

NON-ARCHIMEDEAN COAMOEBAE

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ABSTRACT. A coamoeba is the image of a subvariety of a complex torus under the argument map to the real torus. Similarly, a non-archimedean coamoeba is the image of a subvariety of a torus over a non-archimedean field \mathbb{K} with complex residue field under an argument map. The phase tropical variety is the closure of the image under the pair of maps, tropicalization and argument. We describe the structure of non-archimedean coamoebae and phase tropical varieties in terms of complex coamoebae and their phase limit sets. The argument map depends upon a section of the valuation map, and we explain how this choice (mildly) affects the non-archimedean coamoeba. We also identify a class of varieties whose non-archimedean coamoebae and phase tropical varieties are objects from polyhedral combinatorics.

Dedicated to the memory of Mikael Passare 1.1.1959—15.9.2011.

INTRODUCTION

A coamoeba is the image of a subvariety X of a complex torus $(\mathbb{C}^\times)^n$ under the argument map to the real torus $\mathbb{U}^n := \mathbb{R}^n / (2\pi\mathbb{Z})^n$. It is related to the amoeba, which is the image of X under the coordinatewise logarithm map $z \mapsto \log |z|$ to \mathbb{R}^n . Amoebae were introduced by Gelfand, Kapranov, and Zelevinsky in 1994 [6], and have subsequently been widely studied [10, 11, 16, 18]. In contrast, coamoebae, which were introduced by Passare in a talk in 2004, have only just begun to be studied and they appear to have many beautiful and interesting properties. For example, the closure of the coamoebae of X is the (finite) union of the coamoebae of all its initial schemes, $\text{in}_w X$. Those for $w \neq 0$ form the phase limit set of X [14], whose structure is controlled by the tropical variety of X .

Suppose that \mathbb{K} is an algebraically closed field with a non-trivial valuation $\nu: \mathbb{K}^\times \rightarrow \Gamma$, where Γ is a totally ordered divisible abelian group. A variety X in $(\mathbb{K}^\times)^n$ has a non-archimedean amoeba, or tropicalization, $\text{Trop}(X) \subset \Gamma^n$ [1, 2, 8]. When \mathbb{K} has residue field the complex numbers, there is a version of the argument map to \mathbb{U}^n defined on \mathbb{K}^\times , and the image of $X \subset (\mathbb{K}^\times)^n$ under this argument map is a *non-archimedean coamoeba*. Non-archimedean coamoebae have structure controlled by the tropical variety of X , which generalizes the structure of the phase limit set of a complex variety.

The closure of the image of X in $\Gamma^n \times \mathbb{U}^n$ under the product of the tropicalization and argument maps is the *phase tropical variety* of X , $\mathcal{P}\text{trop}(X)$. For curves and hypersurfaces

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over the field of Puiseux series, this notion was introduced and studied by Mikhalkin [13, §6] and [12].

These objects depend upon a section $\tau: \Gamma \rightarrow \mathbb{K}^\times$ of the valuation map, which allows the identification of all reductions of fibres of tropicalization $(\mathbb{K}^\times)^n \rightarrow \Gamma^n$ with $(\mathbb{C}^\times)^n$. The identifications of a fibre over $w \in \Gamma^n$ for two different sections are related by a translation $\alpha(w) \in (\mathbb{C}^\times)^n$. Translations in different fibres can be significantly different. For example, when $\dim_{\mathbb{Q}} \Gamma > 1$, the map $\alpha: \Gamma^n \rightarrow (\mathbb{C}^\times)^n$ may not be continuous. We explain the (surprisingly mild) effect of a change of section on these objects.

A variety is tropically simple if its tropical reductions are pullbacks along surjections of products of hyperplanes. The non-archimedean coamoeba of a tropically simple variety is essentially an object of polyhedral combinatorics and may be realized as a union of products of zonotope complements in subtori glued together along their phase limit sets. The phase tropical variety of a tropically simple variety has a similar description.

In Section 1, we review the fundamental theorems of tropical geometry in a form that we need, including a discussion of how the choice of section of valuation affects tropical reductions, and review coamoebae. We define non-archimedean coamoebae and phase tropical varieties in Section 2 and establish their basic structure. We close with Section 3 which studies these objects for tropically simple varieties.

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1. TROPICALIZATION AND COAMOEBAE

Let \mathbb{K} be an algebraically closed field equipped with a non-archimedean valuation $\nu: \mathbb{K}^\times \rightarrow \Gamma$, where Γ is a divisible totally ordered group and $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$, the invertible elements of \mathbb{K} . We do not assume that Γ is a subgroup of \mathbb{R} , for example $\Gamma = \mathbb{R}^2$ with lexicographic order is suitable. This is in greater generality than is typical, see [1] for a similar treatment. We extend ν to \mathbb{K} by setting $\nu(0) = \infty$, which exceeds every element of Γ . Then

$$\nu(ab) = \nu(a) + \nu(b) \quad \text{and} \quad \nu(a+b) \geq \min\{\nu(a), \nu(b)\},$$

with equality when $\nu(a) \neq \nu(b)$. Write R for the valuation ring and \mathfrak{m} for its maximal ideal,

$$R := \{x \in \mathbb{K} \mid \nu(x) \geq 0\} \quad \text{and} \quad \mathfrak{m} := \{x \in \mathbb{K} \mid \nu(x) > 0\}.$$

The residue field of \mathbb{K} is $\mathbb{k} := R/\mathfrak{m}$, which is also algebraically closed.

1.1. Tropicalization. Let N be a finitely generated free abelian group and $M := \text{Hom}(N, \mathbb{Z})$ its dual group. Write $\langle \cdot, \cdot \rangle$ for the pairing between M and N . The group ring $\mathbb{Z}[M]$ is the coordinate ring of the torus \mathbb{T}_N . Its \mathbb{K} -points are $\mathbb{T}_N(\mathbb{K}) = \mathbb{K}^\times \otimes_{\mathbb{Z}} N = \text{Hom}(M, \mathbb{K}^\times)$, the set of group homomorphisms $M \rightarrow \mathbb{K}^\times$. The valuation induces a map $\text{Trop}: \mathbb{T}_N(\mathbb{K}) \rightarrow \Gamma_N := \Gamma \otimes_{\mathbb{Z}} N$. Maps $N' \rightarrow N$ of free abelian groups functorially induce maps $\mathbb{T}_{N'} \rightarrow \mathbb{T}_N$ and $\Gamma_{N'} \rightarrow \Gamma_N$.

The points $x \in \mathbb{T}_N(\mathbb{K})$ with valuation w ,

$$\{x: M \rightarrow \mathbb{K} \mid \nu(x(\mathbf{m})) = \langle \mathbf{m}, w \rangle\},$$

form the fibre of $\mathbb{T}_N(\mathbb{K})$ over $w \in \Gamma_N$. For $w = 0$ this is $\mathbb{T}_N(R)$, which has a natural reduction map onto $\mathbb{T}_N(\mathbb{k})$. We follow Payne [17] to understand the other fibres. Write $\xi^{\mathbf{m}}$ for the element of $\mathbb{K}[M]$ corresponding to $\mathbf{m} \in M$. The *tilted group ring* is

$$R[M]^w := \left\{ \sum c_{\mathbf{m}} \xi^{\mathbf{m}} \in \mathbb{K}[M] \mid \nu(c_{\mathbf{m}}) + \langle \mathbf{m}, w \rangle \geq 0 \right\}.$$

If $\mathbb{T}^w := \text{spec } R[M]^w$, then $\mathbb{T}^w(R)$ is the fibre of $\mathbb{T}_N(\mathbb{K})$ over $w \in \Gamma_N$. The residue map $\mathbb{T}^w(R) \rightarrow \mathbb{T}^w(\mathbb{k})$ associates a point $x \in \mathbb{T}^w(R)$ canonically to its *tropical reduction* $\bar{x} \in \mathbb{T}^w(\mathbb{k})$.

The map $c \xi^{\mathbf{m}} \mapsto \xi^{\mathbf{m}} \otimes c \xi^{\mathbf{m}}$ induces a coaction $R[M]^w \rightarrow R[M] \otimes R[M]^w$, and so \mathbb{T}^w is a \mathbb{T}_N -torsor over R . Thus these fibres $\mathbb{T}^w(R)$ and $\mathbb{T}^w(\mathbb{k})$ are non-canonically isomorphic to the tori $\mathbb{T}_N(R)$ and $\mathbb{T}_N(\mathbb{k})$. We also have the *exploded tropicalization* map

$$\mathfrak{Trop} : \mathbb{T}_N(\mathbb{K}) \longrightarrow \coprod_{w \in \Gamma_N} \mathbb{T}^w(\mathbb{k}).$$

This union is disjoint because there is no natural identification between the fibres of \mathfrak{Trop} .

The *tropicalization* $\text{Trop}(X) \subset \Gamma_N$ of a subscheme $X \subset \mathbb{T}_N$ is the image of $X(\mathbb{K})$ under \mathfrak{Trop} . Its *exploded tropicalization* $\mathfrak{Trop}(X)$ is its image under \mathfrak{Trop} . The fibre of $\mathfrak{Trop}(X)$ over a point $w \in \text{Trop}(X)$ is the *tropical reduction* X_w of X , which is a subset of $\mathbb{T}^w(\mathbb{k})$. We consider this at the level of schemes.

The fibre of the R -scheme \mathbb{T}^w over the closed point of $\text{spec } R$ is a scheme over the residue field \mathbb{k} with coordinate ring $\mathbb{k}[M]^w := R[M]^w \otimes_R \mathbb{k}$. This has a concrete description. For $\gamma \in \Gamma$, let \mathfrak{a}^γ be the fractional ideal of R consisting of those $x \in \mathbb{K}$ with valuation $\nu(x) \geq \gamma$. Set $\mathbb{k}^\gamma := \mathfrak{a}^\gamma / (\mathfrak{m} \cdot \mathfrak{a}^\gamma) = \mathfrak{a}^\gamma \otimes_R \mathbb{k}$, which is a one-dimensional \mathbb{k} -vector space. Then

$$R[M]^w = \bigoplus_{\mathbf{m} \in M} \mathfrak{a}^{-\langle \mathbf{m}, w \rangle} \xi^{\mathbf{m}} \quad \text{and} \quad \mathbb{k}[M]^w = \bigoplus_{\mathbf{m} \in M} \mathbb{k}^{-\langle \mathbf{m}, w \rangle} \xi^{\mathbf{m}}.$$

Each point $w \in \Gamma_N$ determines a weight function on monomials in $\mathbb{K}[M]$,

$$c \xi^{\mathbf{m}} \longmapsto \nu(c) + \langle \mathbf{m}, w \rangle,$$

for $c \in \mathbb{K}^\times$ and $\mathbf{m} \in M$. If all terms of $f \in \mathbb{K}[M]$ have non-negative w -weight, then $f \in R[M]^w$, and its image in $\mathbb{k}[M]^w$ is its *tropical reduction* f_w . The *tropical reduction* I_w of an ideal $I \subset \mathbb{K}[M]$ is

$$I_w := \{f_w \mid f \in I \cap R[M]^w\} \subset \mathbb{k}[M]^w.$$

Surjectivity of tropicalization relates tropical reductions to tropicalization.

Proposition 1.1. *Suppose that X is a \mathbb{K} -subscheme of \mathbb{T}_N with ideal $I \subset \mathbb{K}[M]$ and let $w \in \Gamma_N$. Its tropical reduction X_w is the \mathbb{k} -subscheme of \mathbb{T}^w defined by I_w . This is nonempty exactly when $w \in \text{Trop}(X)$.*

This, as well as Propositions 1.2, 1.5, and 1.7 are the fundamental theorems of tropical geometry, and are due to many authors. For a discussion with references, see Section 2.2 of [15]. Another complete source presented in refreshing generality is [7].

1.2. Structure of tropicalization. The usual notions of rational polyhedral complexes in \mathbb{R}^n carry over without essential change to Γ_N . (But when $\Gamma \not\subset \mathbb{R}$, they are not easily interpreted as ordinary polyhedral complexes [2].) For example, a (Γ -rational) *polyhedron* is a subset σ of Γ_N which is the intersection of finitely many half-spaces of the form

$$(1) \quad \{w \in \Gamma_N \mid \langle \mathbf{m}, w \rangle \geq a\},$$

where $a \in \Gamma$ and $\mathbf{m} \in M$. If all constants a in (1) are 0, then σ is a *cone*. A *face* of a polyhedron σ is either σ itself or the intersection of σ with a hyperplane $\{w \mid \langle \mathbf{m}, w \rangle = a\}$, where a, \mathbf{m} define a half-space (1) containing σ . The *relative interior*, $\text{relint}(\sigma)$ of a polyhedron σ is the set-theoretic difference of σ and its proper faces.

The intersection of all hyperplanes containing the differences of elements in a polyhedron σ is a subgroup of Γ_N which has the form $\Gamma_{N'}$ for a sublattice N' of N , written as $\langle \sigma \rangle$. The rank of this sublattice $\langle \sigma \rangle$ is the *dimension* of σ . For any $w \in \sigma$, we have $\sigma \subset w + \Gamma_{\langle \sigma \rangle}$.

A *polyhedral complex* \mathcal{C} is a finite union of polyhedra, called *faces* of \mathcal{C} , such that if $\sigma \in \mathcal{C}$, then all faces of σ lie in \mathcal{C} , and if $\sigma, \sigma' \in \mathcal{C}$, then either $\sigma \cap \sigma'$ is empty or it is a common face of both polyhedra. A *fan* is a polyhedral complex in which all faces are cones. A polyhedral complex has *pure dimension* d if every maximal face has dimension d . The *support* of a polyhedral complex is the union of its faces.

The minimal face containing a point w in the support of \mathcal{C} is the unique face σ with $w \in \text{relint}(\sigma)$. For every face ρ of \mathcal{C} containing w , there is a cone ρ_w such that $\rho_w + w$ is the intersection of those half-spaces containing ρ whose boundary hyperplane contains w . The union of these cones ρ_w for faces ρ of \mathcal{C} containing w forms the *local fan* \mathcal{C}_w of \mathcal{C} at w . These cones all contain the subgroup $\Gamma_{\langle \sigma \rangle}$.

We state part of the basic structure theorem of tropical varieties.

Proposition 1.2. *Let X be an irreducible \mathbb{K} -subscheme of \mathbb{T}_N of dimension d . Then $\text{Trop}(X)$ is the support of a connected polyhedral complex \mathcal{C} in Γ_N of pure dimension d .*

Remark 1.3. Tropical varieties have an additional property, that there are multiplicities on the top-dimensional cones in \mathcal{C} satisfying a *balancing condition*. Here, where we allow the value group Γ to be a divisible totally ordered group, the balancing condition is exactly the same as when $\Gamma \subset \mathbb{R}$, for it comes from the Gröbner fan of the tropical reductions X_w at points w lying in the relative interior of faces of \mathcal{C} of codimension 1.

Remark 1.4. We may partially understand the polyhedral complex \mathcal{C} as follows. A polynomial $f \in \mathbb{K}[M]$ may be written

$$(2) \quad f = \sum_{\mathbf{m} \in \mathcal{A}(f)} c_{\mathbf{m}} \xi^{\mathbf{m}} \quad c_{\mathbf{m}} \in \mathbb{K}^{\times},$$

where the support $\mathcal{A}(f)$ of f is a finite subset of M . For $w \in \Gamma_N$, let $\mathcal{A}(f)_w$ be the points of $\mathcal{A}(f)$ indexing terms of (2) with minimum w -weight, $\nu(c_{\mathbf{m}}) + \langle \mathbf{m}, w \rangle$. When this common minimum is $0 \in \Gamma$, the tropical reduction of f is

$$f_w = \sum_{\mathbf{m} \in \mathcal{A}(f)_w} \overline{c_{\mathbf{m}}} \xi^{\mathbf{m}} \in \mathbb{k}[M]^w.$$

The polyhedral complex \mathcal{C} depends upon an embedding of \mathbb{T}_N as the dense torus in a projective space \mathbb{P}^N . This corresponds to a choice of generators of M , which gives a polynomial subalgebra $\mathbb{K}[\mathbb{N}^n]$ ($n = \text{rank}(N)$) of $\mathbb{K}[M]$. Given a \mathbb{K} -subscheme $X \subset \mathbb{T}_N$ defined by an ideal $I \subset \mathbb{K}[M]$, let \tilde{I} be the homogenization of $I \cap \mathbb{K}[\mathbb{N}^n]$, which defines the closure of X in \mathbb{P}^N . Then polyhedra $\sigma \in \mathcal{C}$ have the property that for $w, u \in \text{relint}(\sigma)$, $\mathcal{A}(f)_w = \mathcal{A}(f)_u$, for every $f \in \tilde{I}$. This implies the same for every $f \in I$. \square

We next relate tropical reductions to the faces of \mathcal{C} . This requires that we compare tropical reductions at different points of Γ_N .

1.3. Trivializations of tropicalization. The fibres $\mathbb{T}^w(\mathbb{k})$ of tropicalization may be identified so that $\mathfrak{Trop}(X)$ becomes a subset of $\Gamma_N \times \mathbb{T}^w(\mathbb{k})$ —but at the price of a choice of a homomorphism $\tau: \Gamma \rightarrow \mathbb{K}^\times$ that is a section of the valuation map. As David Speyer pointed out, such sections always exist. In the exact sequence

$$\{1\} \longrightarrow \ker(\nu) \longrightarrow \mathbb{K}^\times \xrightarrow{\nu} \Gamma \longrightarrow \{0\},$$

the group $\ker(\nu)$ is divisible as \mathbb{K} is algebraically closed. Therefore ν has a section.

A section $\tau: \Gamma \rightarrow \mathbb{K}^\times$ of the valuation map induces a section of tropicalization $\tau_N: \Gamma_N \rightarrow \mathbb{T}_N(\mathbb{K})$ via $\tau_N(w): \mathbf{m} \mapsto \tau(\langle \mathbf{m}, w \rangle)$. This is a homomorphism, $\tau_N(w + u) = \tau_N(w)\tau_N(u)$. Write τ^w for $\tau_N(w)$, which is an element of $\mathbb{T}^w(R)$. We have an identification

$$\varphi_{\tau, w}: \mathbb{T}_N(R) \xrightarrow{\sim} \mathbb{T}^w(R) \quad \text{via} \quad x \longmapsto x \cdot \tau^w.$$

Similarly, $\bar{x} \mapsto \bar{x} \cdot \overline{\tau^w}$ identifies $\mathbb{T}_N(\mathbb{k})$ with $\mathbb{T}^w(\mathbb{k})$. We call this a *trivialization of tropicalization*. We obtain the identification of exploded tropicalization

$$(3) \quad \varphi_\tau^{-1}: \coprod_{w \in \Gamma_N} \mathbb{T}^w(\mathbb{k}) \xrightarrow{\sim} \Gamma_N \times \mathbb{T}_N(\mathbb{k}) \quad \text{via} \quad \mathbb{T}^w(\mathbb{k}) \ni \bar{x} \longmapsto (w, \overline{\tau^{-w}} \cdot \bar{x}).$$

Let us fix a section $\tau: \Gamma \rightarrow \mathbb{K}^\times$, and use it to identify all tropical reductions $\mathbb{T}^w(\mathbb{k})$ with $\mathbb{T}_N(\mathbb{k})$. That is, we set $X_{w, \tau} \subset \mathbb{T}_N(\mathbb{k})$ to be $\overline{\tau^{-w}} \cdot X_w$. A consequence of this choice is the following proposition.

Proposition 1.5. *Let $X \subset \mathbb{T}_N$ be a \mathbb{K} -subscheme and suppose that \mathcal{C} is a polyhedral complex supported on $\text{Trop}(X)$ as in Proposition 1.2 and Remark 1.4. For any face $\sigma \in \mathcal{C}$, the tropical reductions $X_{w, \tau}$ and $X_{u, \tau}$ are equal, for any $w, u \in \text{relint}(\sigma)$.*

We omit the section τ in our notation when it is understood. For σ a face of \mathcal{C} , we write X_σ (or $X_{\sigma, \tau}$) for this common tropical reduction at points of $\text{relint}(\sigma)$.

Remark 1.6. We explain this at the level of schemes. Let $I \subset \mathbb{K}[M]$ be the ideal of X and suppose that $f \in I$ as in (2) with support $\mathcal{A}(f)$. If $w(f)$ is the minimal w -weight of a term of f , then the polynomial $\tau(-w(f)) \cdot f$ lies in $I \cap R[M]^w$ and

$$\overline{\varphi_{\tau,w}^\times(\tau(-w(f)) \cdot f)} = \sum_{\mathbf{m} \in \mathcal{A}(f)_w} \overline{c_{\mathbf{m}}} \cdot \overline{\tau(\langle \mathbf{m}, w \rangle - w(f))} \cdot \xi^{\mathbf{m}} \in \mathbb{K}[M]$$

lies in the ideal of $X_{w,\tau} = \overline{\tau^{-w}} \cdot X_w \subset \mathbb{T}_N(\mathbb{k})$. If $\mathcal{A}(f)_w = \mathcal{A}(f)_u$, then an easy calculation shows that $\langle \mathbf{m}, w \rangle - w(f) = \langle \mathbf{m}, u \rangle - u(f)$, and thus

$$\overline{\varphi_{\tau,w}^\times(\tau(-w(f)) \cdot f)} = \overline{\varphi_{\tau,u}^\times(\tau(-u(f)) \cdot f)}.$$

As faces σ of \mathcal{C} have the property that for any $w, u \in \text{relint}(\sigma)$ and any $f \in I$, we have $\mathcal{A}(f)_w = \mathcal{A}(f)_u$, this explains Proposition 1.5. ◻

1.4. Local structure of tropicalization. Let K be an algebraically closed field. Define the initial form $\text{in}_w f$, initial ideal $\text{in}_w I$, and initial scheme $\text{in}_w X$ for $f \in K[M]$, I an ideal of $K[M]$, and X a K -subscheme of \mathbb{T}_N as in the theory of Gröbner bases [20]. For example, $\text{in}_w f$ is the sum of terms of f on whose exponents the function $\mathbf{m} \mapsto \langle \mathbf{m}, w \rangle$ is minimized.

The *tropicalization* $\text{Trop}(X)$ of a K -subscheme of \mathbb{T}_N is the set of $w \in \Gamma_N$ such that $\text{in}_w X$ is non-empty. This corresponds to the usual tropicalization of $X(\mathbb{K})$, where \mathbb{K} is any non-archimedean valued field containing K whose valuation extends the trivial valuation on K and whose valuation group is Γ . Choosing a projective space closure \mathbb{P}^N of \mathbb{T}_N , $\text{Trop}(X)$ is the support of a rational polyhedral Σ fan in Γ_N , which is a subfan of the Gröbner fan of the closure of \overline{X} . Call this subfan Σ a *Gröber-tropical fan* of X . All elements in the relative interior of a cone σ of Σ induce the same initial scheme, which we write as $\text{in}_\sigma X$.

Let us return to the situation of the previous subsections. That is, X is a \mathbb{K} -subscheme of \mathbb{T}_N and \mathbb{k} is a field with non-archimedean valuation ν , valuation group Γ , and residue field \mathbb{k} . Fix a section τ of the valuation map and a projective space closure \mathbb{P}^N of \mathbb{T}_N . Then there is a polyhedral complex \mathcal{C} in Γ_N whose support is $\text{Trop}(X)$. For $w \in \text{Trop}(X)$, consider the local fan \mathcal{C}_w of \mathcal{C} at w . The tropical reduction X_w is a \mathbb{k} -subscheme of \mathbb{T}_N . Equipping \mathbb{k} with a trivial valuation to Γ , the Gröbner-tropical fan Σ_w of X_w is a second fan associated to $w \in \text{Trop}(X)$, and its support is the tropicalization $\text{Trop}(X_w)$ of X_w .

Proposition 1.7. *We have $\mathcal{C}_w = \Sigma_w$. For any polyhedron ρ of \mathcal{C} containing w , we have*

$$X_\rho = \text{in}_{\rho_w}(X_w).$$

Here, X_ρ is the tropical reduction of X at any point in the relative interior of ρ , while $\text{in}_{\rho_w}(X_w)$ is the initial scheme of the tropical reduction X_w corresponding to the cone ρ_w of the local fan \mathcal{C}_w at w , equivalently, the Gröbner-tropical fan of X_w .

Lastly, we recall a fact from the theory of Gröbner bases. Suppose that σ is a cone of the Gröbner fan of a projective scheme Y . Then $\langle \sigma \rangle$ is a sublattice of N of rank equal to the dimension of σ , and the corresponding torus $\mathbb{T}_{\langle \sigma \rangle} \subset \mathbb{T}_N$ acts on $\text{in}_\sigma Y$. In particular, when $X \subset \mathbb{T}_N$ is irreducible of dimension d and σ is a d -dimensional polyhedron in a polyhedral complex

\mathcal{C} representing $\text{Trop}(X)$, then the tropical reduction X_σ is a d -dimensional subscheme of \mathbb{T}_N on which the d -dimensional subtorus $\mathbb{T}_{\langle\sigma\rangle}$ acts, so that X_σ is a finite union of $\mathbb{T}_{\langle\sigma\rangle}$ -orbits.

1.5. Change of trivialization. We now examine the effect on the tropical reductions of a different choice of section $t: \Gamma \rightarrow \mathbb{K}^\times$ of the valuation homomorphism. This differs from τ by a group homomorphism $\alpha := t/\tau: \Gamma \rightarrow R^\times := R \setminus \mathfrak{m}$. This gives a homomorphism $\alpha_N: \Gamma_N \rightarrow \mathbb{T}_N(R)$ with value $\alpha^w := \alpha_N(w) \in \mathbb{T}_N(R)$ at $w \in \Gamma_N$ defined by $t^w = \alpha^w \cdot \tau^w$. Trivializations induced by t and τ in $\mathbb{T}^w(\mathbb{k})$ are translates of each other by the reduction $\overline{\alpha^w}$ of α^w . Conversely, any homomorphism $\alpha: \Gamma \rightarrow R^\times$ gives a section $\alpha \cdot \tau: \Gamma \rightarrow \mathbb{K}^\times$.

Example 1.8. Suppose that $\Gamma = \mathbb{R}$ and the residue field is \mathbb{C} , which is a subfield of \mathbb{K} . Then any homomorphism $\alpha: \mathbb{R} \rightarrow \mathbb{C}$ gives a section $\alpha \cdot \tau$ of the valuation. Suppose that $N = \mathbb{Z}^n$, then the two identifications (3) of exploded tropicalization with $\mathbb{R}^n \times (\mathbb{C}^\times)^n$ differ by the map

$$\mathbb{R}^n \times (\mathbb{C}^\times)^n \ni (w, x) \mapsto (w, \overline{\alpha^w} \cdot x) \in \mathbb{R}^n \times (\mathbb{C}^\times)^n.$$

When α is discontinuous, this will be discontinuous, and so it is not clear that any reasonable structure can arise from the choice of a section and the corresponding trivialization. ◻

Despite this example, the different trivializations behave reasonably well on fibres of tropicalization.

Let $X \subset \mathbb{T}_N$ be a \mathbb{K} -subscheme. For any $w \in \Gamma_N$, the two tropical reductions are related by the translation

$$(4) \quad X_{w,\tau} = \overline{\alpha^w} \cdot X_{w,t}.$$

Fix a polyhedral complex \mathcal{C} supported on $\text{Trop}(X)$ as in Remark 1.4. For any face σ of \mathcal{C} containing w , we have $\sigma \subset w + \Gamma_{\langle\sigma\rangle}$, and so

$$\overline{\alpha}: \sigma \longrightarrow \overline{\alpha^w} \cdot \overline{\alpha}(\Gamma_{\langle\sigma\rangle}) = \overline{\alpha^w} \cdot \mathbb{T}_{\langle\sigma\rangle}(\mathbb{k}),$$

as $\overline{\alpha}: \Gamma_{\langle\sigma\rangle} \rightarrow \mathbb{T}_{\langle\sigma\rangle}(\mathbb{k})$, by the functoriality of our constructions. Since $\mathbb{T}_{\langle\sigma\rangle}(\mathbb{k})$ acts freely on X_σ , we see that

$$X_{\sigma,\tau} = \overline{\alpha^w} \cdot X_{\sigma,t}$$

for any $w \in \sigma$ (not just those in the relative interior of σ , which is implied by (4)).

Thus the two tropical reductions for every face containing $w \in \text{Trop}(X)$ are related by the same translation. In particular, the tropical reductions for incident faces are related by the same translation, but this is not the case for non-incident faces. Thus we only need to know the translations at minimal faces of \mathcal{C} . Let $\mathcal{M}(\mathcal{C})$ be the set of minimal faces of \mathcal{C} . For each $\sigma \in \mathcal{M}(\mathcal{C})$, we may choose some point $w \in \sigma$ and set $a_\sigma := \overline{\alpha^w}$. We summarize some conclusions of this discussion.

Proposition 1.9. *With these definitions, if $w \in \text{Trop}(X)$, then*

$$X_{w,\tau} = a_\sigma \cdot X_{w,t},$$

where σ is a minimal face of the face containing w in its relative interior.

If $\rho \in \mathcal{C}$, then the elements $\{a_\sigma \mid \sigma \subset \rho \text{ and } \sigma \in \mathcal{M}(\mathcal{C})\}$ lie in a single coset of $\mathbb{T}_{\langle\rho\rangle}$.

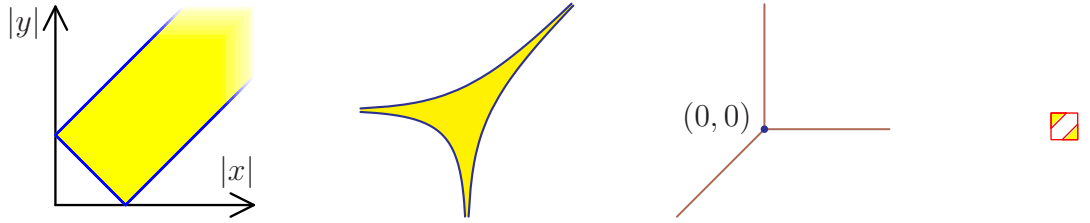
1.6. Complex coamoebae. For this section, we assume we are working over a subfield of the complex numbers. As a real algebraic group, the set \mathbb{C}^\times of invertible complex numbers is isomorphic to $\mathbb{R} \times \mathbb{U}$ under the map $(r, \theta) \mapsto e^{r+\sqrt{-1}\theta}$. Here, \mathbb{U} is set of complex numbers of norm 1 which we identify with $\mathbb{R}/2\pi\mathbb{Z}$. The inverse map $z \mapsto (\log |z|, \arg(z))$ induces an isomorphism $\mathbb{T}_N(\mathbb{C}) \rightarrow \mathbb{R}_N \times \mathbb{U}_N$, where $\mathbb{R}_N := \mathbb{R} \otimes_{\mathbb{Z}} N$ and $\mathbb{U}_N := \mathbb{U} \otimes_{\mathbb{Z}} N$. Composing with the projections to the factors \mathbb{R}_N and \mathbb{U}_N gives the maps Log and Arg , respectively.

The *amoeba* $\mathcal{A}(X)$ of a subscheme $X \subset \mathbb{T}_N$ is the image of its complex points $X(\mathbb{C})$ under the map Log . Let \mathbb{S}_N be the sphere in \mathbb{R}_N , with projection $\pi: \mathbb{R}_N \setminus \{0\} \rightarrow \mathbb{S}_N$. Then the *logarithmic limit set* $\mathcal{L}^\infty(X)$ of X is the set of accumulation points in \mathbb{S}_N of sequences $\{\pi(\text{Log}(x_n))\}$ where $\{x_n\} \subset X$ is unbounded in that its sequence of logarithms $\{\text{Log}(x_n)\}$ is unbounded. This was introduced by Bergman [3], and then Bieri and Groves [4] showed that the cone over it was a rational polyhedral fan. Later work of Kapranov [5], and then Speyer and Sturmfels [19] identified this fan with the negative $-\text{Trop}(X)$ of the tropical variety of X , computed in any valued field \mathbb{K} containing \mathbb{C} with valuation group Γ and residue field \mathbb{C} .

Example 1.10. Let $\ell \subset \mathbb{T}^2$ be defined by $f := x + y + 1 = 0$. By the triangle inequality,

$$|x| + |y| \geq 1, \quad |x| + 1 \geq |y|, \quad \text{and} \quad |y| + 1 \geq |x|.$$

Here is the solution of these inequalities, the amoeba $\mathcal{A}(\ell)$, and the tropical variety $\text{Trop}(\ell)$.



The *coamoeba* $\text{co}\mathcal{A}(X)$ of a subscheme X of \mathbb{T}_N is the image of $X(\mathbb{C})$ under the argument map Arg . Coamoebae are naturally semi-algebraic sets. The *phase limit set* $\mathcal{P}^\infty(X)$ of X is the set of accumulation points of arguments of unbounded sequences in X . For $w \in \mathbb{Z}^n$, the initial scheme $\text{in}_w X \subset (\mathbb{C}^\times)^n$ is defined by the initial ideal of I . Then $\text{in}_w X \neq \emptyset$ exactly when w lies in $\text{Trop}(X)$. For the line ℓ of Example 1.10, $\text{in}_{(0,0)} f = f$, and for $s > 0$,

$$\text{in}_{(s,0)} f = y + 1, \quad \text{in}_{(0,s)} f = x + 1, \quad \text{and} \quad \text{in}_{(-s,-s)} f = x + y,$$

and if w does not lie in one of these rays (which constitute $\text{Trop}(\ell)$), then $\text{in}_w f$ is a monomial, and so $\text{in}_w \ell = \emptyset$.

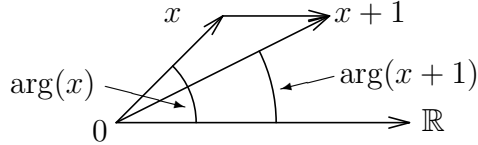
The following structure theorem for the phase limit set was established in [14]. The case when X is a complete intersection was earlier shown by Johansson [9].

Proposition 1.11. *The closure of $\text{co}\mathcal{A}(X)$ is $\text{co}\mathcal{A}(X) \cup \mathcal{P}^\infty(X)$, and*

$$\mathcal{P}^\infty(X) = \bigcup_{w \neq 0} \text{co}\mathcal{A}(\text{in}_w X).$$

This union is finite, for X has only finitely many initial schemes.

Example 1.12. We determine the coamoeba of the line ℓ from Example 1.10. If x is real then so is y . As ℓ meets all but the positive quadrant, its real points give $(\pi, 0)$, (π, π) , and $(0, \pi)$ in $\text{co}\mathcal{A}(\ell)$ (the dots in Figure 1). For $x \notin \mathbb{R}$, as $y = -(x + 1)$, the diagram



shows that $\arg(y) = \pi + \arg(x + 1)$ can take on any value strictly between $\pi + \arg(x)$ (for x near ∞) and 0 (for x near 0). Thus $\text{co}\mathcal{A}(\ell)$ also contains the interiors of the two triangles of Figure 1 displayed in the fundamental domain $[-\pi, \pi]^2$ in \mathbb{R}^2 of \mathbb{U}^2 . (This is modulo 2π , so that $\pi = -\pi$.) Because of this identification, the triangles meet at their vertices, and each shares half of one of the lines $\arg(x) = \pi$, $\arg(y) = \pi$, and $\arg(y) = \pi + \arg(x)$. The phase

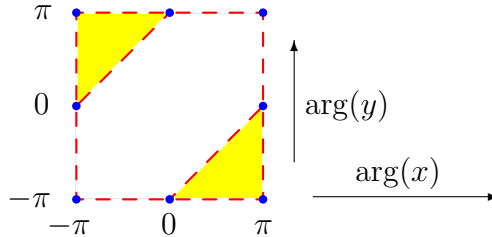


FIGURE 1. The coamoeba of a line

limit set $\mathcal{P}^\infty(\ell)$ consists of the three lines bounding the triangles, which are the coamoebae of the three non-trivial initial schemes of ℓ . ◻

If $X = \mathbb{T}_{N'} \subset \mathbb{T}_N$ is a subtorus, then $\text{co}\mathcal{A}(X) \subset \mathbb{U}_N$ is the corresponding real subtorus, $\mathbb{U}_{N'}$. More generally, if $\mathbb{T}_{N'}$ acts on a scheme X with finitely many orbits (so that X is supported on a union of translates of $\mathbb{T}_{N'}$) then $\text{co}\mathcal{A}(X)$ is a finite union of translates of $\mathbb{U}_{N'}$, and its dimension is equal to the rank of N' , which is the complex dimension of X .

2. NON-ARCHIMEDEAN COAMOEBAE AND PHASE TROPICAL VARIETIES

Suppose now that \mathbb{K} is a valued field with surjective valuation map $\nu: \mathbb{K}^\times \twoheadrightarrow \Gamma$ whose residue field is the complex numbers. Choose a section $\tau: \Gamma \rightarrow \mathbb{K}^\times$ of the valuation. With this choice, we define an argument map $\arg_\tau: \mathbb{K}^\times \rightarrow \mathbb{U}$ as follows. For each element $x \in \mathbb{K}^\times$, there is a unique element $a_x \in R^\times$ such that $x = a_x \cdot \tau(\nu(x))$, and we set $\arg_\tau(x) := \arg(\overline{a_x})$, the argument of the reduction of a_x . This induces an argument map $\text{Arg}_\tau: \mathbb{T}_N(\mathbb{K}) \rightarrow \mathbb{U}_N$, which is also the composition

$$\mathbb{T}_N(\mathbb{K}) \rightarrow \coprod_{w \in \Gamma_N} \mathbb{T}^w(\mathbb{C}) \xrightarrow{\varphi_\tau^{-1}} \Gamma_N \times \mathbb{T}_N(\mathbb{C}) \twoheadrightarrow \mathbb{T}_N(\mathbb{C}) \xrightarrow{\arg} \mathbb{U}_N.$$

We henceforth assume that τ is fixed and drop the dependence on τ from our notation.

2.1. Non-archimedean coamoebae. The *non-archimedean coamoeba* $Nco\mathcal{A}(X)$ of a subscheme X of \mathbb{T}_N is the image of $X(\mathbb{K})$ under the map Arg . It is therefore the union of the complex coamoebae of all tropical reductions of X . This is in fact a finite union. A projective space closure \mathbb{P}^N of \mathbb{T}_N gives a polyhedral complex \mathcal{C} in Γ_N with support $\text{Trop}(X)$ and by Proposition 1.5, we have

$$Nco\mathcal{A}(X) = \bigcup_{\sigma \in \mathcal{C}} co\mathcal{A}(X_\sigma).$$

Example 2.1. Fix a positive element $1 \in \Gamma$, and write t for $\tau^1 \in \mathbb{K}^\times$ (recall that the value of τ at $w \in \Gamma$ is written τ^w). Suppose that ℓ is the line defined in $(\mathbb{K}^\times)^2$ by $f := x + y + t = 0$. Its tropical variety is shown on the left in Figure 2. (For the picture, we assume that $\Gamma \subset \mathbb{R}$.) We have $\text{in}_{(1,1)}f = x + y + 1$, and for $s > 0$,

$$\text{in}_{(1+s,1)}f = y + 1, \quad \text{in}_{(1,1+s)}f = x + 1, \quad \text{and} \quad \text{in}_{(1-s,1-s)}f = x + y,$$

so $Nco\mathcal{A}(\ell)$ is the union of the coamoebae of these tropical reductions, which is the union of the two closed triangles on the right in Figure 2. A different section of the valuation map

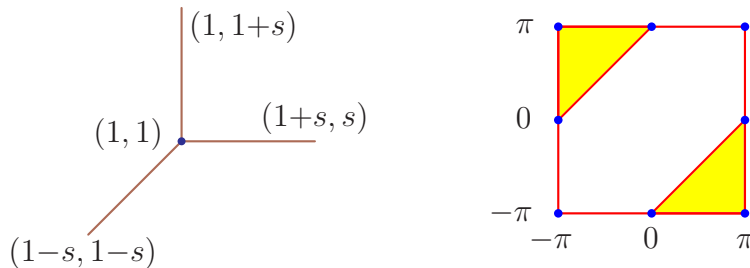


FIGURE 2. Tropical line and its non-archimedean coamoeba

will simply translate this coamoeba, as the tropical variety has a unique minimal face. ◻

Theorem 2.2. *If $X \subset \mathbb{T}_N$ is defined over \mathbb{C} , then the closure of its complex coamoeba is equal to its non-archimedean coamoeba.*

Note that in particular, $Nco\mathcal{A}(X)$ is closed.

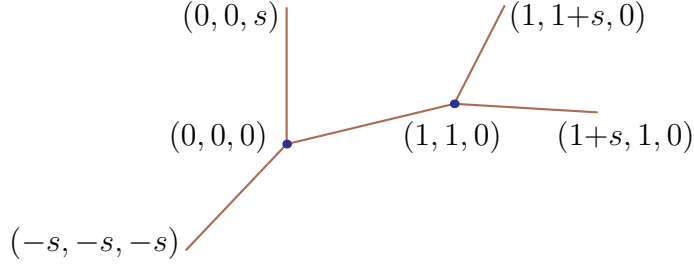
Proof. By definition, $Nco\mathcal{A}(X)$ is the union of the coamoebae of all tropical reductions of X . Since X is a complex scheme, these are its initial schemes, so the result holds by Proposition 1.11. Also, $\text{Trop}(X)$ has a unique minimal face, $0 \in \Gamma_N$, so the only ambiguity in $\text{Trop}(X)$ is a global translation. ◻

We will see that Theorem 2.2 describes the local structure of non-archimedean coamoebae.

Example 2.3. Let ζ be a primitive third root of unity, $\omega := 1 + \sqrt{-1}$, and t be an element of \mathbb{K}^\times with valuation 1 as in Example 2.1. Consider the line

$$x + \zeta y + \zeta^2 t = \sqrt{-1} \cdot x + z - \omega = 0$$

in $(\mathbb{K}^\times)^3$. We display its tropical variety, again assuming that $\Gamma \subset \mathbb{R}$.



This has two vertices $(0, 0, 0)$ and $(1, 1, 0)$ connected by an internal line segment $((s, s, 0)$ for $0 < s < 1$) and four rays as indicated for $s > 0$. For each face of this polyhedral complex, the ideal I of the line has a different initial ideal as follows

$$\begin{aligned} \text{in}_{(0,0,0)}I &= \langle x + \zeta y, \sqrt{-1} \cdot x + z - \omega \rangle & \text{in}_{(1,1,0)}I &= \langle x + \zeta y + \zeta^2, z - \omega \rangle \\ \text{in}_{(s,s,0)}I &= \langle x + \zeta y, z - \omega \rangle & \text{in}_{(1+s,1,0)}I &= \langle \zeta y + \zeta^2, z - \omega \rangle \\ \text{in}_{(0,0,s)}I &= \langle x + \zeta y, \sqrt{-1} \cdot x - \omega \rangle & \text{in}_{(1,1+s,0)}I &= \langle x + \zeta^2, z - \omega \rangle \\ \text{in}_{(-s,-s,-s)}I &= \langle x + \zeta y, \sqrt{-1} \cdot x + z \rangle \end{aligned}$$

Figure 3 shows two views of $Nco\mathcal{A}(\ell)$, which is the union of the seven coamoebae, one for each tropical reduction of ℓ . It is the closure of the two coamoebae corresponding to

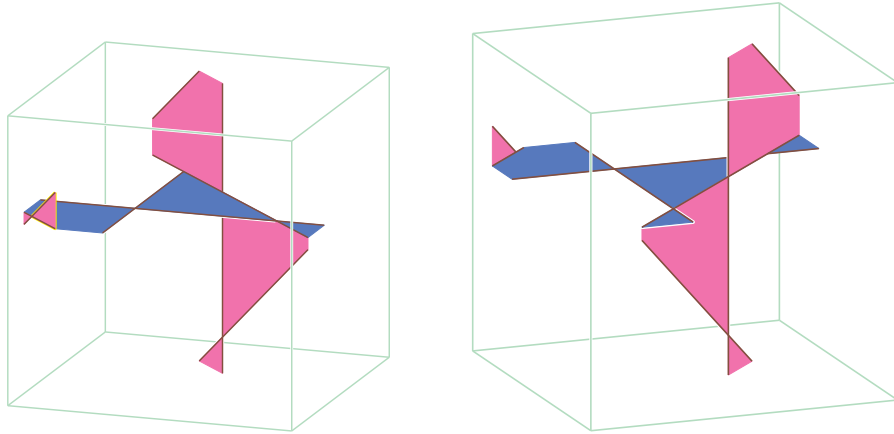


FIGURE 3. Non-archimedean coamoeba of a line

the vertices of $\text{Trop}(\ell)$, each of which is a coamoeba of a line in a plane consisting of two triangles. Here $(0, 0, 0)$ corresponds to the vertical triangles and $(1, 1, 0)$ to the horizontal triangles. These two coamoebae are attached along the coamoeba of the tropical reduction corresponding to the edge between the vertices, and each has two boundary coamoebae corresponding to the unbounded rays at each vertex.

If we change the section of the valuation which we implicitly used to compute this non-archimedean coamoeba, its effect would be to rotate (slide) the horizontal and vertical triangles along the line they share, followed by a global translation. \square

Theorem 2.4. *Let X be an irreducible \mathbb{K} -subscheme of \mathbb{T}_N , where \mathbb{K} is a field with a non-archimedean valuation and residue field \mathbb{C} . Fix a section of the valuation to define the non-archimedean coamoeba of X . Let \mathcal{C} be a polyhedral complex with support $\text{Trop}(X)$ induced by a projective compactification of \mathbb{T}_N . Then*

$$(5) \quad Nco\mathcal{A}(X) = \bigcup_{\substack{\sigma \in \mathcal{C} \\ \text{is minimal}}} \overline{co\mathcal{A}(X_\sigma)} = \bigcup_{\substack{\sigma \in \mathcal{C} \\ \text{is minimal}}} Nco\mathcal{A}(X_\sigma),$$

the union of closures of coamoebae of the tropical reductions for all minimal faces of \mathcal{C} .

If we change the section of valuation, each coamoeba $Nco\mathcal{A}(X_\sigma)$ will be translated by an element $a_\sigma \in \mathbb{U}_N$. For any face σ of \mathcal{C} , the elements a_ρ corresponding to subfaces ρ of σ all lie in a single coset of $\mathbb{U}_{\langle\sigma\rangle}$.

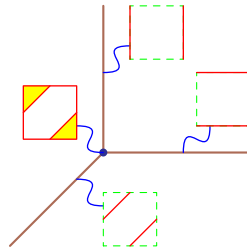
Proof. Let $\rho \in \mathcal{C}$ be a face. Then ρ contains a minimal face σ of \mathcal{C} . Let $w \in \sigma$. By Proposition 1.7, X_ρ is an initial scheme of $X_w = X_\sigma$. By Proposition 1.11, its coamoeba is a component of the phase limit set of X_σ and lies in the closure of $co\mathcal{A}(X_w)$. Similarly, Theorem 2.2 implies that $co\mathcal{A}(X_\rho) \subset Nco\mathcal{A}(X_\sigma)$. These facts imply (5). The statement about the change of section follows from Proposition 1.9. \square

2.2. Phase tropical varieties. The product of the maps Trop and Arg is a map

$$\mathcal{P}\text{trop} : \mathbb{T}_N(\mathbb{K}) \longrightarrow \Gamma_N \times \mathbb{U}_N.$$

The *phase tropical variety* $\mathcal{P}\text{trop}(X)$ of a \mathbb{K} -subscheme X of \mathbb{T}_N is the closure of the image of $X(\mathbb{K})$ under this map. Here, we equip Γ_N with the weakest topology so that the half-spaces (1) are closed. Under the projection to the first coordinate, $\mathcal{P}\text{trop}(X)$ maps to the tropical variety of X with fibre over a point w the closure of the coamoeba of the tropical reduction $\text{in}_w X$ of X . This notion was introduced by Mikhalkin for plane curves [13, §6] and [12].

Example 2.5. Suppose that $\Gamma = \mathbb{R}$ and consider $\mathcal{P}\text{trop}(\ell)$, where ℓ is the line $x + y + 1 = 0$. As we cannot easily represent objects in four dimensions, we draw $\text{Trop}(\ell)$ and indicate the fibres above points of $\text{Trop}(\ell)$ via screens.



This phase tropical variety is homeomorphic to a three-punctured sphere (in fact to ℓ itself as a subset of $(\mathbb{C}^\times)^2$). This indicates why we take the closure, for without the closure, the phase tropical variety would omit the six line segments that are in the closure of the coamoeba of $\text{in}_{(0,0)}\ell$ above $(0, 0)$, but not in the coamoeba. \square

Choose a projective compactification \mathbb{P}^N of \mathbb{T}_N so that for any \mathbb{K} -subscheme X of \mathbb{T}_N , we have a polyhedral complex \mathcal{C} whose support is $\text{Trop}(X)$, as in Remark 1.4. A subset of the form $\rho \times Y$ of $\Gamma_N \times \mathbb{U}_N$, where ρ is a polyhedron (or the relative interior of a polyhedron) and Y is a semi-algebraic subset of \mathbb{U}_N , has *dimension* $\dim(\rho) + \dim(Y)$, where $\dim(\rho)$ is the dimension of the polyhedron as defined in Subsection 1.2 and $\dim(Y)$ is its dimension as a semialgebraic subset of \mathbb{U}_N . Any finite union of such sets has dimension the maximum dimension of the sets in the union.

Theorem 2.6. *Let X be a \mathbb{K} -subscheme of \mathbb{T}_N . Then the phase tropical variety is a subset of $\Gamma_N \times \mathbb{U}_N$ of dimension $2 \dim(X)$. It is the (finite) union*

$$(6) \quad \mathcal{P}\text{trop}(X) = \bigcup_{\sigma \in \mathcal{C}} \sigma \times \overline{\text{co}\mathcal{A}(X_\sigma)} = \prod_{\sigma \in \mathcal{C}} \text{relint } \sigma \times \overline{\text{co}\mathcal{A}(X_\sigma)}.$$

If we change the section of valuation, each coamoeba factor $N\text{co}\mathcal{A}(X_\sigma)$ in this decomposition will be translated by an element $a_\sigma \in \mathbb{U}_N$. For any face σ of \mathcal{C} , the elements a_ρ corresponding to subfaces ρ of σ all lie in a single coset of $\mathbb{U}_{\langle \sigma \rangle}$.

Proof. The decomposition (6) is immediate from the definition of phase tropical varieties. We have the formula for the dimension of a coamoeba of a \mathbb{K} -subscheme X of $\mathbb{T}_N(\mathbb{C})$,

$$\dim(\text{co}\mathcal{A}(X)) \leq \min(\text{rank}(N), 2 \dim(X) - n),$$

when a subtorus \mathbb{T} of $\mathbb{T}_N(\mathbb{C})$ of dimension n acts on X . This is because $\mathbb{T} \simeq \mathbb{U}^n \times \mathbb{R}_{>}^n$, the argument map has fibres containing orbits of $\mathbb{R}_{>}^n$, and the image must have dimension at most $\dim(\mathbb{U}_N) = \text{rank}(N)$.

Suppose that X is irreducible. For a face σ of \mathcal{C} , X_σ has an action of the subtorus $\mathbb{T}_{\langle \sigma \rangle}$, which has dimension $\dim(\sigma)$. Thus $\dim(\sigma \times \overline{\text{co}\mathcal{A}(X_\sigma)})$ is

$$\dim(\sigma) + \dim(\text{co}\mathcal{A}(X_\sigma)) \leq \dim(\sigma) + \min(\text{rank}(N), 2 \dim(X_\sigma) - \dim(\sigma)),$$

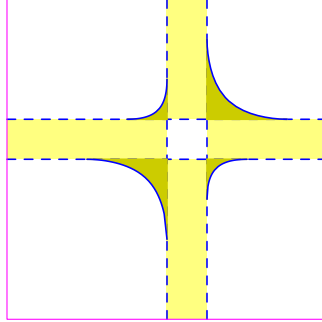
which is at most $2 \dim(X)$, as $\dim(X) = \dim(X_\sigma)$. If σ is a face of maximal dimension, so that $\dim(\sigma) = \dim(X) (\leq \text{rank}(N))$, then $\dim(X_\sigma) = \dim(\sigma)$, and so $\dim(\sigma \times \overline{\text{co}\mathcal{A}(X_\sigma)}) = 2 \dim(\sigma) = 2 \dim(X)$.

The last statement follows from Proposition 1.9. \square

3. COAMOEBAE OF TROPICALLY SIMPLE VARIETIES

Unlike tropical varieties, non-archimedean coamoebae and phase tropical varieties are typically not unions of polyhedra. This is because any complex coamoeba may occur as a

component of a non-archimedean coamoeba, and complex coamoebae are only partially combinatorial, as we may see from the coamoeba of the hyperbola $(1 + \sqrt{-1}) - x - y + xy = 0$,



The coamoeba mostly lies inside the two rectangles enclosed by its phase limit set, which consists of the four lines where the arguments of x and y are 0 and $\pi/4$. It also has some extra pieces with curvilinear boundary that are shaded in this picture.

A \mathbb{K} -subscheme X of \mathbb{T}_N is *tropically simple* if every tropical reduction X_w is defined by polynomials, each of which has affinely independent support, and the spans of these supports are themselves independent. We give a less terse and more geometric description in Subsections 3.3 and 3.4. Non-archimedean coamoebae and phase tropical varieties of tropically simple varieties are unions of polyhedra. We first study some complex varieties whose coamoebae are unions of polyhedra, for they form the local theory of tropically simple varieties.

3.1. Coamoebae of hyperplanes. A hypersurface $H \subset \mathbb{T}_N(\mathbb{C})$ whose Newton polytope is a unimodular simplex is a *hyperplane*. There is an identification of $\mathbb{T}_N(\mathbb{C})$ with $(\mathbb{C}^\times)^{\text{rank}(N)}$ and non-zero complex numbers a_1, \dots, a_n ($n \leq \text{rank}(N)$) such that H is defined by

$$(7) \quad 1 + a_1 z_1 + a_2 z_2 + \dots + a_n z_n = 0,$$

where z_1, z_2, \dots are the coordinate functions on $(\mathbb{C}^\times)^{\text{rank}(N)}$. Then H is the product of the hyperplane in $\mathbb{T}^n(\mathbb{C})$ with equation (7) and a coordinate subtorus of dimension $\text{rank}(N) - n$. The coamoeba of H is the product of the coamoeba of (7) and a coordinate subtorus $\mathbb{T}^{\text{rank}(N)-n}$.

We may change coordinates in (7), replacing $a_i z_i$ by x_i to obtain the equation

$$(8) \quad f := 1 + x_1 + x_2 + \dots + x_n = 0.$$

This simply translates the coamoeba by the vector

$$(\arg(a_1), \arg(a_2), \dots, \arg(a_n)).$$

A hyperplane H in $(\mathbb{C}^\times)^n$ with equation (8) is *standard*. To describe its coamoeba, we work in a fundamental domain of homogeneous coordinates for \mathbb{U}^n , namely

$$\{(\theta_0, \theta_1, \dots, \theta_n) : \theta_i \leq 2\pi + \min\{\theta_0, \dots, \theta_n\}\},$$

where we identify $(\theta_0, \dots, \theta_n)$ with its translate $(\theta_0 + t, \dots, \theta_n + t)$ for any t .

Lemma 3.1. *Let H be a standard hyperplane (8). Then $\overline{\text{co}\mathcal{A}(H)}$ is the complement of*

$$(9) \quad Z := \{(\theta_0, \theta_1, \dots, \theta_n) : |\theta_i - \theta_j| < \pi \text{ for all } 0 \leq i < j \leq n\}.$$

Proof. The set Z is an open zonotope. We first show that no point of Z lies in the coamoeba. Let $(\theta_0, \dots, \theta_n) \in Z$. The inequalities defining Z are invariant under permuting coordinates and simultaneous translation. Thus we may assume that

$$(10) \quad 0 = \theta_0 \leq \theta_1 \leq \dots \leq \theta_n < \pi.$$

If $1 = x_0, x_1, \dots, x_n$ are complex numbers with $\arg(x_i) = \theta_i$ for all i , then they all have nonnegative imaginary parts. They cannot satisfy (8) for either all are positive real numbers or one has a non-zero imaginary part. Thus no point of Z lies in the coamoeba of H .

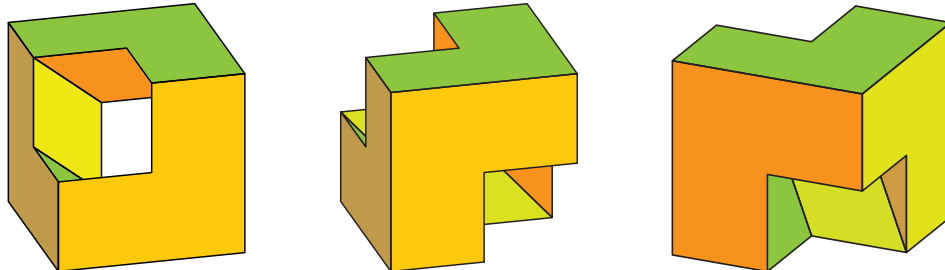
We use the phase limit set for the other inclusion. Let $\text{co}\mathcal{A}(H)$ be the coamoeba of H . Suppose that $\theta = (\theta_0, \theta_1, \dots, \theta_n)$ lies in the complement of Z , so that $|\theta_i - \theta_j| \geq \pi$ for some pair i, j . We may assume that $i = 1, j = 2$, and $\theta_0 = 0$. Then (θ_1, θ_2) lies in the set



which is the closure of the coamoeba of the line given by $1 + x_1 + x_2 = 0$. This is an initial form $\text{in}_w f$ of the polynomial (8), and so by Proposition 1.11, $\overline{\text{co}\mathcal{A}(H)}$ contains $\overline{\text{co}\mathcal{A}(\text{in}_w H)}$, which is the product of the coamoeba (11) and the coordinate torus spanned by the last $n - 2$ coordinates. This shows that $\theta \in \overline{\text{co}\mathcal{A}(\text{in}_w H)} \subset \overline{\text{co}\mathcal{A}(H)}$, which completes the proof. ◻

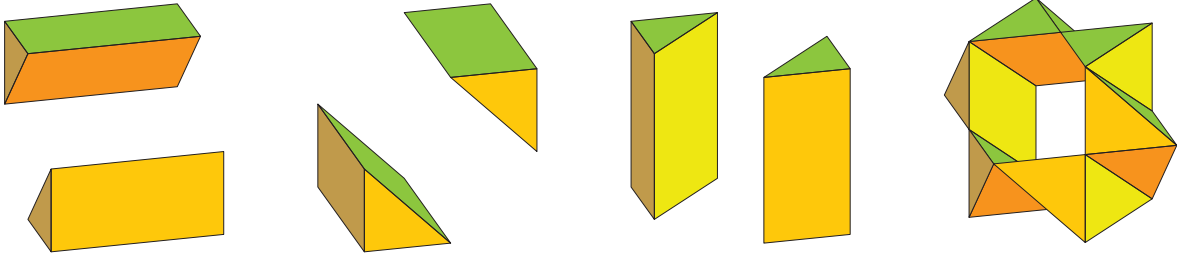
This proof also shows that these portions of the phase limit set, which are cylinders with base the coamoeba of a line in a plane, cover the coamoeba of H . These correspond to two-dimensional (triangular) faces of the Newton polytope of f .

Example 3.2. Here are three views of the closure of the coamoeba of the plane $1 + x_1 + x_2 + x_3 = 0$ in $(\mathbb{C}^\times)^3$, in the fundamental domain $[-\pi, \pi]^3$ for \mathbb{U}^3 .



In this domain, the zonotope consists of those $(\theta_1, \theta_2, \theta_3)$ such that all six quantities $\theta_1, \theta_2, \theta_3, \theta_1 - \theta_2, \theta_2 - \theta_3$, and $\theta_3 - \theta_1$ lie between $-\pi$ and π . Besides the 12 boundary quadrilaterals

lying on the six coordinate planes defined by these inequalities, the phase limit set has four components which are cylinders over coamoebae of planar lines.



The closures of these cylinders cover the coamoeba of H , as we showed in Lemma 3.1. Each of these cylinders is itself the coamoeba of a hyperplane. \square

Theorem 3.3. *Let $H \subset \mathbb{T}_N(\mathbb{C})$ be a hyperplane whose defining polynomial f has Newton polytope the simplex Δ . If $\dim(\Delta) = 1$, then $\text{co}\mathcal{A}(H)$ is a translate of a subtorus of dimension $\text{rank}(N) - 1$. Otherwise, $\text{co}\mathcal{A}(H)$ has dimension $\text{rank}(N)$ and is a cylinder over the coamoeba of a hyperplane in $(\mathbb{C}^\times)^{\dim(\Delta)}$, which is the complement of a zonotope.*

The phase limit set of H is the union of coamoebae of hyperplanes defined by the restrictions of f to the non-vertex faces of Δ . Those corresponding to triangles cover $\text{co}\mathcal{A}(H)$.

The non-archimedean coamoeba of $H_{\mathbb{K}} \subset \mathbb{T}_N(\mathbb{K})$ is the closure of the coamoeba of H and is the complement in \mathbb{U}_N of cylinder over an open zonotope of dimension $\dim(\Delta)$.

The tropical variety $\text{Trop}(H)$ of $H_{\mathbb{K}}$ is the normal fan of Δ with its full-dimensional cones removed. Tropical reductions H_σ for σ a cone of $\text{Trop}(H)$ are the hyperplanes defined by the restriction of f to the face of Δ dual to σ . The phase tropical variety $\mathcal{P}\text{trop}(H)$ of H is the closure of its restriction to the cones of $\text{Trop}(H)$ corresponding to edges and two-dimensional faces of Δ and it is a pure-dimensional polyhedral complex of dimension $2 \dim(H)$.

Proof. These statements are either immediate from the definitions or from Lemma 3.1, or are well-known (e.g. the description of $\text{Trop}(H)$), except perhaps those concerning $\mathcal{P}\text{trop}(H)$.

By Theorem 2.6, $\dim \mathcal{P}\text{trop}(H) = 2 \dim(H)$ with the pieces of the decomposition (6) corresponding to maximal cones σ of $\text{Trop}(X)$ having maximal dimension. Let σ be a non-maximal cone dual to a face \mathcal{F} of Δ of dimension two or greater. Then $\dim(\sigma) = \text{rank}(N) - \dim(\mathcal{F})$, and the coamoeba of the tropical reduction H_σ has dimension $\text{rank}(N)$. Thus the piece of $\mathcal{P}\text{trop}(X)$ corresponding to σ has dimension $2 \text{rank}(N) - \dim(\mathcal{F})$. This is maximized with maximal value $2 \text{rank}(N) - 2 = 2 \dim(H)$ when \mathcal{F} a triangular face of Δ .

Ridges are faces of $\text{Trop}(X)$ corresponding to triangular faces of Δ . Every non-maximal face of $\text{Trop}(X)$ is contained in a ridge, and by the computation in the proof of Lemma 3.1 the coamoeba of the tropical reduction corresponding to a non-maximal face σ is the union of the coamoebae of tropical reductions corresponding to the ridges containing σ . This shows the statement about closure and completes the proof of the theorem. \square

3.2. Coamoeba of simple hypersurfaces. A hypersurface $H \subset \mathbb{T}_N(\mathbb{C})$ is *simple* if the support of its defining polynomial f is an affinely independent subset of M . Multiplying f

by a scalar and a monomial we may assume that it is *monic* in that it contains the term $\xi^0 = 1$, and therefore has the form

$$(12) \quad f = 1 + a_1 \xi^{\mathbf{m}_1} + a_2 \xi^{\mathbf{m}_2} + \cdots + a_n \xi^{\mathbf{m}_n}, \quad n \leq \text{rank}(N) \quad a_i \neq 0,$$

where $\mathcal{A} := \{\mathbf{m}_1, \dots, \mathbf{m}_n\}$ is a linearly independent set. Set $a := (a_1, \dots, a_n) \in (\mathbb{C}^\times)^n$, write T_a for the translation in $(\mathbb{C}^\times)^n$ by a , and let $\varphi_{\mathcal{A}}: \mathbb{T}_N \rightarrow \mathbb{T}^n$ be the homomorphism

$$(13) \quad \varphi_{\mathcal{A}} : x \mapsto (x(\mathbf{m}_1), x(\mathbf{m}_2), \dots, x(\mathbf{m}_n)).$$

Then H is the inverse image of the standard hyperplane (8) along the map $T_a \circ \varphi_{\mathcal{A}}$.

Let $\mathcal{A}^\perp \subset N$ be $\{\mathbf{n} \in N \mid \langle \mathbf{m}, \mathbf{n} \rangle = 0 \text{ for all } \mathbf{m} \in \mathcal{A}\}$. Then $\mathbb{T}_{\mathcal{A}^\perp}$ is the connected component of the identity of the kernel of $\varphi_{\mathcal{A}}$, and so $\mathbb{T}_{\mathcal{A}^\perp}$ acts freely on H . The quotient is a simple hypersurface in $\mathbb{T}_N/\mathbb{T}_{\mathcal{A}^\perp}$, which has the same defining polynomial (12) as H , but in the coordinate ring of $\mathbb{T}_N/\mathbb{T}_{\mathcal{A}^\perp}$, which is $\mathbb{C}[\text{sat}(\mathcal{A})]$. Here, $\text{sat}(\mathcal{A})$ is the saturation of the \mathbb{Z} -span $\mathbb{Z}\mathcal{A}$ of \mathcal{A} , those points of M which lie in the \mathbb{Q} -linear span of \mathcal{A} . The map $\varphi_{\mathcal{A}}: \mathbb{T}_N/\mathbb{T}_{\mathcal{A}^\perp} \rightarrow \mathbb{T}^n$ is surjective of degree the cardinality of its kernel, $\text{Hom}(\text{sat}(\mathcal{A})/\mathbb{Z}\mathcal{A}, \mathbb{C}^\times)$, which we will call $\mathbf{n}\text{-vol}(\mathcal{A})$. When $\text{sat}(\mathcal{A}) = \mathbb{Z}^n$, $\mathbf{n}\text{-vol}(\mathcal{A})$ is the Euclidean volume of the convex hull of $\mathcal{A} \cup \{0\}$, multiplied by $n!$.

These maps T_a and $\varphi_{\mathcal{A}}$ induce linear maps on $\mathbb{R}_N, \mathbb{R}^n$ (via the Log map defining amoebae), and $\mathbb{U}_N, \mathbb{U}^n$ (via the argument map defining coamoebae). A consequence of this discussion is the following description of the amoeba and coamoeba of a simple hypersurface, showing that its coamoeba may be understood largely in terms of polyhedral combinatorics.

Lemma 3.4. *Let H be a simple hypersurface in $\mathbb{T}_N(\mathbb{C})$.*

- (1) *The amoeba of H is the pullback of the amoeba of the standard hyperplane (8) along the map $T_a \circ \varphi_{\mathcal{A}}$.*
- (2) *The coamoeba of H is the pullback of the coamoeba of the standard hyperplane (8) along the map $T_a \circ \varphi_{\mathcal{A}}$. Its closure is equal to $\mathbb{U}_N \setminus \varphi_{\mathcal{A}}^{-1}(T_{-a}Z)$, where Z is the zonotope (9). In particular, the number of components in the complement of $\text{co}\mathcal{A}(H)$ in \mathbb{U}_N is equal to $\mathbf{n}\text{-vol}(\mathcal{A})$.*

Remark 3.5. As a simple hypersurface H and its coamoeba are pullbacks along $T_a \circ \varphi_{\mathcal{A}}$ of the standard hyperplane and its coamoeba, the statements of Theorem 3.3 hold for H , with the obvious modifications as expressed in Lemma 3.4. We leave the precise formulation to the reader, but note that the non-archimedean coamoeba and phase tropical variety of H have a description that is completely combinatorial, up to phase shifts coming from T_a . In particular, $\mathcal{P}\text{trop}(H)$ is a polyhedral complex of pure dimension $2 \dim(H)$.

Example 3.6. Consider the simple curve C in $(\mathbb{C}^\times)^2$ defined by

$$f := 1 + ax^2y + bxy^2 \quad a, b \in \mathbb{C}^\times.$$

Then $\mathcal{A} = \{(2, 1), (1, 2)\}$ and C is the pullback of the standard line of Example 1.10 along the composition $T_{(a,b)} \circ \varphi_{\mathcal{A}}$.

We ignore the rotation and consider only the pullback along $\varphi_{\mathcal{A}}$. On \mathbb{U}^2 , $\varphi_{\mathcal{A}}$ is the map

$$(\alpha, \beta) \mapsto (2\alpha + \beta, \alpha + 2\beta).$$

We lift to the universal cover \mathbb{R}^2 of \mathbb{U}^2 to determine the pullback. Figure 4 shows the image of the fundamental domain of \mathbb{U}^2 shaded and superimposed on nine copies of the coamoeba of the standard line, in the universal cover \mathbb{R}^2 . Thus the pullback is the coamoeba shown in a

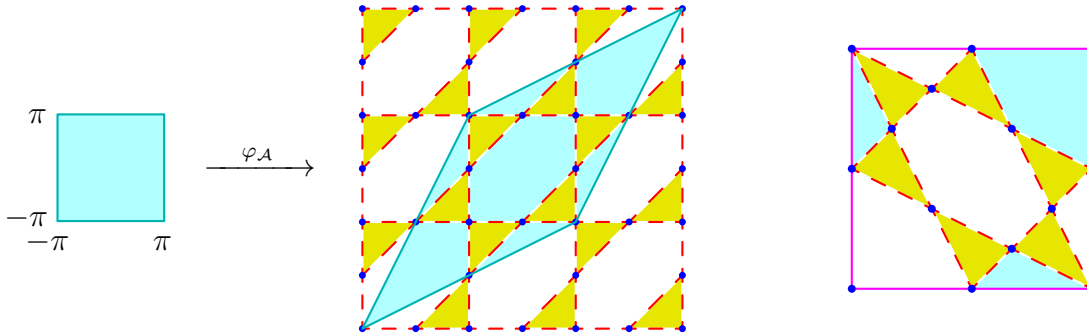


FIGURE 4. The pullback and the coamoeba

single fundamental domain on the right, where we have shaded one of the three components of its complement. ◻

3.3. Simple varieties. A subvariety $X \subset \mathbb{T}_N(\mathbb{C})$ is *simple* if it is the pullback along a surjective homomorphism of a product of hyperplanes. That is, there are hyperplanes $H_i \subset (\mathbb{C}^\times)^{n_i}$ for $1 \leq i \leq m$ and a surjective homomorphism $\varphi: \mathbb{T}_N(\mathbb{C}) \rightarrow (\mathbb{C}^\times)^{n_1} \times \cdots \times (\mathbb{C}^\times)^{n_m}$ such that

$$(14) \quad X = \varphi^{-1}(H_1 \times H_2 \times \cdots \times H_m)$$

Thus, up to a finite cover as in Example 3.6, its coamoeba is the product of coamoebae of hyperplanes, and therefore is an object from polyhedral combinatorics.

Simple subvarieties have a combinatorial characterization. A polynomial $f \in \mathbb{K}[M]$ is *monic* if its support includes $0 \in M$. The *reduced support* of a monic polynomial consists of the non-zero elements of its support. We note that any subscheme $X \subset \mathbb{T}_N(\mathbb{C})$ may be defined by monic polynomials—simply divide each polynomial defining X by one of its terms, which is invertible in $\mathbb{K}[M]$.

Proposition 3.7. *A subscheme $X \subset \mathbb{T}_N(\mathbb{C})$ is a simple subvariety if and only if it is defined by monic polynomials whose reduced supports are linearly independent.*

Proof. Given a simple subvariety $X \subset \mathbb{T}_N(\mathbb{C})$, let $H_i \subset (\mathbb{C}^\times)^{n_i}$ for $i = 1, \dots, m$ be hyperplanes and φ the surjection $\mathbb{T}_N(\mathbb{C}) \rightarrow (\mathbb{C}^\times)^{n_1} \times \cdots \times (\mathbb{C}^\times)^{n_m}$ so that X is the pullback of the product of the H_i . For each $i = 1, \dots, m$, let g_i be a monic polynomial defining the hyperplane $H_i \subset (\mathbb{C}^\times)^{n_i}$ and set $f_i := \varphi^\times(g_i)$. Then f_1, \dots, f_m are monic polynomials defining X .

For each $i = 1, \dots, n$, the reduced support of g_i is a linearly independent subset of \mathbb{Z}^{n_i} . The reduced support \mathcal{A}_i of f_i is the image of $\mathcal{A}'_i \subset \mathbb{Z}^{n_i}$ under the map

$$\Phi : \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_2} \oplus \dots \oplus \mathbb{Z}^{n_m} \hookrightarrow M$$

which corresponds to the surjection φ . It follows that the reduced supports $\mathcal{A}_1, \dots, \mathcal{A}_m$ of the monic polynomials f_1, \dots, f_m defining X are linearly independent.

For the other direction, suppose that we have monic polynomials f_1, \dots, f_m defining X whose reduced supports $\mathcal{A}_1, \dots, \mathcal{A}_m$ are linearly independent. If \mathcal{A} is the union of these reduced supports, then we obtain a surjective homomorphism $\varphi_{\mathcal{A}}$ (13) from $\mathbb{T}_N(\mathbb{C})$ onto the product of tori $(\mathbb{C}^\times)^{n_0} \times (\mathbb{C}^\times)^{n_1} \times \dots \times (\mathbb{C}^\times)^{n_m}$ where n_i is the cardinality of \mathcal{A}_i . Furthermore, each polynomial f_i is the pullback of a polynomial on $(\mathbb{C}^\times)^{n_i}$ defining a hyperplane H_i , and so we obtain the description (14), which completes the proof. \square

Remark 3.8. Up to a finite cover (and/or a product with a torus) the coamoeba of a simple variety is the product of coamoebae of hyperplanes. By Proposition 3.7, nonempty initial schemes of simple subvarieties are simple subvarieties. Thus the structures of coamoebae, non-archimedean coamoebae, and phase tropical varieties of simple varieties may be understood in completely combinatorial terms by adapting the descriptions of Theorem 3.3 (to products of hyperplanes) and the effects of pullbacks along surjective homomorphisms as discussed in Subsection 3.2. In particular, the phase tropical variety of a simple variety X is a polyhedral complex of pure dimension $2 \dim(X)$.

3.4. Tropically simple varieties. A \mathbb{K} -subscheme X of \mathbb{T}_N is *tropically simple* if each of its tropical reductions X_w is a simple subvariety of $\mathbb{T}_N(\mathbb{C})$.

Theorem 3.9. *Let $X \subset \mathbb{T}_N$ be a tropically simple variety. Its non-archimedean coamoeba $Nco\mathcal{A}(X)$ is a union of polyhedra of dimension $\dim(X) + 1$. Its phase tropical variety $\mathcal{P}trop(X)$ is a polyhedral complex of pure dimension $2 \dim(X)$.*

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