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NEWTON POLYTOPES AND WITNESS SETS

JONATHAN D. HAUENSTEIN AND FRANK SOTTILE

Abstract. We present two algorithms that compute the Newton polytope of a polynomial f defining a hypersurface $\mathcal{H}$ in $\mathbb{C}^n$ using numerical computation. The first algorithm assumes that we may only compute values of f—this may occur if f is given as a straight-line program, as a determinant, or as an oracle. The second algorithm assumes that $\mathcal{H}$ is represented numerically via a witness set. That is, it computes the Newton polytope of $\mathcal{H}$ using only the ability to compute numerical representatives of its intersections with lines. Such witness set representations are readily obtained when $\mathcal{H}$ is the image of a map or is a discriminant. We use the second algorithm to compute a face of the Newton polytope of the Lüroth invariant, as well as its restriction to that face.

Introduction

While a hypersurface $\mathcal{H}$ in $\mathbb{C}^n$ is always defined by the vanishing of a single polynomial f, we may not always have access to the monomial representation of f. This occurs, for example, when $\mathcal{H}$ is the image of a map or if f is represented as a straight-line program, and it is a well-understood and challenging problem to determine the polynomial f when $\mathcal{H}$ is represented in this way. Elimination theory gives a symbolic method based on Gröbner bases that can determine f from a representation of $\mathcal{H}$ as the image of a map or as a discriminant [8]. Such computations require that the map be represented symbolically, and they may be infeasible for moderately-sized input.

The set of monomials in f, or more simply the convex hull of their exponent vectors (the Newton polytope of f), is an important combinatorial invariant of the hypersurface. The Newton polytope encodes asymptotic information about $\mathcal{H}$ and determining it from $\mathcal{H}$ is a step towards determining the polynomial f. For example, numerical linear algebra [7, 13] may be used to find f given its Newton polytope. Similarly, the Newton polytope of an image of a map may be computed from Newton polytopes of the polynomials defining the map [12, 14, 15, 31, 32], and computed using algorithms from tropical geometry [33].

We propose numerical methods to compute the Newton polytope of f in two cases when f is not known explicitly. We first show how to compute the Newton polytope when we are able to evaluate f. This occurs, for example, if f is represented as a straight-line program or as a determinant (neither of which we want to expand as a sum of monomials), or perhaps as a compiled program. For the other case, we suppose that f defines a

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hypersurface $\mathcal{H}$ that is represented numerically as a witness set. Our basic idea is similar to ideas from tropical geometry. The tropical variety of a hypersurface $\mathcal{H}$ in $(\mathbb{C}^*)^n$ is the normal fan to the Newton polytope of a defining polynomial f, augmented with the edge lengths. The underlying fan coincides with the logarithmic limit set [4, 5] of $\mathcal{H}$, which records the asymptotic behavior of $\mathcal{H}$ in $(\mathbb{C}^*)^n$. We use numerical algebraic geometry to study the asymptotic behavior of $\mathcal{H}$ in $(\mathbb{C}^*)^n$ and use this to recover the Newton polytope of a defining equation of $\mathcal{H}$. As the second algorithm is based on path following, it is easily parallelizable, therefore this numerical approach to Newton polytopes should allow the computation of significantly larger examples than are possible with algorithms that employ symbolic methods.

This paper is organized as follows. In Section 1, we explain symbolic and geometric-combinatorial preliminaries, including representations of polytopes, Newton polytopes, and straight-line programs. In Section 2, we discuss the essentials of numerical algebraic geometry, in particular explaining the fundamental data structure of witness sets. Our main results are in the next two sections. In Section 3 we explain (in Theorem 5 and Remark 6) how to compute the Newton polytope of f, given only that we may numerically evaluate f, and in Section 4, we explain (in Theorems 9 and 10, and Remark 11) how to use witness sets to compute the Newton polytope of f. Examples are presented in these sections, and we note that the algorithms are presented in the remarks. In Section 5, we combine our approach with other techniques in numerical algebraic geometry to determine the Newton polytope of and then to explicitly compute the hypersurface of Lüroth quartics in the subspace of Ciani (even) quartics. Roughly a month after an initial draft of the present article was posted on the `arXiv`, this result was verified using different techniques in [1].

1. Polynomials and Polytopes

We explain necessary background from geometric combinatorics and algebra.

1.1. **Polytopes.** A polytope P is the convex hull of finitely many points $\mathcal{A} \subset \mathbb{R}^n$,

$$P \;=\; \operatorname{conv}(\mathcal{A}) \;:=\Bigl\{\sum_{\alpha\in\mathcal{A}} \lambda_\alpha \alpha \;:\; \lambda_\alpha \geq 0\,,\; \sum_{\alpha} \lambda_\alpha = 1\Bigr\}\,. \tag{1.1}$$

Dually, a polytope is the intersection of finitely many halfspaces in $\mathbb{R}^n$,

$$P \;=\; \{x \in \mathbb{R}^n \;:\; w_i \cdot x \leq b_i \quad \text{for } i = 1, \ldots, N\}\,, \tag{1.2}$$

where $w_1, \ldots, w_N \in \mathbb{R}^n$ and $b_1, \ldots, b_N \in \mathbb{R}$. These are two of the most common representations of a polytope. The first (1.1) is the *convex hull* representation and the second (1.2) is the *halfspace* representation. The classical algorithm of Fourier-Motzkin elimination converts between these two representations.

The *affine hull* of a polytope P is the smallest affine-linear space containing P. The boundary of P (in its affine hull) is a union of polytopes of smaller dimension than P, called *faces* of P. A *facet* of P is a maximal proper face, while a *vertex* is a minimal face of P (which is necessarily a point). An *edge* is a 1-dimensional face.

In addition to the two representations given above, polytopes also have a *tropical representation*, which consists of the edge lengths, together with the normal fan to the edges. (This normal fan encodes the edge-face incidences.) Jensen and Yu [22] gave an algorithm for converting a tropical representation into a convex hull representation.

Every linear function $x \mapsto w \cdot x$ on $\mathbb{R}^n$ (here, $w \in \mathbb{R}^n$) achieves a maximum value on a polytope P. The subset P_w of P where this maximum value is achieved is a face of P, called the *face exposed by w*. Let $h_P(w)$ be this maximum value of $w \cdot x$ on P. The function $w \mapsto h_P(w)$ is called the *support function* of P. The support function encodes the halfspace representation as

$$P \;=\; \{x \in \mathbb{R}^n \;:\; w \cdot x \leq h_P(w) \quad \text{for } w \in \mathbb{R}^n\}\,.$$

The *oracle* representation is a fourth natural representation of a polytope P. There are two versions. For the first, given $w \in \mathbb{R}^n$, if the face P_w exposed by w is a vertex, then it returns that vertex, and if P_w is not a vertex, it either returns a vertex on P_w or detects that P_w is not a vertex. Alternatively, it returns the value $h_P(w)$ of the support function at w. The classical beneath-beyond algorithm [16, §5.2] uses an oracle representation of a polytope to simultaneously construct its convex-hull and halfspace representations. It iteratively builds a description of the polytope, including the faces and facet-supporting hyperplanes, adding one vertex at a time. The software package iB4e [21] implements this algorithm. Another algorithm converting the oracle representation to the convex hull and halfspace representation is "gift-wrapping" [6].

Our numerical algorithms return oracle representations.

1.2. **Polynomials and their Newton polytopes.** Let $\mathbb{N} = \{0, 1, \dots\}$ be the nonnegative integers and write $\mathbb{C}^*$ for the nonzero complex numbers. Of the many ways to represent a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$, perhaps the most familiar is in terms of monomials. For $\alpha \in \mathbb{N}^n$, we have the monomial

$$x^\alpha \;:=\; x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\,,$$

which has *degree* $|\alpha| := \alpha_1 + \cdots + \alpha_n$. A polynomial f is a linear combination of monomials

$$f \;=\; \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha \qquad c_\alpha \in \mathbb{C}\,, \tag{1.3}$$

where only finitely many coefficients c_α are nonzero. The set $\{\alpha \in \mathbb{N}^n \;:\; c_\alpha \neq 0\}$ is the *support* of f, which we will write as $\mathcal{A}(f)$, or simply $\mathcal{A}$ when f is understood.

A coarser invariant of the polynomial f is its *Newton polytope*, $\mathcal{N}(f)$. This is the convex hull of its support

$$\mathcal{N}(f) \;:=\; \operatorname{conv}(\mathcal{A}(f))\,.$$

For $w \in \mathbb{R}^n$, the *restriction f_w* of f to the face $\mathcal{N}(f)_w$ of $\mathcal{N}(f)$ exposed by w is

$$f_w \;:=\; \sum_{\alpha \in \mathcal{A} \cap \mathcal{N}(f)_w} c_\alpha x^\alpha\,, \tag{1.4}$$

the sum over all terms $c_\alpha x^\alpha$ of f where $w \cdot \alpha$ is maximal (and thus equal to $h_{\mathcal{N}(f)}(w)$.)

A *hypersurface* $\mathcal{H} \subset \mathbb{C}^n$ is defined by the vanishing of a single polynomial, $\mathcal{H} = \mathcal{V}(f)$. This polynomial f is well-defined up to multiplication by non-zero scalars if we require it to be of minimal degree among all polynomials vanishing on $\mathcal{H}$. We define the *Newton polytope, $\mathcal{N}(\mathcal{H})$*, of $\mathcal{H}$ to be the Newton polytope of any minimal degree polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ defining $\mathcal{H}$.

Polynomials are not always given as a linear combination of monomials (1.3). For example, a polynomial may be given as a determinant whose entries are themselves polynomials. It may be prohibitive to expand this into a sum of monomials, but it is computationally efficient to evaluate the determinant. Additionally, a polynomial may be given as an oracle or as a compiled program. Other representations are possible. In [1], the Lüroth invariant is expressed in terms of the fundamental and secondary invariants of plane quartics.

An efficient encoding of a polynomial is as a *straight line program*. For a polynomial $f\colon \mathbb{C}^n \to \mathbb{C}$, this is a list

$$(f_{-n}, \ldots, f_{-1}, f_0, f_1, \ldots, f_l)$$

of polynomials where $f = f_l$ and we have the initial values $f_{-i} = x_i$ for $i = 1, \ldots, n$, and for every $k \geq 0$, f_k is one of

$$f_i + f_j\,,\ \ f_i \cdot f_j\,,\ \text{ or } c\,,$$

where $i, j < k$ and $c \in \mathbb{Q}[\sqrt{-1}]$ is a Gaussian rational number. (Gaussian rational numbers are used for simplicity as they are representable on a computer.)

Our goal is twofold, we present an algorithm to compute the Newton polytope of a polynomial f that we can only evaluate numerically, and we present an algorithm to recover the Newton polytope of a polynomial f defining a hypersurface $\mathcal{H}$ that is represented numerically as a witness set (defined in Section 2 below).

In the first case, we explain how to compute the support function $h_{\mathcal{N}(f)}$ of the Newton polytope of f, and to compute $\mathcal{N}(f)_w$, when this is a vertex. This becomes an algorithm, at least for general w, when we have additional information about f, such as a finite superset $\mathcal{B} \subset \mathbb{Z}^n$ of its support and bounds on the magnitudes of its coefficients. This is discussed in Remark 6.

In the second case, we show how to compute $\mathcal{N}(f)_w$, when this is a vertex. This is discussed in Remark 11.

2. Numerical algebraic geometry and witness sets

Numerical algebraic geometry provides methods based on numerical continuation for studying algebraic varieties on a computer. The fundamental data structure in this field is a witness set, which is a geometric representation based on linear sections and generic points.

Given a polynomial system $F\colon \mathbb{C}^m \to \mathbb{C}^n$, suppose that $V \subset \mathcal{V}(F) := F^{-1}(0)$ is an irreducible component of its zero set of dimension k and degree d. Let $\mathcal{L}\colon \mathbb{C}^m \to \mathbb{C}^k$ be a system of general affine-linear polynomials so that $\mathcal{V}(\mathcal{L})$ is a general codimension k affine subspace of $\mathbb{C}^m$. Then $W := V \cap \mathcal{V}(\mathcal{L})$ will consist of d distinct points, and we call the triple $(F, \mathcal{L}, W)$ (or simply W for short) a *witness set* for V. The set W represents a general linear section of V. Numerical continuation may be used to follow the points of W as $\mathcal{L}$ (and hence $\mathcal{V}(\mathcal{L})$) varies continuously. This allows us to sample points from V.

Ideally, V is a *generically reduced* component of the scheme $\mathcal{V}(F)$ in that the Jacobian of F at a general point $w \in W \subset V$ of V has a k-dimensional null space. Otherwise the scheme $\mathcal{V}(F)$ is not reduced along V. When V is a generically reduced component of $\mathcal{V}(F)$, the points of W are nonsingular zeroes of the polynomial system $\left[\begin{smallmatrix} F \\ \mathcal{L} \end{smallmatrix}\right]$. When $\mathcal{V}(F)$ is not reduced along V, the points of W are singular zeroes of this system, and it is numerically challenging, but feasible, to compute such singular points.

The method of deflation [9, 19, 23, 24, 27, 28, 30] can compute W when $\mathcal{V}(F)$ is not reduced along V. In particular, the strong deflation method of [19] yields a system $F' \colon \mathbb{C}^m \to \mathbb{C}^{n'}$ where $n' \geq n$ such that V is a generically reduced component of $\mathcal{V}(F')$. Replacing F with F', we will assume that V is a generically reduced component of $\mathcal{V}(F)$.

The notion of a witness set for the image of an irreducible variety under a linear map was developed in [17]. Suppose that we have a polynomial system $F \colon \mathbb{C}^m \to \mathbb{C}^n$, a generically reduced component V of $\mathcal{V}(F)$ of dimension k and degree d, and a linear map $\omega \colon \mathbb{C}^m \to \mathbb{C}^p$ defined by $\omega(x) = Ax$ for $A \in \mathbb{C}^{p \times m}$. Suppose that the algebraic set $U = \overline{\omega(V)} \subset \mathbb{C}^p$ has dimension k' and degree d'. A witness set for the projection U requires an affine-linear map $\mathcal{L}$ adapted to the projection ω. Let B be a matrix $\left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$ where the rows of the matrix $B_1 \in \mathbb{C}^{k' \times m}$ are general vectors in the row space of A and the rows of $B_2 \in \mathbb{C}^{(k-k') \times m}$ are general vectors in $\mathbb{C}^m$. Define $\mathcal{L} \colon \mathbb{C}^m \to \mathbb{C}^k$ by $\mathcal{L}(x) = Bx - 1$ and set $W := V \cap \mathcal{V}(\mathcal{L})$. Then the quadruple $(F, \omega, \mathcal{L}, W)$ is a *witness set for the projection U*. By our choice of B, the number of points in $\omega(W)$ is the degree d' of U and for any fixed $u \in \omega(W)$, the number of points in $W \cap \omega^{-1}(u)$ is the degree of the general fiber of ω restricted to V. Note that $k - k'$ is the dimension of the general fiber.

Example 1. Consider the discriminant hypersurface $\mathcal{H} \subset \mathbb{C}^3$ for univariate quadratic polynomials, that is, $\mathcal{H} := \mathcal{V}(f)$ where $f(a, b, c) = b^2 - 4ac$. The triple $(f, \mathcal{L}, W)$ where, for simplicity in the presentation, we take the sufficiently generic linear system

$$\mathcal{L}(a, b, c) \ := \ \begin{bmatrix} 2a - 2b + 3c - 1 \\ 3a + b - 5c - 1 \end{bmatrix}$$

and $W = \mathcal{H} \cap \mathcal{V}(\mathcal{L})$, which consists of the two points, (a, b, c),

$$\{(0.3816,\ -0.1071,\ 0.00752)\,,\ (1.2243,\ 2.1801,\ 0.97058)\}\,,$$

is a witness set for $\mathcal{H}$.

This discriminant also has the form $\mathcal{H} = \overline{\omega(V)}$ where ω is the linear projection mapping (a, b, c, x) to (a, b, c) and $V = \mathcal{V}(F)$ where

$$F(a, b, c, x) \ = \ \begin{bmatrix} ax^2 + bx + c \\ 2ax + b \end{bmatrix}.$$

This variety V has dimension 2 and degree 3, and ω is defined by the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The quadruple $(F, \omega, \mathcal{L}', W')$ where $\mathcal{L}'(a,b,c,x) = \mathcal{L}(a,b,c)$ and $W' = V \cap \mathcal{V}(\mathcal{L}')$, which also consists of two points, (a,b,c,x),

$$\{(0.3816,\ -0.1071,\ 0.00752,\ 0.1403882)\,,\ (1.2243,\ 2.1801,\ 0.97058,\ -0.8903882)\}\,,$$

is also a witness set for $\mathcal{H}$. In particular, $\omega(W') = W$ and we see that ω restricted to V is generically one-to-one. □

Remark 2. The computations performed in the two algorithms that we present involve the evaluation of polynomials of very high degree or solving systems of equations with extreme coefficients. This at the least requires computation using high precision arithmetic. While it may seem to restrict the computation to exact inputs or lead to numerical instabilities, we note that the asymptotic information we extract from this computation is expected to be stable under perturbation.

3. Newton polytopes via evaluation

We address the problem of computing the Newton polytope of a polynomial $f : \mathbb{C}^n \to \mathbb{C}$ when we have a method to evaluate f. This is improved when we have some additional information about the polynomial f.

While the following discussion is elementary (and contained implicitly in [20]) we present it for completeness. For t a positive real number and $w \in \mathbb{R}^n$, set $t^w := (t^{w_1}, t^{w_2}, \ldots, t^{w_n})$. Consider the monomial expansion of the polynomial f,

$$f \;=\; \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha \qquad \text{where} \quad c_\alpha \in \mathbb{C}^*\,.$$

For $x \in \mathbb{C}^n$, we define the action of t^w on x by

$$t^w.x \;:=\; (t^{w_1}x_1\,,\, t^{w_2}x_2\,,\, \ldots\,,\, t^{w_n}x_n)\,,$$

the coordinatewise product, and consider the evaluation,

$$f(t^w.x) \;=\; \sum_{\alpha \in \mathcal{A}} c_\alpha t^{w\cdot\alpha} x^\alpha\,. \tag{3.1}$$

Let $\mathcal{F} := \mathcal{N}(f)_w$ be the face of $\mathcal{N}(f)$ that is exposed by w. If $\alpha \in \mathcal{F}$, then we have $w \cdot \alpha = h_{\mathcal{N}(f)}(w)$. There is a positive real number d_w such that if $\alpha \in \mathcal{A} \setminus \mathcal{F}$, then $w \cdot \alpha \leq h_{\mathcal{N}(f)}(w) - d_w$. Thus (3.1) becomes

$$\begin{aligned} f(t^w.x) \;&=\; \sum_{\alpha \in \mathcal{A}\cap\mathcal{F}} c_\alpha t^{w\cdot\alpha} x^\alpha \;+\; \sum_{\alpha \in \mathcal{A}\smallsetminus\mathcal{F}} c_\alpha t^{w\cdot\alpha} x^\alpha \\ &=\; t^{h_{\mathcal{N}(f)}(w)}\Big(f_w(x) \;+\; \sum_{\alpha \in \mathcal{A}\smallsetminus\mathcal{F}} c_\alpha t^{w\cdot\alpha - h_{\mathcal{N}(f)}(w)} x^\alpha\Big)\,, \end{aligned}$$

where f_w is the restriction of f to the face $\mathcal{F}$. Observe that no exponent of t which occurs in the sum exceeds $-d_w$. This gives an asymptotic expression for $t \gg 0$,

$$\log|f(t^w.x)| \;=\; h_{\mathcal{N}(f)}(w)\log(t) \;+\; \log|f_w(x)| \;+\; O(t^{-d_w})\,, \tag{3.2}$$

from which we deduce the following limit.

Lemma 3. *If $f_w(x) \neq 0$, then*

$$h_{\mathcal{N}(f)}(w) \;=\; \lim_{t\to\infty} \frac{\log|f(t^w.x)|}{\log(t)}\,.$$

Thus we may approximate the support function of $\mathcal{N}(f)$ by evaluating f numerically.

Remark 4. To turn Lemma 3 into an algorithm for computing $h_{\mathcal{N}(f)}$, we need more information about f, so that we may estimate the rate of convergence. For example, if we have a bound, in the form of a finite superset $\mathcal{B} \subset \mathbb{N}^n$ of $\mathcal{A}$, then $\{w\cdot\alpha \mid \alpha\in\mathcal{B}\}$ is a discrete set which contains the value of $h_{\mathcal{N}(f)}(w)$, and therefore the limit in Lemma 3. □

When w is generic in that $\alpha \mapsto w\cdot\alpha$ is injective on $\mathcal{A}$, then the face $\mathcal{N}(f)_w$ of $\mathcal{N}(f)$ exposed by w is a vertex so that $f_w(x)\neq 0$ for any $x\in(\mathbb{C}^*)^n$. We may dispense with the limit given an *a priori* estimate on the magnitude of the coefficients of f.

Theorem 5. *Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial with monomial expansion,*

$$f(x) \;=\; \sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha\,, \qquad c_\alpha \in \mathbb{C}^*\,.$$

Suppose that $\delta,\lambda \geq 1$ and $\mathcal{B}\subset\mathbb{N}^n$ are such that

(1) $\log|c_\alpha| \leq \delta$ *for all* $\alpha\in\mathcal{A}$,
(2) $\log|c_\alpha| \leq \lambda + \log|c_\beta|$ *for all* $\alpha,\beta\in\mathcal{A}$, *and*
(3) $\mathcal{A}\subset\mathcal{B}$ *with* $|\mathcal{B}|<\infty$.

Let $w\in\mathbb{R}^N$ be general in that

$$d_w \;:=\; \min_{\alpha\neq\beta\in\mathcal{B}} |w\cdot\alpha - w\cdot\beta| \;>\; 0\,.$$

Then, the face of $\mathcal{N}(f)$ exposed by w is a vertex $\beta\in\mathcal{B}$ which, for any $t>0$ such that $\log(t)$ *exceeds* $\max\{2\lambda, 2(\delta+e^{-1}), \lambda+\log|\mathcal{B}|+1\}/d_w$*, is the unique element of $\mathcal{B}$ satisfying*

$$\left| w\cdot\beta - \frac{\log|f(t^w)|}{\log(t)} \right| \;<\; \frac{d_w}{2}.$$

Similarly, the face exposed by $-w$ is a vertex $\beta\in\mathcal{B}$ which, for any $t>0$ such that $\log(t)$ *exceeds* $\max\{2\lambda, 2(\delta+e^{-1}), \lambda+\log|\mathcal{B}|+1\}/d_w$*, is the unique element of $\mathcal{B}$ satisfying*

$$\left| -w\cdot\beta - \frac{\log|f(t^{-w})|}{\log(t)} \right| \;<\; \frac{d_w}{2}.$$

Remark 6. Suppose that we know or may estimate the quantities $\mathcal{B}$, δ, and λ of Theorem 5. Then, for general $w\in\mathbb{R}^n$ we may compute d_w, and therefore evaluating $\log|f(t^w)|/\log(t)$ for $t^{d_w} > \max\{e^{2\lambda}, e^{2+2\delta}, |\mathcal{B}|e^{\lambda+1}\}$ will yield $w\cdot\beta$ and hence β. Therefore, Theorem 5 yields an algorithm in the formal sense.

Even without this knowledge, we may still compute the support function $h_{\mathcal{N}(f)}(w)$ for $w\in\mathbb{Q}^n$ as follows. For $0\neq w\in\mathbb{Q}^n$ the map $\mathbb{Z}^n\to\mathbb{Q}$ given by $\beta\mapsto w\cdot\beta$ has image a free group $\mathbb{Z}d_w$ for some $d_w>0$. For $x\in\mathbb{C}^n$ with $f_w(x)\neq 0$ and $t:=e^\tau$ with $\tau>0$, we have

$$\left| \frac{\log|f(e^{\tau w}.x)|}{\tau} \;-\; h_{\mathcal{N}(f)}(w) \right| \;\approx\; \frac{\log|f_w(x)|}{\tau} \;+\; O(e^{-d_w\tau})\,.$$

Since $h_{\mathcal{N}(f)}(w) \in \mathbb{Z}d_w$, we may do the following. Pick a general $x \in \mathbb{C}^n$ (so that $f_w(x) \neq 0$), and compute the quantity

$$\frac{\log |f(e^{\tau w}.x)|}{\tau} \tag{3.3}$$

for τ in some increasing sequence of positive numbers. We monitor (3.3) for $\frac{1}{\tau}$-convergence to some $\kappa d_w \in \mathbb{Z}d_w$. Then $h_{\mathcal{N}(f)}(w) = \kappa d_w$.

Every such computation gives a halfspace

$$\{x \in \mathbb{R}^n \mid w \cdot x \leq h_{\mathcal{N}(f)}(w)\}$$

containing $\mathcal{N}(f)$. Since $\mathcal{N}(f)$ lies in the positive orthant, we may repeat this one or more times to obtain a bounded polytope P containing $\mathcal{N}(f)$. Having done so, set $\mathcal{B} := P \cap \mathbb{N}^n$.

Suppose that $w \in \mathbb{R}^n$ is general in that the values $w \cdot \alpha$ for $\alpha \in \mathcal{B}$ are distinct. This implies that w exposes a vertex β of $\mathcal{N}(f)$. Then, a similar (but simpler as $f_w(t^w) = c_\beta t^{h_{\mathcal{N}(f)}(w)}$) scheme as described above will result in the computation of the support function $h_{\mathcal{N}(f)}(w)$ and vertex β. This provides an algorithm in the practical sense. $\square$

Proof of Theorem 5. By the choice of w, the face of $\mathcal{N}(f)$ it exposes is a vertex, say $\beta \in \mathcal{A}$, and we have $w \cdot \beta = h_{\mathcal{N}(f)}(w)$. We may write

$$f(t^w) \;=\; c_\beta t^{w\cdot\beta} \;+\; \left(f(t^w) - c_\beta t^{w\cdot\beta}\right) \;=\; c_\beta t^{w\cdot\beta}\left(1 \;+\; \frac{f(t^w) - c_\beta t^{w\cdot\beta}}{c_\beta t^{w\cdot\beta}}\right).$$

Taking absolute value and logarithms, and using that $\log|c_\beta| < \delta$ and $w \cdot \beta = h(w)$,

$$\begin{aligned} \log|f(t^w)| \;&=\; \log\left|c_\beta t^{w\cdot\beta}\right| + \log\left|1 + \frac{f(t^w) - c_\beta t^{w\cdot\beta}}{c_\beta t^{w\cdot\beta}}\right| \\ &\leq\; \delta + w\cdot\beta\,\log(t) + \log\left|1 + \frac{f(t^w) - c_\beta t^{w\cdot\beta}}{c_\beta t^{w\cdot\beta}}\right|. \end{aligned} \tag{3.4}$$

Let us estimate the last term. As $\mathcal{A} \subset \mathcal{B}$, we have

$$\left|\frac{f(t^w) - c_\beta t^{w\cdot\beta}}{c_\beta t^{w\cdot\beta}}\right| \;=\; \left|\sum_{\substack{\alpha\in\mathcal{A}\\ \alpha\neq\beta}} \frac{c_\alpha}{c_\beta} t^{w\cdot\alpha - w\cdot\beta}\right| \;\leq\; \sum_{\substack{\alpha\in\mathcal{A}\\ \alpha\neq\beta}} e^{\lambda} t^{-d_w} \;\leq\; |\mathcal{B}|\, e^{\lambda - \log(t) d_w}.$$

Since $\log(t) > (\lambda + \log|\mathcal{B}| + 1)/d_w$, we have $|\mathcal{B}|e^{\lambda - \log(t)d_w} < e^{-1}$. Since $\log|1+x| \leq |x|$, we have

$$\log\left|1 + \frac{f(t^w) - c_\beta t^{w\cdot\beta}}{c_\beta t^{w\cdot\beta}}\right| \;\leq\; \left|\frac{f(t^w) - c_\beta t^{w\cdot\beta}}{c_\beta t^{w\cdot\beta}}\right| \;\leq\; |\mathcal{B}|\, e^{\lambda - d_w \log(t)} \;<\; e^{-1}.$$

Finally, as we have $\log(t) > 2(\delta + e^{-1})/d_w$, we obtain

$$\frac{\log|f(t^w)|}{\log(t)} \leq w\cdot\beta + (\delta + e^{-1})\frac{1}{\log(t)} \;<\; w\cdot\beta + \frac{d_w}{2}. \tag{3.5}$$

For the other inequality, using (3.4) and Condition (2) of the theorem,

$$\log|f(t^w)| \;\geq\; \delta - \lambda + \log(t)\, w\cdot\beta + \log\left|1 + \frac{f(t^w) - c_\beta t^{w\cdot\beta}}{c_\beta t^{w\cdot\beta}}\right| . \tag{3.6}$$

Since

$$\left|\frac{f(t^w) - c_\beta t^{w\cdot\beta}}{c_\beta t^{w\cdot\beta}}\right| \;<\; e^{-1}\,,$$

the logarithm on the right of (3.6) exceeds -1. As $\delta - \lambda \geq 1 - \lambda$, we have

$$\frac{\log|f(t^w)|}{\log(t)} \;>\; -\frac{\lambda}{\log(t)} + w\cdot\beta \;\geq\; w\cdot\beta - \frac{d_w}{2}\,,$$

since $d_w \log(t) \geq 2\lambda$. Combining this with (3.5) proves the first statement about $f(t^w)$. The statement about $f(t^{-w})$ has the same proof, replacing w with $-w$. □

Example 7. Reconsider the polynomial $f(a,b,c) = b^2 - 4ac$ from Ex. 1 with the vector $w = (-1.2, 0.4, 3.7)$. Suppose that we take $\lambda = \delta = 2$ and $\mathcal{B} = \{a^2, ab, ac, b^2, bc, c^2\}$ which are the columns of the matrix

$$\mathcal{B} \;=\; \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix} .$$

Then the dot products are $w\cdot\mathcal{B} = (-2.4, -0.8, 2.5, 0.8, 4.1, 7.4)$, so that $d_w = 1.6$. Since we need $\log(t) > 3.75$, we can take $t = 45$, and so $t^w = (45^{-1.2}, 45^{0.4}, 45^{3.7})$. We compute

$$\frac{\log|f(t^w)|}{\log(t)} \;=\; 2.864 \qquad \text{and} \qquad -\frac{\log|f(t^{-w})|}{\log(t)} \;=\; 0.8016\,.$$

Thus, the monomials ac and b^2 are the vertices $\mathcal{N}(f)_w$ and $\mathcal{N}(f)_{-w}$, respectively. □

4. Newton polytopes via witness sets

Let $\mathcal{H} \subset \mathbb{C}^n$ be an irreducible hypersurface and suppose that we have a witness set representation for $\mathcal{H}$. As discussed in Section 2, this means that we may compute the intersections of $\mathcal{H}\cap\ell$ where ℓ is a general line in $\mathbb{C}^n$. We explain how to use this information to compute an oracle representation of the Newton polytope of $\mathcal{H}$.

The hypersurface $\mathcal{H} \subset \mathbb{C}^n$ is defined by a single irreducible polynomial

$$f \;=\; \sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha \qquad c_\alpha \in \mathbb{C}^*\,, \tag{4.1}$$

which is determined by $\mathcal{H}$ up to multiplication by a scalar.

Let $a, b \in \mathbb{C}^n$ be general points, and consider the parametrized line

$$\ell_{a,b} \;=\; \ell(s) \;:=\; \{sa - b \mid s \in \mathbb{C}\}\,.$$

Then the solutions to $f(\ell(s)) = 0$ parameterize the intersection of $\mathcal{H}$ with the line $\ell_{a,b}$, which is a witness set for $\mathcal{H}$.

Let $w \in \mathbb{R}^n$. For t a positive real number, consider $f(t^w.\ell(s))$, which is

$$\sum_{\alpha\in\mathcal{A}} c_\alpha (sa_1 - b_1)^{\alpha_1}(sa_2 - b_2)^{\alpha_2}\cdots(sa_n - b_n)^{\alpha_n}\,.t^{w\cdot\alpha} \tag{4.2}$$

Write $(sa-b)^\alpha$ for the product of terms $(sa_i - b_i)^{\alpha_i}$ appearing in the sum.

Let $\mathcal{F} := \mathcal{N}(\mathcal{H})_w$ be the face of the Newton polytope of $\mathcal{H}$ exposed by w. If $\alpha \in \mathcal{F}$, then $w\cdot\alpha = h(w)$, where h is the support function of $\mathcal{N}(\mathcal{H})$. There is a positive number d_w such that if $\alpha \in \mathcal{A} \smallsetminus \mathcal{F}$, then $w \cdot \alpha \leq h(w) - d_w$. We may rewrite (4.2),

$$f(t^w.\ell(s)) \;=\; t^{h(w)} \sum_{\alpha\in\mathcal{A}\cap\mathcal{F}} c_\alpha (as-b)^\alpha \;+\; \sum_{\alpha\in\mathcal{A}\smallsetminus\mathcal{F}} c_\alpha (as-b)^\alpha t^{w\cdot\alpha}\,.$$

Multiplying by $t^{-h(w)}$ and rewriting using the definition (1.4) of f_w gives

$$t^{-h(w)} f(t^w.\ell(s)) \;=\; f_w(\ell(s)) \;+\; \sum_{\alpha\in\mathcal{A}\smallsetminus\mathcal{F}} c_\alpha (as-b)^\alpha t^{w\cdot\alpha - h(w)}\,. \tag{4.3}$$

Observe that the exponent of t in each term of the sum over $\mathcal{A} \smallsetminus \mathcal{F}$ is at most $-d_w$.

As $s \mapsto \ell(s)$ and $s \mapsto t^w.\ell(s)$ are general parametrized lines in $\mathbb{C}^n$, the zeroes (in s) of $f(t^w.\ell(s))$ and $f_w(\ell(s))$ parameterize witness sets for f and f_w, respectively. The following summarizes this discussion.

Lemma 8. *In the limit as $t \to \infty$, there are $\deg(f) - \deg(f_w)$ points of the witness set $f(t^w.\ell(s)) = 0$ which diverge to ∞ (in s) and the remaining points converge to the witness set $f_w(\ell(s)) = 0$.*

When $\mathcal{N}(\mathcal{H})_w$ is a vertex β, then $f_w = c_\beta x^\beta$ (and $\deg(f_w) = |\beta|$), and

$$f_w(\ell(s)) \;=\; c_\beta (sa_1 - b_1)^{\beta_1}(sa_2 - b_2)^{\beta_2}\cdots(sa_n - b_n)^{\beta_n} \;=\; c_\beta (sa-b)^\beta\,.$$

In particular, there will be β_i points of $f(t^w.\ell(s)) = 0$ converging to b_i/a_i as $t \to \infty$, and so Lemma 8 gives a method to compute the vertices β of $\mathcal{N}(\mathcal{H})$. We give some definitions to make these notions more precise.

Let $a \in (\mathbb{C}^*)^n$ and $b \in \mathbb{C}^n$ be general in that the univariate polynomial $f(\ell_{a,b}(s))$ has $d = \deg(\mathcal{H})$ nondegenerate roots, and if $i \neq j$, then $b_i/a_i \neq b_j/a_j$. For any $w \in \mathbb{R}^n$ with $\mathcal{N}(\mathcal{H})_w = \{\beta\}$, consider the bivariate function $g_{a,b,w}(s,t) = g(s,t) := f(t^w.\ell_{a,b}(s))$. Since $g(s,1)$ has d simple zeroes, there are at most finitely many positive numbers t for which $g(s,t)$ does not have d simple zeroes. Therefore, there is a $t_0 > 0$ and d disjoint analytic curves $s(t) \in \mathbb{C}$ for $t > t_0$ which parameterize the zeroes of $g(s,t)$ for $t > t_0$ (that is, $g(s(t),t) \equiv 0$ for $t > t_0$).

By Lemma 8 and our choice of a, b, for each $i = 1,\ldots,n$, exactly β_i of these curves will converge to b_i/a_i as $t \to \infty$, for each $i = 1,\ldots,n$, while the remaining $d - |\beta|$ curves will diverge to infinity. We give an estimate of the rates of these convergences/divergences.

Let $w \in \mathbb{R}^n$ be general in that $\mathcal{N}(\mathcal{H})_w$ is a vertex, β. Let d_w be as above, and set

$$C \;:=\; \frac{\max\{|c_\alpha| \;:\; \alpha\in\mathcal{A}\}}{\min\{|c_\alpha| \;:\; \alpha\in\mathcal{A}\}}\,.$$

Furthermore, set $a_{\min} := \min\{1, |a_i| \ : \ i = 1, \dots, n\}$, $a_{\max} := \max\{1, |a_i| \ : \ i = 1, \dots, n\}$, and the same, $b_{\min}$ and $b_{\max}$, for b. Finally, for each $i = 1, \dots, n$, define

$$\begin{aligned} \gamma_i &:= \min\left\{ a_{\min}\,,\ \frac{1}{2}\left|\frac{b_i}{a_i} - \frac{b_j}{a_j}\right| \ : \ i \neq j \right\}\,, \qquad \text{and} \\ \Gamma_i &:= \max\left\{ \frac{2}{a_{\max}}\,,\ \left|\frac{b_i}{a_i} - \frac{b_j}{a_j}\right| \ : \ i \neq j \right\}\,. \end{aligned}$$

We give two results about the rate of convergence/divergence of the analytic curves $s(t)$ of zeroes of $g(s,t)$, and then discuss how these may be used to compute $\mathcal{N}(\mathcal{H})$.

Theorem 9. *With the above definitions, suppose that* $s\colon (t_0, \infty) \to \mathbb{C}$ *is a continuous function such that* $g_{a,b,w}(s(t), t) \equiv 0$ *for* $t > t_0$ *and that* $s(t)$ *converges to* b_i/a_i *as* $t \to \infty$. *Let* $t_1 \geq t_0$ *be a number such that if* $t > t_1$ *then*

$$\left| s(t) \ - \ \frac{b_i}{a_i} \right| \ \leq \ \gamma_i\,. \tag{4.4}$$

Then, for all $t > t_1$,

$$\left| s(t) \ - \ \frac{b_i}{a_i} \right|^{\beta_i} \ \leq \ t^{-d_w} \cdot C \cdot |\mathcal{A}| \cdot \left(\frac{a_{\max}}{a_{\min}} \left(1 + \frac{\Gamma_i}{\gamma_i} \right) \right)^d \,. \tag{4.5}$$

Theorem 10. *With the above definitions, suppose that* $s\colon (t_0, \infty) \to \mathbb{C}$ *is a continuous function such that* $g_{a,b,w}(s(t), t) = 0$ *for* $t > t_0$ *and that* $s(t)$ *diverges to* ∞ *as* $t \to \infty$. *Let* $t_1 \geq t_0$ *be a number such that if* $t > t_1$ *then*

$$|s(t)| \ > \ \frac{2b_{\max}}{a_{\min}} \ \geq \ 2\,. \tag{4.6}$$

Then, for all $t > t_1$,

$$|s(t)|^{d-|\beta|} \ \geq \ \frac{t^{d_w}}{C \cdot |\mathcal{A}|} \cdot \left(\frac{a_{\min}}{2(a_{\max} + a_{\min})} \right)^d \,. \tag{4.7}$$

Remark 11. Theorems 9 and 10 lead to an algorithm in the practical sense to determine vertices of $\mathcal{N}(\mathcal{H})$. First, choose $a, b \in \mathbb{C}^n$ as above and compute γ_i, $b_{\max}$, and $a_{\min}$. For a general $w \in \mathbb{R}^n$, follow points in the witness set $\mathcal{H} \cap (t^w.\ell_{a,b}(s))$ as t increases until the inequalities (4.4) and (4.6) are satisfied by the different points of the witness sets, at some t_1. This will give likely values for the integer components of the vertex β exposed by w. Next, continue following these points until the subexponential convergence in (4.5) and (4.7) is observed, which will confirm the value of β.

Presently, we are unable to obtain an upper bound on t_1 that depends on, say, C, γ_i, and Γ_i. Such an upper bound would yield an algorithm in the formal sense.

If we do not observe clustering of points of the witness set at $s = b_i/a_i$ and $s = \infty$, then we discard w, as it is not sufficiently general. That is, either it exposes a positive dimensional face of $\mathcal{N}(\mathcal{H})$ or else it is very close to doing so in that d_w is too small. □

Proof of Theorem 9. Fix $t > t_1$. Since $0 = g_{a,b,w}(s(t),t) = f(t^w.\ell_{a,b}(s(t)))$ and $f_w(x) = c_\beta x^\beta$, (4.3) gives

$$\begin{aligned} |(s(t)a-b)^\beta| &\leq \sum_{\alpha\in\mathcal{A}\smallsetminus\{\beta\}} t^{w\cdot\alpha - w\cdot\beta}\cdot\frac{|c_\alpha|}{|c_\beta|}\cdot|(s(t)a-b)^\alpha| \\ &\leq t^{-d_w}\cdot C\cdot\sum_{\alpha\in\mathcal{A}\smallsetminus\{\beta\}}|(s(t)a-b)^\alpha|\,. \end{aligned} \tag{4.8}$$

For any i and j we have

$$|s(t)a_j - b_j| \;=\; |a_j|\cdot|s(t)-\tfrac{b_j}{a_j}| \;\leq\; a_{\max}\bigl|s(t)-\tfrac{b_i}{a_i} \;+\; \tfrac{b_i}{a_i}-\tfrac{b_j}{a_j}\bigr| \;\leq\; a_{\max}(\gamma_i+\Gamma_i)\,.$$

Since $2\leq a_{\max}\Gamma_i$ and if $\alpha\in\mathcal{A}$, then $|\alpha|\leq d$, we have

$$|(s(t)a-b)^\alpha| \;\leq\; \bigl(a_{\max}(\gamma_i+\Gamma_i)\bigr)^d\,. \tag{4.9}$$

With (4.8), this becomes

$$|(s(t)a-b)^\beta| \;\leq\; t^{-d_w}\cdot C\cdot|\mathcal{A}|\cdot\bigl(a_{\max}(\gamma_i+\Gamma_i)\bigr)^d\,. \tag{4.10}$$

If $j\neq i$, then

$$\begin{aligned} |s(t)a_j-b_j| \;=\; |a_j|\cdot\bigl|s(t)-\tfrac{b_j}{a_j}\bigr| &\;=\; |a_j|\cdot\bigl|s(t)-\tfrac{b_i}{a_i}+\tfrac{b_i}{a_i}-\tfrac{b_j}{a_j}\bigr| \\ &\;\geq\; a_{\min}\cdot\Bigl|\bigl|\tfrac{b_i}{a_i}-\tfrac{b_j}{a_j}\bigr| \;-\; \bigl|s(t)-\tfrac{b_i}{a_i}\bigr|\Bigr| \\ &\;\geq\; a_{\min}\cdot(2\gamma_i \;-\; \gamma_i) \;=\; a_{\min}\gamma_i\,. \end{aligned}$$

Since $a_{\min}\gamma_i\leq 1$ and $|\beta|\leq d$, we have

$$\prod_{j\neq i}\bigl|(s(t)a_j-b_j)^{\beta_j}\bigr| \;\geq\; (a_{\min}\gamma_i)^{d-\beta_i}\,. \tag{4.11}$$

Observe that we have

$$\left|s(t) \;-\; \frac{b_i}{a_i}\right|^{\beta_i} \;=\; \frac{1}{|a_i|^{\beta_i}}\cdot|s(t)a_i-b_i|^{\beta_i} \;=\; \frac{1}{|a_i|^{\beta_i}}\cdot\frac{|(s(t)a-b)^\beta|}{\prod_{j\neq i}\bigl|(s(t)a_j-b_j)^{\beta_j}\bigr|}\,.$$

Combining this with (4.10) and (4.11) gives

$$\left|s(t) \;-\; \frac{b_i}{a_i}\right|^{\beta_i} \;\leq\; t^{-d_w}\cdot C\cdot|\mathcal{A}|\cdot\bigl(a_{\max}(\gamma_i+\Gamma_i)\bigr)^d\cdot\frac{1}{a_{\min}^{\beta_i}}\cdot\frac{1}{(a_{\min}\gamma_i)^{d-\beta_i}}\,.$$

Since $1\geq a_{\min}\geq\gamma_i$ and $d\geq\beta_i\geq 0$, we have

$$\left|s(t) \;-\; \frac{b_i}{a_i}\right|^{\beta_i} \;\leq\; t^{-d_w}\cdot C\cdot|\mathcal{A}|\cdot\left(\frac{a_{\max}}{a_{\min}}\left(1+\frac{\Gamma_i}{\gamma_i}\right)\right)^d\,,$$

which completes the proof. □

Proof of Theorem 10. Fix $t > t_1$. Then $|s(t)| > 2$. Since $g_{a,b,w}(s(t),t) = 0$, we have

$$|c_\beta t^{w\cdot\beta}(s(t)a-b)^\beta| \;=\; \Big|\sum_{\alpha\in\mathcal{A}\smallsetminus\{\beta\}} c_\alpha t^{w\cdot\alpha}(s(t)a-b)^\alpha\Big| \;\le\; \sum_{\alpha\in\mathcal{A}\smallsetminus\{\beta\}} |c_\alpha t^{w\cdot\alpha}(s(t)a-b)^\alpha|\,.$$

Factoring out powers of $|s(t)|$, we obtain

$$t^{w\cdot\beta}|s(t)|^{|\beta|}|c_\beta|\,|(a-b\,s(t)^{-1})^\beta| \;\le\; \sum_{\alpha\in\mathcal{A}\smallsetminus\{\beta\}} t^{w\cdot\alpha}|s(t)|^{|\alpha|}|c_\alpha|\,|(a-b\,s(t)^{-1})^\alpha|\,.$$

Since $s(t)\to\infty$ as $t\to\infty$, we must have $d-|\beta|>0$. Dividing by most of the left hand side and by $|s(t)|^d$ and using the definition of d_w, we obtain

$$\begin{aligned}
|s(t)|^{|\beta|-d} \;&\le\; \sum_{\alpha\in\mathcal{A}\smallsetminus\{\beta\}} t^{w\cdot\alpha-w\cdot\beta}|s(t)|^{|\alpha|-d}\frac{|c_\alpha|}{|c_\beta|}\,\frac{|(a-b\,s(t)^{-1})^\alpha|}{|(a-b\,s(t)^{-1})^\beta|}\\
&\le\; t^{-d_w}\cdot C\cdot \sum_{\alpha\in\mathcal{A}\smallsetminus\{\beta\}} |s(t)|^{|\alpha|-d}\frac{|(a-b\,s(t)^{-1})^\alpha|}{|(a-b\,s(t)^{-1})^\beta|}\,. \qquad (4.12)
\end{aligned}$$

We estimate the terms in this last sum. As $|s(t)|\ge 2$, for any i we have

$$|a_i - b_i\,s(t)^{-1}| \;\le\; |a_i| + |b_i\,s(t)^{-1}| \;\le\; a_{\max}+b_{\max}\,,$$

and so $|(a-b\,s(t)^{-1})^\alpha| \le (a_{\max}+b_{\max})^{|\alpha|}$. Similarly, for any i we have

$$|a_i - b_i\,s(t)^{-1}| \;\ge\; |a_i| - |b_i\,s(t)^{-1}| \;\ge\; a_{\min} \;-\; b_{\max}\cdot\frac{a_{\min}}{2b_{\max}} \;=\; \frac{a_{\min}}{2}\,.$$

Thus

$$\frac{|(a-b\,s(t)^{-1})^\alpha|}{|(a-b\,s(t)^{-1})^\beta|} \;\le\; (a_{\max}+b_{\max})^{|\alpha|}\Big(\frac{2}{a_{\min}}\Big)^{|\beta|} \;<\; \Big(\frac{2(a_{\max}+b_{\max})}{a_{\min}}\Big)^d.$$

Substituting this into (4.12) completes the proof of the theorem. □

Example 12. We demonstrate the convergence and divergence bounds by considering the polynomial $f(x,y) = x^2+3x+2y-5$ with the hypersurface $\mathcal{H} := \mathcal{V}(f)$ it defines. We have $\mathcal{A} = \{1,x,y,x^2\}$ with $|\mathcal{A}| = 4$ and will take $C = 5$, $a = (2+\sqrt{-1}, 3-2\sqrt{-1})$, $b = (-1-\sqrt{-1}, 2-3\sqrt{-1})$, $a_{\min} = 1$, $a_{\max} = \sqrt{13}$, $b_{\min} = 1$, and $b_{\max} = \sqrt{13}$. Additionally, $\gamma_i = \Gamma_i \approx 1.5342$ for $i = 1,2$.

First, consider the vector $w = (1,1)$ for which $\mathcal{N}(\mathcal{H})_w = (2,0)$ and $d_w = 1$. We have $g_{a,b}(s,t) = f(t\cdot(sa_1-b_1), t\cdot(sa_2-b_2))$, and $g_{a,b}(s,t) = 0$ has two nonsingular solutions for all $t>0$. Since $\mathcal{N}(\mathcal{H})_w = (2,0)$ both solution paths converge to b_1/a_1 as $t\to\infty$. The following table compares the actual values for the two solution paths, $s_1(t)$ and $s_2(t)$, with the upper bound (4.5) in Theorem 9. In particular, this table shows $|s_i(t)-b_1/a_1|^2 \approx 2.2t^{-1}$ whereas the upper bound is $1040t^{-1}$.

t	$\lvert s_1(t)-b_1/a_1\rvert^2$	$\lvert s_2(t)-b_1/a_1\rvert^2$	Upper bound (4.5)
$1e2$	0.26	0.19	10.4
$1e4$	2.2e-4	2.2e-4	0.104
$1e6$	2.2e-6	2.2e-6	1.04e-3
$1e8$	2.2e-8	2.2e-8	1.04e-5

We now consider the vector $w = (-1, -1)$ for which $\mathcal{N}(\mathcal{H})_w = (0, 0)$ and $d_w = 2$. With the same a, b as above, $g_{a,b}(s, t) = f(t^{-1} \cdot (sa_1 - b_1), t^{-1} \cdot (sa_2 - b_2))$ and $g_{a,b}(s, t) = 0$ has two nonsingular solutions for all $t > 0$. Since $\mathcal{N}(\mathcal{H})_w = (0, 0)$, both solution paths diverge to ∞ as $t \to \infty$. The following table compares the actual values for the two solution paths, $s_1(t)$ and $s_2(t)$, and the lower bound (4.7) in Theorem 10. This table shows $|s_i(t)|^2 \approx t^2/8.71$ whereas the lower bound is $t^2/4160$.

t	$\lvert s_1(t)\rvert^2$	$\lvert s_2(t)\rvert^2$	Lower bound (4.7)
$1e2$	1.17e3	1.13e3	2.40
$1e4$	1.15e7	1.15e7	2.40e4
$1e6$	1.15e11	1.15e11	2.40e8
$1e8$	1.15e15	1.15e15	2.40e12

□

5. Even Lüroth quartics

Associating a plane quartic curve to a defining equation identifies the set of plane quartics with $\mathbb{P}^{14}$. This projective space has an interesting Lüroth hypersurface whose general point is a *Lüroth quartic*, which is a quartic that contains the ten vertices of some pentalateral (arrangement of five lines). The equation for this hypersurface is the *Lüroth invariant*, which has degree 54 [26] and is invariant under the induced action of $PGL(3)$ on $\mathbb{P}(S^4\mathbb{C}^3) \simeq \mathbb{P}^{14}$. A discussion of this remarkable hypersurface, with references, is given in [10, Remark 6.3.31].

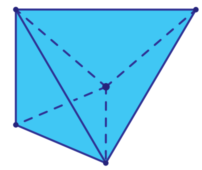
About a month after an initial draft of the present article was posted on the `arXiv`, Basson, et al. [1] posted an initial draft where they computed the Lüroth invariant in terms of the fundamental and secondary invariants of plane quartics. This is an expression in monomials of these invariants which is well-defined up to the ideal of relations among the invariants. Upon restricting this expression to the Ciani (even) quartics, they confirmed the following computations related to Ciani quartics. The authors additionally remark that they "could not figure out....the Newton polytope [of the Lüroth invariant]".

We use the algorithm of Section 4 to investigate the *Lüroth polytope*, the Newton polytope of the Lüroth invariant. While we are not yet able to compute the full Lüroth polytope, we can compute some of its vertices, including all those on a particular three-dimensional face. This face is the Newton polytope of the Lüroth hypersurface in the five-dimensional family of *Ciani quartics* whose monomials are squares,

$$\mathcal{E} \ := \ \{q_{400}x^4 + q_{040}y^4 + q_{004}z^4 + 2q_{220}x^2y^2 + 2q_{202}x^2z^2 + 2q_{022}y^2z^2 \ : \ [q_{400}, \ldots, q_{022}] \in \mathbb{P}^5\}\,.$$

(Note the coefficients of 2 on the last three terms. This scaling tempers the coefficients in the equation f_5 in Figure 1 for the even Lüroth quartics.) We show that this Newton polytope is a bipyramid that is affinely isomorphic to

$$\operatorname{conv}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \quad = \tag{5.1}$$



We furthermore use the numerical method of [3] to compute the equation for the hypersurface in $\mathcal{E}$ of even Lüroth quartics.

If $\ell_1,\ldots,\ell_5$ are general linear forms on $\mathbb{P}^2$, then the quartic with equation

$$\ell_1\ell_2\ell_3\ell_4\ell_5\cdot\left(\tfrac{1}{\ell_1}+\tfrac{1}{\ell_2}+\tfrac{1}{\ell_3}+\tfrac{1}{\ell_4}+\tfrac{1}{\ell_5}\right) \;=\; 0 \tag{5.2}$$

contains the ten points of pairwise intersection of the five lines defined by $\ell_1,\ldots,\ell_5$. Counting constants suggests that there is a 14-dimensional family of such quartics, but Lüroth showed [25] that the set of such quartics forms a hypersurface in $\mathbb{P}^{14}$.

The formula (5.2) exhibits the *Lüroth hypersurface* $\mathcal{LH}$ as the closure of the intersection of a general affine hyperplane $\mathfrak{M}\subset\mathbb{C}^{15}$ with the image of the map

$$\begin{array}{rcl} g\;:\;(\mathbb{C}^3)^5 & \longrightarrow & \mathbb{C}^{15} \\ (\ell_1,\ldots,\ell_5) & \longmapsto & \displaystyle\prod_{i=1}^{5}\ell_i\cdot\sum_{i=1}^{5}\frac{1}{\ell_i} \end{array} \tag{5.3}$$

The codimension of $\mathcal{LH}$ and the dimension of the general fiber (both 1) are easily verified using this parameterization [17, Lemma 3]. In particular, we used the method of [17] described in Section 2 with Bertini [2] to compute a witness set for $\mathcal{LH}=\overline{g(\mathbb{C}^{15})\cap\mathfrak{M}}$. This witness set verifies that the degree of $\mathcal{LH}$ is 54. As shown in [18], this witness set also provides the ability to test membership in $\mathcal{LH}$ by tracking at most 54 paths.

The space $\mathbb{C}^{15}$ of quartic polynomials has coordinates given by the coefficients of the monomials in a quartic,

$$\begin{aligned}\sum_{i+j+k=4} q_{ijk}x^iy^jz^k \;=\;& q_{400}x^4+q_{310}x^3y+q_{301}x^3z+q_{220}x^2y^2+q_{211}x^2yz\\ &+q_{202}x^2z^2+q_{130}xy^3+q_{121}xy^2z+q_{112}xyz^2+q_{103}xz^3\\ &+q_{040}y^2+q_{031}y^3z+q_{022}y^2z^2+q_{013}yz^3+q_{004}z^4\,.\end{aligned}$$

In theory, we may use the algorithm of Remark 11 to determine the Newton polytope of $\mathcal{LH}$. While difficult in practice, we may compute some vertices. For example,

$$q_{400}^6q_{301}^6q_{121}^{30}q_{013}^{12}\;\leftrightarrow\;(6,0,6,0,0,0,0,30,0,0,0,0,0,12,0)\,,$$

is the extreme monomial in the direction

$$(3,-5,3,2,3,-2,-1,4,-3,-2,3,1,-5,3,-5)\,.$$

By symmetry, this gives five other vertices,

$$q_{400}^6q_{310}^6q_{112}^{30}q_{031}^{12}\,,\;q_{040}^6q_{031}^6q_{211}^{30}q_{103}^{12}\,,\;q_{040}^6q_{130}^6q_{112}^{30}q_{301}^{12}\,,\;q_{004}^6q_{013}^6q_{211}^{30}q_{130}^{12}\,,\;q_{004}^6q_{103}^6q_{121}^{30}q_{310}^{12}\,.$$

It is dramatically more feasible to compute the Newton polytope of the hypersurface of Lüroth quartics in the space $\mathcal{E}$ of even quartics. This is the face of the Lüroth polytope that is extreme in the direction of v, where

$$v\cdot(q_{400},q_{310},\dots,q_{004}) \;=\; -\sum\{q_{ijk}\mid \text{at least one of } i,\ j,\ k \text{ is odd}\}\,.$$

Obtaining a witness set for the even Lüroth quartics, $\mathcal{EH} := \mathcal{E}\cap\mathcal{LH}$, is straightforward; we reparameterize using the 2's in the definition of $\mathcal{E}$ and include the linear equations

$$q_{ijk} \;=\; 0 \qquad \text{where at least one of } i,\ j,\ k \text{ is odd}$$

among the affine linear equations $\mathcal{L}\colon \mathbb{C}^{15}\to\mathbb{C}^{13}$ used for the witness set computation. When performing this specialization, some of the 54 points from $\mathcal{LH}$ coalesce. More precisely, six points of $\mathcal{EH}$ arise as the coalescence of four points each, nine points of $\mathcal{EH}$ arise as the coalescence of two points each, and the remaining twelve points remain distinct. This implies that $\mathcal{EH}$ is reducible with non-reduced components.

Numerical irreducible decomposition shows that $\mathcal{EH}$ consists of eight components, only one of which is reduced. However, as we are using witness sets for images of maps [17] (as described in Section 2) the numerical computations are not performed in $\mathcal{E}$, but rather on the smooth incidence variety of the map g (5.3).

We first determine the Newton polytope of each component and then use [3] to recover the defining equation for each component. For $f_1,\dots,f_5$ as given in Figure 1, $\mathcal{EH}$ is defined by

$$q_{400}^4\cdot q_{040}^4\cdot q_{004}^4\cdot f_1^4\cdot f_2^2\cdot f_3^2\cdot f_4^2\cdot f_5 \;=0\,. \tag{5.4}$$

For completeness, we used the algorithm of [18] to verify that a random element of each hypersurface $\mathcal{V}(f_i)$ lies on $\mathcal{LH}$.

Observe that f_1, f_2, f_3, and f_4 all have the same support and therefore the same Newton polytope, Δ. Every integer point of Δ corresponds to a monomial in these polynomials and all are extreme. The Newton polytope of f_5 is 4Δ and it has 65 nonzero terms, which correspond to all the integer points in 4Δ. Thus the Newton polytope of $\mathcal{EH}$ is $14\Delta+\alpha$, where α is the exponent vector of $q_{400}^4q_{040}^4q_{004}^4$. To complete the identification of $\mathcal{N}(\mathcal{EH})$, consider the integer points $\{O,A,B,C,D\}$ of Δ, which are on the left in Table 1 Replacing $\{O,\dots,D\}$ by their differences with O gives the points o,a,b,c,d on the right

Table 1. Vertices of Δ

	q_{400}	q_{040}	q_{004}	q_{022}	q_{202}	q_{220}
O	0	0	0	1	1	1
A	1	0	0	2	0	0
B	0	1	0	0	2	0
C	0	0	1	0	0	2
D	1	1	1	0	0	0

	q_{400}	q_{040}	q_{004}	q_{022}	q_{202}	q_{220}
o	0	0	0	0	0	0
a	1	0	0	1	−1	−1
b	0	1	0	−1	1	−1
c	0	0	1	−1	−1	1
d	1	1	1	−1	−1	−1

in Table 1. Note that $a+b+c=d$. Projecting to the first three coordinates is an isomorphism of the integer span of a,b,c with $\mathbb{Z}^3$, and shows that Δ is affinely isomorphic to the bipyramid (5.1)

$$
\begin{aligned}
f_1 &= q_{400}q_{040}q_{004} - q_{400}q_{022}^2 - q_{040}q_{202}^2 - q_{004}q_{220}^2 - 2q_{220}q_{202}q_{022} \\
f_2 &= q_{400}q_{040}q_{004} - q_{400}q_{022}^2 + 3q_{040}q_{202}^2 - q_{004}q_{220}^2 + 2q_{220}q_{202}q_{022} \\
f_3 &= q_{400}q_{040}q_{004} + 3q_{400}q_{022}^2 - q_{040}q_{202}^2 - q_{004}q_{220}^2 + 2q_{220}q_{202}q_{022} \\
f_4 &= q_{400}q_{040}q_{004} - q_{400}q_{022}^2 - q_{040}q_{202}^2 + 3q_{004}q_{220}^2 + 2q_{220}q_{202}q_{022} \\
f_5 &= 2401q_{400}^4q_{040}^4q_{004}^4 - 196q_{400}^4q_{040}^3q_{004}^3q_{022}^2 + 102q_{400}^4q_{040}^2q_{004}^2q_{022}^4 - 4q_{400}^4q_{040}q_{004}q_{022}^6 \\
&\quad + q_{400}^4q_{022}^8 - 196q_{400}^3q_{040}^4q_{004}^3q_{202}^2 - 196q_{400}^3q_{040}^3q_{004}^4q_{220}^2 + 840q_{400}^3q_{040}^3q_{004}^3q_{220}q_{202}q_{022} \\
&\quad - 820q_{400}^3q_{040}^3q_{004}^2q_{202}^2q_{022}^2 - 820q_{400}^3q_{040}^2q_{004}^3q_{220}^2q_{022}^2 + 232q_{400}^3q_{040}^2q_{004}^2q_{220}q_{202}q_{022}^3 \\
&\quad - 12q_{400}^3q_{040}^2q_{004}q_{202}^2q_{022}^4 - 12q_{400}^3q_{040}q_{004}^2q_{220}^2q_{022}^4 - 40q_{400}^3q_{040}q_{004}q_{220}q_{202}q_{022}^5 \\
&\quad + 4q_{400}^3q_{040}q_{202}^2q_{022}^6 + 4q_{400}^3q_{004}q_{220}^2q_{022}^6 - 8q_{400}^3q_{220}q_{202}q_{022}^7 + 102q_{400}^2q_{040}^4q_{004}^2q_{202}^4 \\
&\quad - 820q_{400}^2q_{040}^3q_{004}^3q_{220}^2q_{202}^2 + 232q_{400}^2q_{040}^3q_{004}^2q_{220}q_{202}^3q_{022} - 12q_{400}^2q_{040}^3q_{004}q_{202}^4q_{022}^2 \\
&\quad + 102q_{400}^2q_{040}^2q_{004}^4q_{220}^4 + 232q_{400}^2q_{040}^2q_{004}^3q_{220}^3q_{202}q_{022} + 128q_{400}^2q_{040}^2q_{004}^2q_{220}^2q_{202}^2q_{022}^2 \\
&\quad - 80q_{400}^2q_{040}^2q_{004}q_{220}q_{202}^3q_{022}^3 + 6q_{400}^2q_{040}^2q_{202}^4q_{022}^4 - 12q_{400}^2q_{040}q_{004}^3q_{220}^4q_{022}^2 \\
&\quad - 80q_{400}^2q_{040}q_{004}^2q_{220}^3q_{202}q_{022}^3 + 220q_{400}^2q_{040}q_{004}q_{220}^2q_{202}^2q_{022}^4 - 24q_{400}^2q_{040}q_{220}q_{202}^3q_{022}^5 \\
&\quad + 6q_{400}^2q_{004}^2q_{220}^4q_{022}^4 - 24q_{400}^2q_{004}q_{220}^3q_{202}q_{022}^5 + 24q_{400}^2q_{220}^2q_{202}^2q_{022}^6 \\
&\quad - 4q_{400}q_{040}^4q_{004}q_{202}^6 - 12q_{400}q_{040}^3q_{004}^2q_{220}^2q_{202}^4 - 40q_{400}q_{040}^3q_{004}q_{220}q_{202}^5q_{022} \\
&\quad + 4q_{400}q_{040}^3q_{202}^6q_{022}^2 - 12q_{400}q_{040}^2q_{004}^3q_{220}^4q_{202}^2 - 80q_{400}q_{040}^2q_{004}^2q_{220}^3q_{202}^3q_{022} \\
&\quad + 220q_{400}q_{040}^2q_{004}q_{220}^2q_{202}^4q_{022}^2 - 24q_{400}q_{040}^2q_{220}q_{202}^5q_{022}^3 - 4q_{400}q_{040}q_{004}^4q_{220}^6 \\
&\quad - 40q_{400}q_{040}q_{004}^3q_{220}^5q_{202}q_{022} + 220q_{400}q_{040}q_{004}^2q_{220}^4q_{202}^2q_{022}^2 - 272q_{400}q_{040}q_{004}q_{220}^3q_{202}^3q_{022}^3 \\
&\quad + 48q_{400}q_{040}q_{220}^2q_{202}^4q_{022}^4 + 4q_{400}q_{004}^3q_{220}^6q_{022}^2 - 24q_{400}q_{004}^2q_{220}^5q_{202}q_{022}^3 \\
&\quad + 48q_{400}q_{004}q_{220}^4q_{202}^2q_{022}^4 - 32q_{400}q_{220}^3q_{202}^3q_{022}^5 + q_{040}^4q_{202}^8 + 4q_{040}^3q_{004}q_{220}^2q_{202}^6 \\
&\quad - 8q_{040}^3q_{220}q_{202}^7q_{022} + 6q_{040}^2q_{004}^2q_{220}^4q_{202}^4 - 24q_{040}^2q_{004}q_{220}^3q_{202}^5q_{022} + 24q_{040}^2q_{220}^2q_{202}^6q_{022}^2 \\
&\quad + 4q_{040}q_{004}^3q_{220}^6q_{202}^2 - 24q_{040}q_{004}^2q_{220}^5q_{202}^3q_{022} + 48q_{040}q_{004}q_{220}^4q_{202}^4q_{022}^2 \\
&\quad -32\, q_{040}q_{220}^3q_{202}^5q_{022}^3 + q_{004}^4q_{220}^8 - 8q_{004}^3q_{220}^7q_{202}q_{022} + 24q_{004}^2q_{220}^6q_{202}^2q_{022}^2 \\
&\quad - 32q_{004}q_{220}^5q_{202}^3q_{022}^3 + 16q_{220}^4q_{202}^4q_{022}^4 \,.
\end{aligned}
$$

FIGURE 1. Polynomials defining $\mathcal{EH}$

Using (5.4), we can determine which Edge quartics [11, 29] are Lüroth quartics since the family of Edge quartics $\mathcal{ED}$ is contained in $\mathcal{E}$ with

$$\mathcal{ED} \ := \ \{\mathcal{V}(s(x^4+y^4+z^4) - t(y^2z^2+x^2z^2+x^2y^2)) \ : \ [s,t] \in \mathbb{P}^1\} \,.$$

Identifying $\mathcal{ED}$ with $\mathbb{P}^1$, and evaluating at (5.4) gives the equation for $\mathcal{ED} \cap \mathcal{LH}$,

$$s^{12}(s+t)^4(2s-t)^{16}(7s+t)(2s^2+st+t^2)^6(28s^3+8s^2t+3st^2+t^3)^3 \ = \ 0 \,.$$

Set $\omega := \sqrt[3]{297+24\sqrt{159}}$. Besides the point $[0,1]$, the eight points $[1,t]$ corresponding to Edge quartics that are Lüroth quartics are

$$
\begin{gathered}
t_1 \;=\; -1\,,\quad t_2 \;=\; 2\,,\quad t_3 \;=\; -7\,,\\
t_4 \;=\; \tfrac{1}{2}(\sqrt{-7}-1)\,,\quad t_5 \;=\; \tfrac{-1}{2}(1+\sqrt{-7})\,,\quad t_6 \;=\; \tfrac{1}{3\omega}(15-3\omega-\omega^2)\,,\\
t_7 \;=\; \tfrac{1}{6\omega}(\omega^2-6\omega-15+\sqrt{-3}\,(\omega^2+15))\,,\quad t_8 \;=\; \tfrac{1}{6\omega}(\omega^2-6\omega-15-\sqrt{-3}\,(\omega^2+15))\,.
\end{gathered}
$$

In particular, there are four real values t_1,t_2,t_3,t_6 and four nonreal values t_4,t_5,t_7,t_8. The Edge Lüroth quartic corresponding to $[0,1]$ has three real points, each of which is singular. Also, except for $t=t_2=2$, which is the union of four lines

$$x-y+z \;=\; x-y-z \;=\; x+y-z \;=\; x+y+z \;=\; 0\,,$$

the Edge Lüroth quartics corresponding to $[1,t_i]$ are smooth with no real points.

6. Conclusion

We presented two algorithms for computing the Newton polytope of a hypersurface $\mathcal{H}$ given numerically. The first assumes that we may evaluate a polynomial defining $\mathcal{H}$ while the second uses a witness set representation of $\mathcal{H}$. The second is illustrated through the determination of the polynomial defining the hypersurface of even Lüroth quartics (which gives a face of the Lüroth poytope), along with some other vertices of the Lüroth polytope. Implementing these algorithms remains a future project.

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Jonathan D. Hauenstein, Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695, USA
E-mail address: hauenstein@ncsu.edu
URL: http://www.math.ncsu.edu/~jdhauens

Frank Sottile, Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
E-mail address: sottile@math.tamu.edu
URL: http://www.math.tamu.edu/~sottile