

THE MONOTONE SECANT CONJECTURE IN THE REAL SCHUBERT CALCULUS

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ABSTRACT. The monotone secant conjecture posits a rich class of polynomial systems, all of whose solutions are real. These systems come from the Schubert calculus on flag manifolds, and the monotone secant conjecture is a compelling generalization of the Shapiro conjecture for Grassmannians (Theorem of Mukhin, Tarasov, and Varchenko). We present some theoretical evidence for this conjecture, as well as computational evidence obtained by 1.9 teraHertz-years of computing, and we discuss some of the phenomena we observed in our data.

1. INTRODUCTION

A system of real polynomial equations with finitely many solutions has some, but likely not all, of its solutions real. In fact, sometimes the structure of the equations implies an upper bound on the number of real solutions [2, 12], ensuring that not all solutions are real. The monotone secant conjecture posits a family of systems of polynomial equations with the extreme property that all of their solutions are real.

The Shapiro conjecture asserts that a zero-dimensional intersection of Schubert subvarieties of a Grassmannian consists only of real points provided that the Schubert varieties are given by flags tangent to a real rational normal curve. While the statement concerns the Schubert calculus on Grassmannians, its proofs involve complex analysis [5, 6] or integrable systems and representation theory [14, 15]. A complete story of this conjecture and its proof can be found in [20].

The Shapiro conjecture is false for non-Grassmannian flag manifolds, but in a very interesting manner. This failure was first noticed in [18] and systematic computer experimentation suggested a correction, the monotone conjecture [17, 19], that appears to be valid for flag manifolds of type A. Eremenko, Gabrielov, Shapiro, and Vainshtein [7] proved a result that implies the monotone conjecture for some manifolds of two-step flags and concerns codimension-two subspaces that meet flags which are secant to the rational normal curve

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along disjoint intervals. This suggested the secant conjecture, which asserts that an intersection of Schubert varieties in a Grassmannian is transverse with all points real, provided that the Schubert varieties are defined by flags secant to a rational normal curve along disjoint intervals. This was posed and evidence was presented for its validity in [10].

The monotone secant conjecture is a common extension of both the monotone conjecture and the secant conjecture. It is also the last of the conjectures our group has made concerning reality in Schubert calculus of osculating flags. In addition to those mentioned, there is a version of the Shapiro conjecture for the orthogonal Grassmannian which was proven by Purbhoo [16], and a version for the Lagrangian Grassmannian described in [21, Ch. 14.2]. Exploratory computations in other flag manifolds suggest there is no regularity in the number of real solutions to Schubert calculus problems given by osculating flags.

We give here an open instance of the monotone secant conjecture, expressed as a system of polynomial equations in local coordinates for the variety of flags $E_2 \subset E_3$ in \mathbb{C}^5 , where $\dim E_i = i$. Let x_1, \dots, x_8 be indeterminates and consider the polynomials

$$(1.1) \quad f(s,t,u;x) := \det \begin{pmatrix} 1 & 0 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 & x_6 \\ 1 & s & s^2 & s^3 & s^4 \\ 1 & t & t^2 & t^3 & t^4 \\ 1 & u & u^2 & u^3 & u^4 \end{pmatrix}, \quad g(v,w;x) := \det \begin{pmatrix} 1 & 0 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 & x_6 \\ 0 & 0 & 1 & x_7 & x_8 \\ 1 & v & v^2 & v^3 & v^4 \\ 1 & w & w^2 & w^3 & w^4 \end{pmatrix},$$

which depend upon parameters s,t,u and v,w respectively.

Conjecture 1.1. *Let $s_1 < t_1 < u_1 < \dots < s_4 < t_4 < u_4 < v_1 < w_1 < \dots < v_4 < w_4$ be real numbers. Then the system of polynomial equations*

$$(1.2) \quad \begin{aligned} f(s_1, t_1, u_1; x) &= f(s_2, t_2, u_2; x) = f(s_3, t_3, u_3; x) = f(s_4, t_4, u_4; x) = 0 \\ g(v_1, w_1; x) &= g(v_2, w_2; x) = g(v_3, w_3; x) = g(v_4, w_4; x) = 0 \end{aligned}$$

has twelve solutions, and all of them are real.

These equations have geometric meaning. Let E_2 be the span of the first two rows of either matrix and E_3 the span of the first three rows of the second matrix that defines g . Then $E_2 \subset E_3$ is a general flag. If we let $F_3(s,t,u)$ be the span of the last three rows of the matrix for f , then this is a 3-plane that is secant to the rational normal curve $\gamma: y \mapsto (1, y, y^2, y^3, y^4)$ at the points $\gamma(s), \gamma(t), \gamma(u)$. The equation $f(s,t,u;x) = 0$ is the condition that E_2 meet $F_3(s,t,u)$ non-trivially. Similarly, if $F_2(u,v)$ is the span of the last two rows of the second matrix, which is 2-plane secant to γ at $\gamma(u)$ and $\gamma(v)$, then the equation $g(v,w;x) = 0$ is the condition that E_3 meet $F_2(u,v)$ non-trivially.

The monotonicity hypothesis is that the four 3-planes given by s_i, t_i, u_i are secant along intervals $[s_i, u_i]$ which are pairwise disjoint and occur before the pairwise disjoint intervals $[v_i, w_i]$ where the 2-planes are secant. If the order of the intervals $[s_4, u_4]$ and $[v_1, w_1]$ is reversed, the evaluation is no longer monotone. We tested 3,000,000 instances of Conjecture 1.1, finding only real solutions. In contrast, we tested 21,000,000 with the monotonicity condition violated, of which 18,085,537 had some non real solutions.

We formulate the monotone secant conjecture, explain its relation to the other reality conjectures, describe data supporting it from a large computational experiment, and discuss some features observed in our data that go beyond the monotone secant conjecture. This experiment verified the monotone secant conjecture in each of the 768,846,000 instances it tested. We have created a website [9] for viewing the data online. This includes pages for browsing the data and viewing the results for each Schubert problem. We only sketch the other reality conjectures, for they are described in the cited literature, and we also only sketch the design and execution of this experiment, for the purpose of the paper [11] was to present the software environment we developed for such distributed computational experiments.

This paper is organized as follows. In Section 2 we illustrate the main point of the monotone secant conjecture through the classical problem of four lines. Section 3 is a primer on flag manifolds and contains a precise statement of the monotone secant conjecture while also explaining its relation to the Shapiro, secant, and monotone conjectures. In Section 4 we expand on the relation between the monotone secant and monotone conjectures, discuss the experimental evidence for the monotone secant conjecture, and some phenomena we observed in our data. Lastly, in Section 5 we sketch the methods we used to test the conjecture.

2. THE PROBLEM OF FOUR LINES

The classical problem of four lines asks for the finitely many lines m that meet four given general lines $\ell_1, \ell_2, \ell_3, \ell_4$ in (projective) three-space. This has a pleasing synthetic solution, which leads to the first interesting case of the monotone secant conjecture.

Three general lines ℓ_1, ℓ_2, ℓ_3 lie in one ruling of a doubly-ruled quadric surface Q , with the other ruling consisting of all lines that meet the first three. The line ℓ_4 meets Q in two points, and through each of these points there is a line in the second ruling. These two lines, m_1 and m_2 , are the solutions to this problem. If the lines $\ell_1, \ell_2, \ell_3, \ell_4$ are real, then so is Q , but the intersection of Q with ℓ_4 is either two real points or a pair of complex conjugate points. In the first case, the problem of four lines has two real solutions, while in the second, it has no real solutions.

Let us consider a variant in the manifold of flags consisting of a line m lying on a plane M in 3-space, $m \subset M$. Consider the Schubert problem in which m meets three lines ℓ_1, ℓ_2, ℓ_3 and M contains two points, p and q . Then M contains the affine span $\overline{p, q}$ of p and q . Since $m \subset M$, we must have that m also meets $\overline{p, q}$ and is therefore a solution to the problem of four lines given by ℓ_1, ℓ_2, ℓ_3 and $\overline{p, q}$. As M is spanned by m and $\overline{p, q}$, we see that solving this auxiliary problem of four lines solves our original Schubert problem. Furthermore, if the lines and points are real, then a solution $m \subset M$ is real if and only if m is real.

Suppose that the three lines are secant to a rational normal curve γ along disjoint intervals and the points are $p = \gamma(s)$ and $q = \gamma(t)$, which do not lie in any interval of secancy. There are two possible combinatorial placements of the two points. Removing the three intervals of secancy from γ results in three disjoint intervals along $\gamma \simeq \mathbb{RP}^1$. Either both points $\gamma(s)$ and $\gamma(t)$ lie in the same interval or they lie in different intervals. We examine each case.

Fixing secant lines ℓ_1, ℓ_2, ℓ_3 , the quadric Q described above is a hyperboloid of one sheet. This is displayed in Figures 1 and 2, along with γ and the lines. Suppose that $\gamma(s)$ and $\gamma(t)$ lie in the same interval, say I , as indicated in Figure 1. Then the secant line they span,

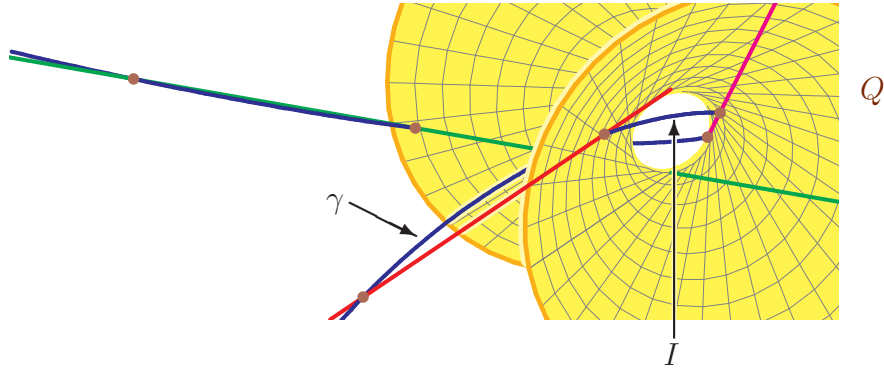


FIGURE 1. The problem of four secant lines.

$\ell(s,t)$, lies in the direction of I and meets the hyperboloid Q in two real points. Thus, in this first case, our Schubert problem has two real solutions. (This is also an instance of the secant conjecture, which holds for this problem of four lines [10, §4].)

In the second case, where the points $\gamma(s)$ and $\gamma(t)$ do not lie in the same interval, it is possible to have no real solutions. Consider the choice of points $\gamma(s)$ and $\gamma(t)$ as shown in Figure 2, so that the secant line $\ell(s,t)$ does not meet the quadric Q . By our previous

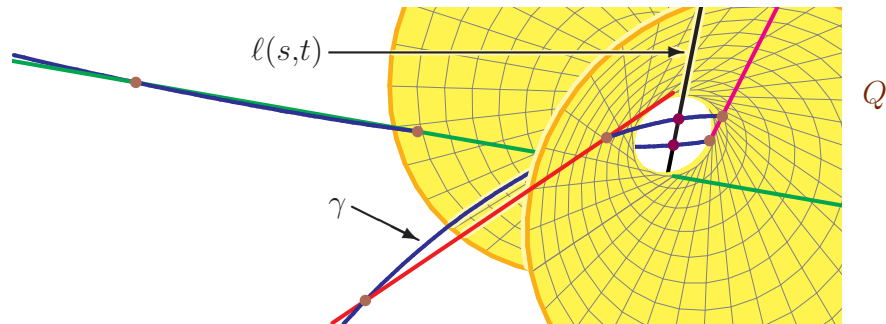


FIGURE 2. A non-monotone evaluation.

analysis, there will be no real lines m meeting these four secant lines, and therefore no real solutions $m \subset M$ to our Schubert problem.

We conclude that the positions of the points $\gamma(s), \gamma(t)$ relative to the other intervals of secancy may affect whether or not the solutions are real. The schematic in Figure 3 illustrates the relative positions of the secancies along γ (which is homeomorphic to the circle). The idea behind the monotone secant conjecture is to attach to each interval the dimension of that part of the flag which it affects. This is 1 for m and 2 for M . The schematic on the left has labels 1,1,1,2,2, reading clockwise, starting just past the point s , while that on the right reads 1,1,2,1,2. The labels increase monotonically in the first and do not in the second.

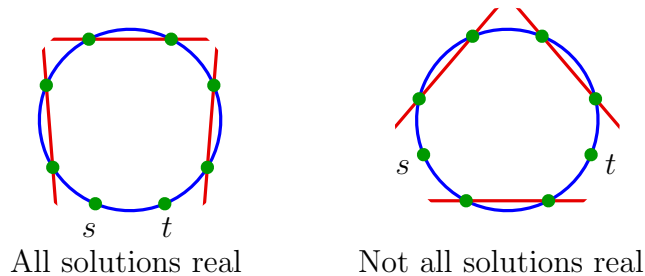


FIGURE 3. Schematic for the secancies.

3. BACKGROUND

We develop the background for the statement of the monotone secant conjecture, defining flag varieties and their Schubert problems. More may be found in the book of Fulton [8]. Fix positive integers $a_\bullet := (a_1 < \dots < a_k)$ and n with $a_k < n$. A *flag* E_\bullet of type a_\bullet is a sequence of subspaces

$$E_\bullet : \{0\} \subset E_{a_1} \subset E_{a_2} \subset \dots \subset E_{a_k} \subset \mathbb{C}^n, \quad \text{where } \dim(E_{a_i}) = a_i.$$

The set of all such flags forms the *flag manifold* $\mathbb{F}\ell(a_\bullet; n)$, which has dimension $\dim(a_\bullet) := \sum_{i=1}^k (n - a_i)(a_i - a_{i-1})$, where $a_0 := 0$. When $a_\bullet = (a)$ is a singleton, $\mathbb{F}\ell(a_\bullet; n)$ is the *Grassmannian* of a -planes in \mathbb{C}^n , written $\text{Gr}(a, n)$. Flags of type $1 < 2 < \dots < n - 1$ in \mathbb{C}^n are *complete*.

The positions of flags relative to a fixed complete flag F_\bullet stratify $\mathbb{F}\ell(a_\bullet; n)$ into topological cells whose closures are *Schubert varieties*. These positions are indexed by certain permutations. The *descent set* $\delta(\sigma)$ of a permutation $\sigma \in S_n$ is the set of numbers i such that $\sigma(i) > \sigma(i+1)$. For a permutation $\sigma \in S_n$ with $\delta(\sigma) \subset a_\bullet$, the Schubert variety $X_\sigma F_\bullet$ is

$$X_\sigma F_\bullet = \{E_\bullet \in \mathbb{F}\ell(a_\bullet; n) \mid \dim E_{a_i} \cap F_j \geq \#\{l \leq i \mid j + \sigma(l) > n\} \quad \forall i, j\}.$$

Flags E_\bullet in $X_\sigma F_\bullet$ have position σ relative to F_\bullet . A permutation σ with descent set contained in a_\bullet is a *Schubert condition* on flags of type a_\bullet . The Schubert variety $X_\sigma F_\bullet$ is irreducible with codimension $\ell(\sigma) := |\{i < j \mid \sigma(i) > \sigma(j)\}|$. A *Schubert problem* for $\mathbb{F}\ell(a_\bullet; n)$ is a list $\sigma := (\sigma_1, \dots, \sigma_m)$ of Schubert conditions for $\mathbb{F}\ell(a_\bullet; n)$ satisfying $\ell(\sigma_1) + \dots + \ell(\sigma_m) = \dim(a_\bullet)$.

Given a Schubert problem σ for $\mathbb{F}\ell(a_\bullet; n)$ and complete flags $F_\bullet^1, \dots, F_\bullet^m$, the intersection

$$(3.1) \quad X_{\sigma_1} F_\bullet^1 \cap \dots \cap X_{\sigma_m} F_\bullet^m$$

is an *instance* of σ . When the flags are in general position, this intersection is transverse and zero-dimensional [13], and it consists of all flags $E_\bullet \in \mathbb{F}\ell(a_\bullet; n)$ having position σ_i relative to F_\bullet^i , for each $i = 1, \dots, m$. Such a flag E_\bullet is a *solution* to this instance of σ .

The degree of a zero-dimensional intersection (3.1) is independent of the choice of the flags and we call this number $d(\sigma)$ the *degree* of the Schubert problem σ . When the intersection is transverse, the number of solutions to σ equals its degree.

When the flags $F_\bullet^1, \dots, F_\bullet^m$ are real, the solutions to the Schubert problem need not be real. The monotone secant conjecture posits a method to select the flags F_\bullet so that all solutions are real, for a certain class of Schubert problems.

Let $\gamma: \mathbb{R} \rightarrow \mathbb{R}^n$ be a rational normal curve, which is any curve affinely equivalent to the moment curve $t \mapsto (1, t, t^2, \dots, t^{n-1})$. Consider this projectively, so that γ is homeomorphic to \mathbb{RP}^1 , which is a circle. A flag F_\bullet is *secant along an interval* I of γ if every subspace in the flag is spanned by its intersection with I . A list of flags $F_\bullet^1, \dots, F_\bullet^m$ secant to γ is *disjoint* if the intervals of secancy are pairwise disjoint. Disjoint flags are naturally ordered by the order in which their intervals of secancy lie within \mathbb{RP}^1 . We remark that this order is to be taken cyclically as in Figure 3, and with respect to one of the two orientations of \mathbb{RP}^1 .

A permutation σ is *Grassmannian* if it has a unique descent, for these, let $\delta(\sigma)$ be the descent. A *Grassmannian Schubert problem* is one that involves only Grassmannian Schubert conditions. A list of disjoint secant flags $F_\bullet^1, \dots, F_\bullet^m$ is *monotone* with respect to a Grassmannian Schubert problem $(\sigma_1, \dots, \sigma_m)$ if the function $F_\bullet^i \mapsto \delta(\sigma_i)$ is monotone with respect to one of the two orientations of \mathbb{RP}^1 . In other words, if

$$\delta(\sigma_i) < \delta(\sigma_j) \implies F^i < F^j, \quad \text{for all } i, j,$$

where $<$ is induced by one of the two cyclic orderings of \mathbb{RP}^1 .

Monotone Secant Conjecture 3.1. *For any Grassmannian Schubert problem $(\sigma_1, \dots, \sigma_m)$ on the flag manifold $\mathbb{F}\ell(a_\bullet; n)$ and any disjoint secant flags $F_\bullet^1, \dots, F_\bullet^m$ that are monotone with respect to the Schubert problem, the intersection*

$$X_{\sigma_1} F_\bullet^1 \cap X_{\sigma_2} F_\bullet^2 \cap \dots \cap X_{\sigma_m} F_\bullet^m$$

is transverse with all points real.

Conjecture 1.1 is the monotone secant conjecture for a Schubert problem on $\mathbb{F}\ell(2, 3; 5)$ involving the Schubert conditions $\sigma := 13245$ and $\tau := 12435$, where we write permutations in one-line notation, so that $\sigma(2) = 3$ and $\tau(2) = 2$. Then $\delta(\sigma) = 2$, $\delta(\tau) = 3$, and $\ell(\sigma) = \ell(\tau) = 1$, so that $(\sigma, \sigma, \sigma, \sigma, \tau, \tau, \tau)$ is a Schubert problem for $\mathbb{F}\ell(2, 3; 5)$, as $\dim(\mathbb{F}\ell(2, 3; 5)) = 8$. We use exponential notation for repeated conditions, so that this Schubert problem is written as (σ^4, τ^4) . The corresponding Schubert varieties are

$$\begin{aligned} X_\sigma F_\bullet &= \{E_\bullet \in \mathbb{F}\ell(2, 3; 5) \mid \dim E_2 \cap F_3 \geq 1\}, \\ X_\tau F_\bullet &= \{E_\bullet \in \mathbb{F}\ell(2, 3; 5) \mid \dim E_3 \cap F_2 \geq 1\}, \end{aligned}$$

that is, the set of flags E_\bullet whose 2-plane E_2 meets a fixed 3-plane F_3 non-trivially, and the set of E_\bullet where E_3 meets a fixed 2-plane F_2 non-trivially, respectively. Consequently, we write $X_\sigma F_3$ for $X_\sigma F_\bullet$ and $X_\tau F_2$ for $X_\tau F_\bullet$.

For $s, t, u, v, w \in \mathbb{R}$, let $F_3(s, t, u)$ be the linear span of $\gamma(s), \gamma(t)$, and $\gamma(u)$ and $F_2(v, w)$ be the linear span of $\gamma(v)$ and $\gamma(w)$; these are a secant 3-plane and a secant 2-plane to the rational normal curve, respectively. The condition $f(s, t, u; x) = 0$ of Conjecture 1.1 is equivalent to the membership $E_\bullet \in X_\sigma F_3(s, t, u)$. Similarly, the condition $g(v, w; x) = 0$ is equivalent to the membership $E_\bullet \in X_\tau F_2(v, w)$. Lastly, the condition on the ordering of

the points s_i, t_i, u_i, v_j, w_j in Conjecture 1.1 implies that the relevant subspaces $F_3(s_i, t_i, u_i)$ and $F_2(v_j, w_j)$ for $i, j = 1, \dots, 4$ lie in disjoint secant flags that are monotone with respect to this Schubert problem.

Three conjectures that have driven progress in enumerative real algebraic geometry are specializations of the monotone secant conjecture. For the Grassmannian $\text{Gr}(a; n)$, any list of disjoint secant flags $F_\bullet^1, \dots, F_\bullet^m$ is monotone with respect to any Schubert problem $(\sigma_1, \dots, \sigma_m)$, as all the conditions have the same descent. In this way, the monotone secant conjecture reduces to the secant conjecture.

Secant Conjecture 3.2. *For any Schubert problem $(\sigma_1, \dots, \sigma_m)$ on any Grassmannian and any disjoint secant flags $F_\bullet^1, \dots, F_\bullet^m$, the intersection*

$$X_{\sigma_1} F_\bullet^1 \cap X_{\sigma_2} F_\bullet^2 \cap \dots \cap X_{\sigma_m} F_\bullet^m$$

is transverse with all points real.

We studied this conjecture in a large-scale experiment whose results are described in [10], solving 1,855,810,000 instances of 703 Schubert problems on 13 different Grassmannians, verifying the secant conjecture in each of the 448,381,157 instances checked. This took 1.065 teraHertz-years of computing.

The *osculating flag* $F_\bullet(t)$ is the flag whose j -dimensional subspace $F_j(t)$ is the span of the first j derivatives $\gamma(t), \gamma'(t), \dots, \gamma^{(j-1)}(t)$ of γ at t . This subspace $F_j(t)$ is the unique j -dimensional subspace having maximal order of contact, namely j , with γ at $\gamma(t)$. It follows that the limit of any family of flags whose intervals of secancy shrink to a point $\gamma(t)$ is this osculating flag $F_\bullet(t)$. In this way, the limit of the monotone secant conjecture, as the secant flags become osculating flags, is a similar conjecture where we replace monotone secant flags by monotone osculating flags.

Monotone Conjecture 3.3. *For any Schubert problem $(\sigma_1, \dots, \sigma_m)$ on the flag manifold $\mathbb{F}\ell(a_\bullet; n)$ and any flags $F_\bullet^1, \dots, F_\bullet^m$ osculating a rational normal curve γ at real points that are monotone with respect to the Schubert problem, the intersection*

$$X_{\sigma_1} F_\bullet^1 \cap X_{\sigma_2} F_\bullet^2 \cap \dots \cap X_{\sigma_m} F_\bullet^m$$

is transverse with all points real.

Ruffo, et al. [17] formulated and studied this conjecture, establishing special cases and giving substantial experimental evidence in support of it.

The Shapiro conjecture is a specialization of the monotone secant conjecture that both restricts to the Grassmannian and to osculating flags. This was posed around 1995 by Boris Shapiro and Michael Shapiro and studied in [18]. Proofs were given by Eremenko and Gabrielov for $\text{Gr}(n-2; n)$ [5, 6] using complex analysis and in complete generality by Mukhin, Tarasov, and Varchenko [14, 15] using integrable systems and representation theory.

Shapiro Conjecture 3.4. *For any Schubert problem $(\sigma_1, \dots, \sigma_m)$ in a Grassmannian and any distinct real numbers t_1, \dots, t_m , the intersection*

$$X_{\sigma_1} F_\bullet(t_1) \cap X_{\sigma_2} F_\bullet(t_2) \cap \dots \cap X_{\sigma_m} F_\bullet(t_m)$$

is transverse with all points real.

4. RESULTS

A consequence of the example discussed in Section 2 is that the secant conjecture (like the Shapiro conjecture before it) does not hold for flag manifolds $\mathbb{F}\ell(a_\bullet; n)$ that are not Grassmannians. The monotonicity condition appears to correct this failure in both conjectures. We give more details on the relation of the monotone conjecture to the monotone secant conjecture and give a conjecture that interpolates between the two. Then we discuss some of our data in an experiment that tested both conjectures.

4.1. The monotone conjecture is the limit of the monotone secant conjecture. The osculating plane $F_i(s)$ is the unique i -dimensional subspace having maximal order of contact with the rational normal curve γ at the point $\gamma(s)$, and therefore it is a limit of secant planes. We give a more precise statement of this fact.

Proposition 4.1. *Let $\{s_1^{(j)}, \dots, s_i^{(j)}\}$ for $j = 1, 2, \dots$ be a sequence of lists of i distinct complex numbers with the property that for each $p = 1, \dots, i$, we have*

$$\lim_{j \rightarrow \infty} s_p^{(j)} = s,$$

for some number s . Then,

$$\lim_{j \rightarrow \infty} \text{span}\{\gamma(s_1^{(j)}), \dots, \gamma(s_i^{(j)})\} = F_i(s).$$

As explained in the previous section, by this proposition, the monotone secant conjecture implies the monotone conjecture. This has a partial converse which follows from a standard limiting argument.

Theorem 4.2. *Let $(\sigma_1, \dots, \sigma_m)$ be a Schubert problem on $\mathbb{F}\ell(a; n)$ for which the monotone conjecture holds. Then, for any distinct real numbers that are monotone with respect to $(\sigma_1, \dots, \sigma_m)$, there exists a number $\epsilon > 0$ such that, if for each $i = 1, \dots, m$, F_\bullet^i is a flag secant to γ along an interval of length ϵ containing t_i , then the intersection*

$$X_{\sigma_1} F_\bullet^1 \cap X_{\sigma_2} F_\bullet^2 \cap \dots \cap X_{\sigma_m} F_\bullet^m$$

is transverse with all points real.

4.2. Generalized monotone secant conjecture. We generalize the monotone secant conjecture, replacing secant flags by flags which are spanned by osculating subspaces, as in [10, § 3.3]. By Proposition 4.1, such flags are intermediate between secant and osculating flags, so this new conjecture interpolates between the monotone secant and monotone conjectures. A *generalized secant subspace* to the rational normal curve γ is a subspace that is spanned by subspaces osculating γ at real points. This notion includes secant subspaces as well as osculating subspaces, as a point of γ generates a one-dimensional subspace osculating γ . A *generalized secant flag* is one consisting of generalized secant subspaces. A generalized secant

flag is *secant along an interval* I if the osculating subspaces spanning its subspaces osculate γ at points of I .

Conjecture 4.3 (Generalized monotone secant conjecture). *For any Grassmannian Schubert problem $(\sigma_1, \dots, \sigma_m)$ on the flag manifold $\mathbb{F}\ell(a_\bullet; n)$ and any disjoint generalized secant flags $F_\bullet^1, \dots, F_\bullet^m$ that are monotone with respect to the Schubert problem, the intersection*

$$X_{\sigma_1} F_\bullet^1 \cap X_{\sigma_2} F_\bullet^2 \cap \dots \cap X_{\sigma_m} F_\bullet^m$$

is transverse with all points real.

This conjecture contains the monotone and monotone secant conjectures as special cases, and interpolates between the two.

4.3. Experimental evidence for the monotone secant conjecture. While its relation to existing conjectures led to the monotone secant conjecture, we believe the immense weight of empirical evidence is the strongest support for it. We conducted an experiment that tested 11,141,897,000 instances of 1300 Schubert problems on 19 flag manifolds. Of these, 768,846,000 were instances of the monotone secant conjecture, which was verified in every case tested. We also tested 918,902,000 instances of the monotone conjecture for comparison. The remaining instances involved non-monotone evaluations of either disjoint secant flags or osculating flags. Our data consistently displayed a striking inner border, and a number of Schubert problems exhibited lower bounds on their numbers of real solutions. This experiment used 1.9 teraHertz-years of computing.

Table 1 shows the data we obtained for the Schubert problem (σ^4, τ^4) with 12 solutions on the flag manifold $\mathbb{F}\ell(2,3; 5)$ introduced in Conjecture 1.1. We computed 24,000,000 instances

		Real Solutions						Total	
		0	2	4	6	8	10	12	
Necklace	22223333							3000000	3000000
	22232333			31	205380	545269	1245017	1004303	3000000
	22322333			196	403485	1071579	967226	557514	3000000
	22332233			391	801525	1200700	651183	346201	3000000
	22323323		1025	70009	300121	938430	1123770	566645	3000000
	22323233		33488	950203	1256341	560205	164020	35743	3000000
	22232333		93232	284316	460016	1010425	750171	401840	3000000
	23232323	885953	854550	830122	298843	104574	23741	2217	3000000
	Total	885953	982295	2135268	3725711	5431182	4925128	5914463	24000000

TABLE 1. Necklaces vs. real solutions for (σ^4, τ^4) in $\mathbb{F}\ell(2,3; 5)$.

of this problem, all involving flags that were secant to the rational normal curve along disjoint intervals. This took 17.618 gigaHertz-years. The columns are indexed by even integers from 0 to 12, indicating the number of real solutions. The rows are indexed by *necklaces*, which are sequences $\delta(\sigma_1), \dots, \delta(\sigma_m)$, where $\delta(\sigma_i)$ denotes the descent of the Grassmannian permutation σ_i , as described in Section 3. In the table a 2 represents the condition on the two-plane E_2

given by the permutation $\sigma = 13245$, and a $\mathfrak{3}$ represents the condition on E_3 given by the permutation $\tau = 12435$.

An entry in a column labeled r in a row corresponding to a necklace represents the number of instances observed with r real solutions when the flags were arranged as indicated by the necklace. In Table 1, the first row labeled with 22223333 represents tests of the monotone secant conjecture. Since the only nonzero entry in this row is in the last column which records observed instances with 12 real solutions, the monotone secant conjecture was verified in 3,000,000 instances. This is the only row with only real solutions.

Compare this to the 16,000,000 instances of the same Schubert problem, but with osculating flags, which we present in Table 2. This computation took 85.203 gigahertz-days. Both tables are similar with nearly identical “inner borders”, except for the shaded cell in

		Real Solutions							
		0	2	4	6	8	10	12	Total
Necklace	22223333							2000000	2000000
	22233233			1041	246876	581972	582865	587246	2000000
	22322333			1480	263981	621920	584508	528111	2000000
	22332233			8882	217100	861124	503562	409332	2000000
	22323323		7329	120195	402799	665766	549653	261587	2000000
	22323233		7329	255114	431074	664551	420699	221233	2000000
	22232333		25552	137116	227922	415553	424582	769275	2000000
	23232323	49197	125725	557851	395992	516675	244212	110348	2000000
Total	49197	158606	1081679	2185744	4327561	3310081	4887132	16000000	

TABLE 2. Necklaces vs. real solutions for (σ^4, τ^4) in $\mathbb{F}\ell(2,3;5)$.

Table 2. In fact, by a standard argument similar to that which implied Theorem 4.2, we may conclude that every number of real solutions to a Schubert problem observed for a given necklace with osculating flags also occurs for that Schubert problem and necklace with secant flags, where the points of secancy are sufficiently clustered. That is, the support of a table for the monotone conjecture will be a subset of the support of the corresponding table for the monotone secant conjecture. There are some Schubert problems for which we did not observe this containment; the reason for this is that we apparently did not compute an instance with secant flags whose points of secancy were sufficiently clustered.

4.4. Lower bounds and inner borders. The most enigmatic phenomenon that we observe in our data is the presence of an “inner border” for many geometric problems, as we have pointed out in example of Table 1. That is, for some necklaces (besides the monotone ones), there appears to be a lower bound on the number of real solutions. We do not understand this phenomenon, even conjecturally. Our software that displays the tables is designed to highlight this feature of our data.

Another common phenomenon is that for many problems, there are always at least some real solutions, for any necklace. (Note that the last rows of Tables 1 and 2 had instances with no real solutions). Table 3 displays an example of this for a Schubert problem (σ^3, τ^5) on

$\mathbb{F}\ell(2,3;6)$ with 21 solutions, where $\sigma := 142356$ has $\delta(\sigma) = 2$ and $\ell(\sigma) = 2$ and $\tau := 124356$

		Real Solutions											
		1	3	5	7	9	11	13	15	17	19	21	Total
Necklace	22233333											80000	80000
	23233333							921	16549	26267	14475	21788	80000
	22332333						39	1208	24559	39013	13947	1234	80000
	23233233						612	9544	43256	23583	2927	78	80000
	23232333						3244	19887	31931	13688	3632	7618	80000
	Total						3895	31560	116295	102551	34981	110718	400000

TABLE 3. Enumerative Problem $W^3X^5 = 21$ on $\mathbb{F}\ell(2,3;6)$

has $\delta(\tau) = 3$ and $\ell(\tau) = 1$. Very prominently, it appears that at least 11 of the solutions will always be real. This computation took 7.67 gigaHertz-years.

Such lower bounds and inner borders were observed when studying the monotone conjecture [17]. Eremenko and Gabrielov established lower bounds for the Wronski map [4] in Schubert calculus for the Grassmannian, more recently Azar and Gabrielov [1] established lower bounds for some instances of the monotone conjecture which were observed in [17].

5. METHODS

Our experimentation was possible as instances of Schubert problems are simple to model on a computer. The procedure we use may be semi-automated and run on supercomputers. We will not describe how this automation is done, for that is the subject of the paper [11]; instead, we explain here the computations we performed.

For a Schubert condition σ on a flag variety $\mathbb{F}\ell(a_\bullet; n)$, let $j(\sigma)$ be the dimension of the largest subspace in a flag F_\bullet that is needed to define $X_\sigma F_\bullet$. For example, we have seen that $j(13245) = 3$ and $j(12435) = 2$.

To compute an instance of a Schubert problem $(\sigma_1, \dots, \sigma_m)$ corresponding to a necklace ν , we first select $N := N(\sigma_1) + \dots + N(\sigma_m)$ real numbers and then group them into disjoint subsets $s^{(1)}, \dots, s^{(m)}$ where $s^{(i)}$ consists of $N(\sigma_i)$ consecutive numbers. Furthermore, the relative ordering of the subsets corresponds to the necklace ν . Having done this, each subset $s^{(i)}$ defines a secant (partial) flag $F_\bullet(s^{(i)})$. We use these flags to formulate the instance of the Schubert problem

$$X_{\sigma_1} F_\bullet(s^{(1)}) \cap X_{\sigma_2} F_\bullet(s^{(2)}) \cap \dots \cap X_{\sigma_m} F_\bullet(s^{(m)})$$

as a system of polynomials in $\dim(a_\bullet)$ local coordinates for $\mathbb{F}\ell(a_\bullet; n)$, whose common zeroes represent the solutions to this instance of the Schubert problem. This was illustrated in the Introduction when Conjecture 1.1 was presented. See [8, 17, 18] for details.

Given this system of polynomials, we use Gröbner bases to compute a polynomial in one variable of minimal degree in the ideal of these equations. This univariate polynomial is called an *eliminant*. If the eliminant is square-free and has degree equal to the expected number of complex solutions (this is easily verified) then the number of real roots of the eliminant

equals the number of real solutions to the Schubert problem. This follows from the form of a lexicographic Gröbner basis for the ideal, as described by the Shape Lemma [3]. This is given in more detail in §2.2 of [21]. We determine the number of real solutions to the Schubert problem by computing the number of real roots of the eliminant. For this, we use MAPLE's `realroot` command, which uses symbolic methods based on Sturm sequences to determine the number of real roots of a univariate polynomial. If the software is reliably implemented, which we believe, then this computation provides a proof that the given instance has the computed number of real solutions to the original Schubert problem.

In our computations, for a given Schubert problem, we first make a choice of N real numbers, and then use these same N numbers for all necklaces for that problem. Then we make another choice, and so on, ultimately making thousands to millions of such choices.

For each Schubert problem we studied, we not only tested many instances of the monotone secant conjecture, but also of the monotone conjecture, comparing the two as we did for the Schubert problem (σ^4, τ^4) in $\mathbb{F}\ell(2,3;5)$, where $\sigma = 13245$ and $\tau = 12435$. To compute instances of the monotone conjecture, we choose real numbers s_1, \dots, s_m and use osculating flags $F_\bullet(s_1), \dots, F_\bullet(s_m)$. This is also described in [17, § 5].

For many Schubert problems, it was infeasible to compute instances of the monotone secant conjecture, and we instead computed instances of the generalized monotone secant conjecture. For these, one of the flags was the flag $F_\bullet(\infty)$ osculating the rational normal curve at infinity. Then we used local coordinates for $X_{\sigma_1} F_\bullet(\infty)$, in place of the local coordinates for $\mathbb{F}\ell(a_\bullet; n)$; this uses $\ell(\sigma_i)$ fewer local coordinates.

For some Schubert problems we wanted to study, there were several hundred to many thousands of necklaces, and for these we uniformly chose a much smaller set of necklaces to compute, which we called coarse necklaces. In our on-line tables, we encoded these choices in a variable called computation type. Computation types 4 and 7 were for instances of the monotone conjecture, 5 and 8 for the monotone secant conjecture, and 6 and 9 for the generalized monotone secant conjecture. In each of these, the first number indicates that we used all necklaces, while the second we used coarse necklaces.

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