^2 QW}{\partial c_i \partial c_j}\right) \cdot \left[\det\left(\frac{\partial c_i}{\partial x_j}\right)\right]^2\,.$$

The Hessian of QW with respect to the variables x_i is the diagonal matrix with entry nx_i^{n-1} in position (i,i) and by Lemma 5.4 below, $\det(\partial c_i/\partial x_j) = \Delta$. Since $\delta(\alpha^{(a)})$ is the sum over p-tuples $(x_1, \ldots, x_p)$ of nth roots of $(-1)^{p+1}$, we compute the value of the Jacobian $J = \det(\partial^2 QW/\partial c_i \partial c_j)$ at the p-tuple $(x_1, \ldots, x_p)$ to be

$$J \;=\; \frac{n^p(x_1 \cdots x_p)^{n-1}}{\Delta^2} \;=\; \frac{n^p(x_1 \cdots x_p)^n}{(x_1 \cdots x_p)\Delta^2} \;=\; \frac{n^p}{(x_1 \cdots x_p)\Delta^2}\,,$$

as $(x_1 \cdots x_p)^n = (-1)^{p(p+1)} = 1$.

Since each summand involves the Vandermonde, we may restrict the sum to be over the set I of all p-tuples of distinct roots, which we will always take to be in an order compatible with a fixed ordering of the nth roots $y_1, \ldots, y_n$ of $(-1)^{p+1}$. We may put these calculations together and obtain the following formula for $\delta(\alpha^{(a)})$

$$(5.13) \qquad \frac{(-1)^{\binom{p}{2}}}{n^p} \sum_I (x_1 \cdots x_p)(x_1 + \cdots + x_p)^{|\alpha^{(a)}|} \det(x_j^{n-\alpha_i}) \det(x_i^{p-j})\,.$$

Let $(k_1, \ldots, k_p) = J(\alpha^{(a)})$. If we write $a = pl + r$ with $0 \le r < p$, then this is the sequence

$$(ln + \alpha_{r+1}, \ldots, ln + \alpha_p, (l+1)n + \alpha_1, \ldots, (l+1)n + \alpha_r)\,.$$

The vector $(x_i^{n-k_j})$ is

$$((-1)^{(p+1)l}x_i^{n-\alpha_{r+1}}, \ldots, (-1)^{(p+1)l}x_i^{n-\alpha_p},$$
$$(-1)^{(p+1)(l+1)}x_i^{n-\alpha_1}, \ldots, (-1)^{(p+1)(l+1)}x_i^{n-\alpha_r})\,.$$

Thus we see that

$$\det(x_i^{n-k_j}) \;=\; (-1)^{(p+1)a+r(p-r)} \det(x_i^{n-\alpha_j}) \;=\; \det(x_i^{n-\alpha_j})\,,$$

since, as in the calculation at the end of Section 2, $r(p-r) \equiv pa + a$ modulo 2.

Since $|\alpha^{(a)}| = \sum_j k_j - j$, we may substitute the last formula into (5.13) and obtain

$$
\begin{aligned}
\delta(\alpha^{(a)}) &= \frac{(-1)^{\binom{p}{2}}}{n^p} \sum_I (x_1 \cdots x_p)(x_1 + \cdots + x_p)^{\sum_j k_j - j} \det(x_j^{n-k_i}) \det(x_i^{p-j}) \\
&= D(J(\alpha^{(a)})),
\end{aligned}
$$

as claimed.

We complete the proof of Lemma 5.3(ii) and hence of Theorem 5.2 with the calculation below.

LEMMA 5.4.

$$
\det\left(\frac{\partial c_i}{\partial x_j}\right) = \prod_{i<j}(x_i - x_j) = \Delta.
$$

PROOF. Let $F_p(x_1, \ldots, x_p)$ be this determinant. Since

$$
\frac{\partial c_i}{\partial x_j} = c_{i-1}(x_1, \ldots, \widehat{x_j}, \ldots, x_p),
$$

where $\widehat{x_j}$ indicates that x_j is omitted, we seek the determinant of the matrix whose (i,j)th entry is $c_{i-1}(x_1, \ldots, \widehat{x_j}, \ldots, x_p)$. If we subtract the first column from each the rest, we obtain a matrix in block form

$$
\begin{pmatrix} 1 & 0 \\ * & A \end{pmatrix},
$$

where the entries of A in position (i,j) (note the shift from the original matrix) are

$$
c_i(x_1, \ldots, \widehat{x_{j+1}}, \ldots, x_p) - c_i(\widehat{x_1}, \ldots, x_p) = (x_1 - x_{j+1})c_{i-1}(x_2, \ldots, \widehat{x_{j+1}}, \ldots, x_p).
$$

Dividing the common factors of $(x_1 - x_{j+1})$ from the columns of A gives the matrix with entries $c_{i-1}(x_2, \ldots, \widehat{x_{j+1}}, \ldots, x_p)$, and so we have the recursive formula

$$
F_p(x_1, \ldots, x_p) = \prod_{j=2}^{p}(x_1 - x_j) \cdot F_{p-1}(x_2, \ldots, x_p).
$$

Since $F_1(x_p) = 1$, this completes the Lemma. $\qquad \square$

5.4. The quantum cohomology ring of the Grassmannian. We discuss an alternative view of the quantum cohomology ring of the Grassmannian, mention how this ring arose in representation theory, and give some open problems.

The presentation (5.6) of $QH^*(Grass(p, \mathbb{C}^{m+p}))$ is not what one ordinarily sees in algebraic geometry, but rather an integral form with a parameter q

$$
\mathbb{Z}[q][c_1, c_2, \ldots, c_p]/\langle h_{m+1}, \ldots, h_{m+p-1}, h_{m+p} + (-1)^p q \rangle.
$$

Then the genus zero Gromov-Witten invariant $\langle X_1, X_2, \ldots, X_l \rangle_0$ of cycles represented by classes $\xi_1, \xi_2, \ldots, \xi_l \in \mathbb{Z}[c_1, c_2, \ldots, c_p]$ is the coefficient of $c_p^m \cdot q^d$ in the product $\xi_1 \cdot \xi_2 \cdots \xi_l$. Here d is the degree of the curves this invariant enumerates. In this presentation, the variable q (which is the variable β of Section 5.2) keeps track of the degrees of curves. This ring is graded, if q has cohomological degree $2(m+p)$.

The cohomology of the Grassmannian has a basis of Schubert classes, σ_α, given by the Giambelli formula (Jacobi-Trudi for combinatorists).

(5.14) $$\sigma_\alpha = \det(h_{\alpha_i - j})_{1 \le i,j \le p}.$$

The class σ_α is Poincaré dual to the Schubert variety $\Omega_{\alpha^\vee}$. The quantum coho-
mology ring of $Grass(p, \mathbb{C}^{m+p})$ may be viewed additively as polynomials in q with
coefficients in the cohomology of the Grassmannian, $H^*(Grass(p, \mathbb{C}^{m+p}))[q]$, but
with a deformed product, $*$, defined by

$$(5.15) \qquad \sigma_\alpha * \cdots * \sigma_\beta \;=\; \sum_{\gamma, d \geq 0} q^d \, \langle \Omega_{\alpha^\vee}, \ldots, \Omega_{\beta^\vee}, \Omega_\gamma \rangle_0^d \, \sigma_\gamma \,,$$

where $\langle \cdots \rangle_0^d$ is the genus 0 Gromov-Witten invariant for degree d curves. (The
degree d is determined by the cohomological degrees of the Schubert classes.)

Bertram [5] studied this ring, and showed that the Giambelli formula (5.14)
remains valid with the quantum multiplication. He also established a Pieri formula

$$\sigma_\alpha * h_a \;=\; \sum \sigma_\beta + q \sum \sigma_\gamma \,,$$

the sum over all β, γ with $|\beta| = |\alpha| + a$ and $|\gamma| = |\alpha| + a - m - p$, and satisfying

$$\alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \cdots < \alpha_p \leq \beta_p \,,$$
$$\gamma_1 \leq \alpha_1 - 1 < \gamma_2 \leq \alpha_2 - 1 < \cdots < \gamma_p \leq \alpha_p - 1 \,.$$

Like the classical Giambelli and Pieri formulae [33], these determine the ring struc-
ture of quantum cohomology with respect to the basis of Schubert classes.

In particular, the structure constants $N_{\alpha\,\beta}^\gamma(m, p)$ defined by the formula

$$(5.16) \qquad \sigma_\alpha * \sigma_\beta \;=\; \sum_{\gamma, d} q^d N_{\alpha\,\beta}^\gamma(m, p) \sigma_\gamma$$

are completely determined. (Here, the summation is over γ, d with $|\gamma| + d(m+p) = |\alpha| + |\beta|$.) These are the analogs of the classical Littlewood-Richardson coeffi-
cients. Like them, these numbers $N_{\alpha\,\beta}^\gamma(m, p)$ are non-negative. Unlike the classical
coefficients, there is as yet no quantum Littlewood-Richardson formula for these
constants which proves their non-negativity. These are certain three-point Gromov
Witten invariants, as combining (5.15) and (5.16) shows

$$N_{\alpha\,\beta}^\gamma(m, p) \;=\; \langle \Omega_{\alpha^\vee}, \Omega_{\beta^\vee}, \Omega_\gamma \rangle_0 \,.$$

These are known in the case of the Pieri formula and when $q = 0$; for then they
are the classical Littlewood-Richardson coefficients. The only case for which there
is such a positive formula is due to Tudose [61], when the minimum of m or p is 2.
A formula for $N_{\alpha\,\beta}^\gamma(m, p)$ which involves signs (like the formula (1.2) for $d(q; m, p)$)
was given by Bertram, Ciocan-Fontanine, and Fulton [6]. Interestingly, a similar
formula was given previously in two different contexts.

The Verlinde algebra is a variant of the representation ring of $\mathfrak{sl}_p$ where the
usual product is replaced by the fusion product, which is the tensor product of
two representations reduced at level m. Witten [67] explained the isomorphism
between the Verlinde algebra and the quantum cohomology ring of the Grassman-
nian, and this was rigorously established by Agnihotri [1]. This isomorphism is an
analog of the relation between the cohomology rings of the Grassmann varieties
$Grass(p, \mathbb{C}^{m+p})$, as m varies, and the representation ring of $\mathfrak{sl}_p$. A formula similar
to that of Bertram, Ciocan-Fontanine, and Fulton was given by Kac [31, Exercise
13.35] and Walton [63] in this context, where further details may be found.

The cohomology ring of the Grassmannian is likewise isomorphic to an external
representation ring of the symmetric groups (see [21, 48]). Similarly, there is
a family of quotients of Hecke algebras at a primitive $(m + p)$th root of unity

whose external representation ring is isomorphic to the quantum cohomology of the Grassmannian. This was studied by Goodman and Wenzl [**22**], and they also gave a formula for $N_{\alpha\,\beta}^{\gamma}(p,m)$ identical to that of Kac and Walton.

They also gave another presentation of the quantum cohomology ring

$$\mathbb{Z}[c_1, c_2, \ldots, c_p]/I_{m,p}\,,$$

where $I_{m,p}$ is the ideal generated by

$$\{S_K \mid k_p - k_1 = m + p\}\,,$$

here, $K : 0 < k_1 < k_2 < \cdots < k_p$ is an increasing sequence of positive integers and S_K is defined by the Giambelli formula

$$S_K \;:=\; (h_{k_i - j})_{1 \le i, j \le p}\,,$$

where the polynomials h_i are defined recursively by (5.2). Goodman and Wenzl proved that this quotient ring has an integral basis consisting of the classes

$$\{S_I \mid I : 0 < i_1 < i_2 < \cdots < i_p < i_1 + m + p\}\,.$$

This is just the set of sequences $J(\alpha^{(a)})$ for all $\alpha^{(a)} \in \mathcal{C}_{m,p} := \bigcup_b \mathcal{C}_{m,p}^b$, where $J(\,\cdot\,)$ is the map (2.12) of Section 2.4. The relation between this basis and the basis $q^a \sigma_\alpha$ of $H^*(Grass(p, \mathbb{C}^{m+p}))[q]$ is just

$$q^a \sigma_\alpha \;=\; S_{J(\alpha^{(a)})}\,.$$

Bertram's Pieri formula has a nice expression in this basis:

$$h_a * S_I \;=\; \sum S_J\,,$$

the sum over all sequences J with $|J| = |I| + a$, where

$$i_1 \le j_1 < i_2 \le j_2 < \cdots < i_p \le j_p < i_1 + m + p\,.$$

We close with two additional problems concerning the quantum Littlewood-Richardson coefficients.

Let $f^{\alpha^{(a)}} = \deg Z_{\alpha^{(a)}}$ be the number of saturated chains in the poset $\mathcal{C}_{m,p}$ from the minimal element to $\alpha^{(a)}$. More generally, given $\beta^{(b)}, \alpha^{(a)} \in \mathcal{C}_{m,p}$, let $f_{\beta^{(b)}}^{\alpha^{(a)}}$ be the number of saturated chains that begin at $\beta^{(b)}$ and end at $\alpha^{(a)}$. Write $\sigma_{\alpha^{(a)}}$ for the class $q^a \sigma_\alpha$ and then the expansion (5.16) becomes

$$\sigma_{\alpha^{(a)}} * \sigma_{\beta^{(b)}} \;=\; \sum_{|\gamma^{(c)}| = |\alpha^{(a)}| + |\beta^{(b)}|} N_{\alpha\,\beta}^{\gamma}(m,p)\, \sigma_{\gamma^{(c)}}\,.$$

Iterating the Pieri formula with $a = 1$ implies that

$$h_1^{*l} * \sigma_{\beta^{(b)}} \;=\; \sum_{|\gamma^{(c)}| = l + |\beta^{(b)}|} f_{\beta^{(b)}}^{\gamma^{(c)}}\, \sigma_{\gamma^{(c)}}\,.$$

Since $h_1^{*l} \;=\; \sum_{|\alpha^{(a)}| = l} f^{\alpha^{(a)}} \sigma_{\alpha^{(a)}}$, we may expand the left hand side to obtain

$$\sum_{|\alpha^{(a)}| = l} f^{\alpha^{(a)}} \sigma_{\alpha^{(a)}} * \sigma_{\beta^{(b)}} \;=\; \sum_{|\alpha^{(a)}| = l}\ \sum_{|\gamma^{(c)}|} f^{\alpha^{(a)}} N_{\alpha\,\beta}^{\gamma}(m,p)\, \sigma_{\gamma^{(c)}}\,.$$

Equating the coefficients of $\sigma_{\gamma^{(c)}}$, we obtain

(5.17)
$$\sum_{|\alpha^{(a)}| = l} f^{\alpha^{(a)}} N_{\alpha\,\beta}^{\gamma}(m,p) \;=\; f_{\beta^{(b)}}^{\gamma^{(c)}}\,.$$

The eventual combinatorial formula for the quantum Littlewood-Richardson coefficients should also explain this identity. That is, there should be some algorithm to convert a path in the poset $\mathcal{C}_{m,p}$ from $\beta^{(b)}$ to $\gamma^{(c)}$ into a path of the same length that starts at the minimal element, where the multiplicity of the occurrence of any path to $\alpha^{(a)}$ is the quantum Littlewood-Richardson coefficient $N^\gamma_{\alpha\,\beta}(m,p)$. In short, we ask for a a quantum version of Schensted insertion.

Another open problem is to show the (apparent) inequalities:

$$N^\gamma_{\alpha\,\beta}(m,p) \ \leq \ N^\gamma_{\alpha\,\beta}(m+1,p) \ \leq \ N^\gamma_{\alpha\,\beta}(m+2,p) \ \leq \ \cdots \ ,$$

which were conjectured by Walton [63].

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