A SAGBI BASIS FOR THE QUANTUM GRASSMANNIAN

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Dedicated to the memory of Gian-Carlo Rota

ABSTRACT. The maximal minors of a $p \times (m+p)$-matrix of univariate polynomials of degree n with indeterminate coefficients are themselves polynomials of degree np. The subalgebra generated by their coefficients is the coordinate ring of the quantum Grassmannian, a singular compactification of the space of rational curves of degree np in the Grassmannian of p-planes in $(m+p)$-space. These subalgebra generators are shown to form a sagbi basis. The resulting flat deformation from the quantum Grassmannian to a toric variety gives a new "Gröbner basis style" proof of the Ravi-Rosenthal-Wang formulas in quantum Schubert calculus. The coordinate ring of the quantum Grassmannian is an algebra with straightening law, which is normal, Cohen-Macaulay, and Koszul, and the ideal of quantum Plücker relations has a quadratic Gröbner basis. This holds more generally for skew quantum Schubert varieties. These results are well-known for the classical Schubert varieties ($n=0$). We also show that the row-consecutive $p \times p$-minors of a generic matrix form a sagbi basis and we give a quadratic Gröbner basis for their algebraic relations.

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1. STATEMENT OF THE MAIN RESULT

Let $\mathcal{M}(t)$ be the $p \times (m+p)$-matrix whose i,jth entry is the degree n polynomial in t,

$$x_{i,j}^{(n)} \cdot t^n \; + x_{i,j}^{(n-1)} \cdot t^{n-1} + \; \ldots \; + x_{i,j}^{(2)} \cdot t^2 \; + x_{i,j}^{(1)} \cdot t \; + x_{i,j}^{(0)}.$$

The coefficients $x_{i,j}^{(l)}$ are indeterminates. We write $k[X]$ for the polynomial ring over a field k generated by these indeterminates, for $i = 1, \ldots, p$, $j = 1, \ldots, m+p$, and $l = 0, \ldots, n$. Lexicographic order on the triples l, i, j gives a total order of these variables. For example,

$$x_{1,2}^{(0)} \; < \; x_{1,2}^{(1)} \; < \; x_{1,5}^{(1)} \; < \; x_{2,3}^{(1)} \; < \; x_{2,4}^{(1)} \; < \; x_{1,3}^{(2)} \, .$$

Let $\prec$ be the resulting degree reverse lexicographic term order on the polynomial ring $k[X]$.

For each $\alpha \in \binom{[m+p]}{p}$ and $a \geq 0$, let $\alpha^{(a)}$ be a variable which, when $a \leq np$, formally represents the coefficient of t^a in the maximal minor of $\mathcal{M}(t)$ given by the columns indexed by $\alpha_1, \alpha_2, \ldots, \alpha_p$. These variables $\alpha^{(a)}$ have a natural partial order, denoted $\mathcal{C}_{p,m}$, which is defined as follows:

$$\alpha^{(a)} \; \leq \; \beta^{(b)} \quad \Longleftrightarrow \quad a \leq b \;\text{ and }\; \alpha_i \leq \beta_{b-a+i} \;\text{ for } i = 1, 2, \ldots, p-b+a.$$

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Fix $0 \le q \le np$ and let $\mathcal{C}^q_{p,m}$ denote the truncation of the infinite poset $\mathcal{C}_{p,m}$ to the finite subset $\left\{ \alpha^{(a)} \mid \alpha \in \binom{[m+p]}{p} \text{ and } a \le q \right\}$. The posets $\mathcal{C}^q_{p,m}$ are graded distributive lattices in that all saturated chains have the same length and any two elements $\alpha^{(a)}$ and $\beta^{(b)}$ have a meet $\alpha^{(a)} \wedge \beta^{(b)}$ (greatest lower bound) and join $\alpha^{(a)} \vee \beta^{(b)}$ (least upper bound). Figure 1 shows $\mathcal{C}^1_{2,3}$.

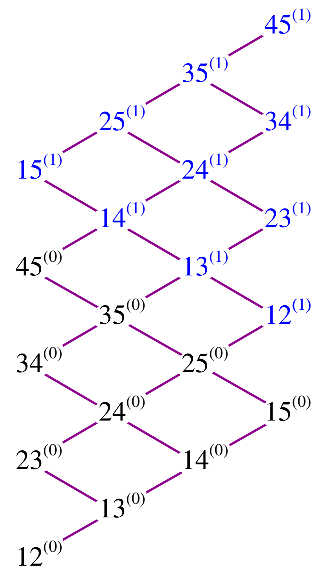

FIGURE 1. The distributive lattice $\mathcal{C}^1_{2,3}$.

Let $\varphi : k[\mathcal{C}^q_{p,m}] \to k[X]$ denote the k-algebra homomorphism which sends the formal variable $\alpha^{(a)}$ to the coefficient of t^a in the αth maximal minor of the matrix $\mathcal{M}(t)$. For example when $n = 1$,

$$\varphi\big(456^{(2)}\big) \quad = \quad \text{coefficient of } t^2 \text{ in } \det \begin{bmatrix} x^{(0)}_{1,4} + x^{(1)}_{1,4}\cdot t & x^{(0)}_{1,5} + x^{(1)}_{1,5}\cdot t & x^{(0)}_{1,6} + x^{(1)}_{1,6}\cdot t \\ x^{(0)}_{2,4} + x^{(1)}_{2,4}\cdot t & x^{(0)}_{2,5} + x^{(1)}_{2,5}\cdot t & x^{(0)}_{2,6} + x^{(1)}_{2,6}\cdot t \\ x^{(0)}_{3,4} + x^{(1)}_{3,4}\cdot t & x^{(0)}_{3,5} + x^{(1)}_{3,5}\cdot t & x^{(0)}_{3,6} + x^{(1)}_{3,6}\cdot t \end{bmatrix}$$

$$\begin{aligned} = \quad & -\underline{x^{(0)}_{3,6}x^{(1)}_{1,5}x^{(1)}_{2,4}} + x^{(0)}_{3,5}x^{(1)}_{1,6}x^{(1)}_{2,4} + x^{(0)}_{3,6}x^{(1)}_{1,4}x^{(1)}_{2,5} - x^{(0)}_{3,4}x^{(1)}_{1,6}x^{(1)}_{2,5} - x^{(0)}_{3,5}x^{(1)}_{1,4}x^{(1)}_{2,6} + x^{(0)}_{3,4}x^{(1)}_{1,5}x^{(1)}_{2,6} \\ & + x^{(0)}_{2,6}x^{(1)}_{1,5}x^{(1)}_{3,4} - x^{(0)}_{2,5}x^{(1)}_{1,6}x^{(1)}_{3,4} - x^{(0)}_{2,6}x^{(1)}_{1,4}x^{(1)}_{3,5} + x^{(0)}_{2,4}x^{(1)}_{1,6}x^{(1)}_{3,5} + x^{(0)}_{2,5}x^{(1)}_{1,4}x^{(1)}_{3,6} - x^{(0)}_{2,4}x^{(1)}_{1,5}x^{(1)}_{3,6} \\ & - x^{(0)}_{1,6}x^{(1)}_{2,5}x^{(1)}_{3,4} + x^{(0)}_{1,5}x^{(1)}_{2,6}x^{(1)}_{3,4} + x^{(0)}_{1,6}x^{(1)}_{2,4}x^{(1)}_{3,5} - x^{(0)}_{1,4}x^{(1)}_{2,6}x^{(1)}_{3,5} - x^{(0)}_{1,5}x^{(1)}_{2,4}x^{(1)}_{3,6} + x^{(0)}_{1,4}x^{(1)}_{2,5}x^{(1)}_{3,6}\,. \end{aligned}$$

Our notation intentionally disregards the dependence of φ on n, as the main results are independent of n, as long as $q \le np$. This is because in that case, the quantum Grassmannian $K^q_{p,m}$ is a quantum Schubert subvariety of $K^{np}_{p,m}$, as we show in Section 3.

Theorem 1. *The set of polynomials $\varphi(\alpha^{(a)})$ as $\alpha^{(a)}$ runs over the poset $\mathcal{C}^q_{p,m}$ forms a sagbi basis with respect to the reverse lexicographic term order $\prec$ on $k[X]$ defined above.*

Our second theorem states that the subalgebra *image*(φ) of $k[X]$ generated by this sagbi basis is an *algebra with straightening law* on the poset $\mathcal{C}^q_{p,m}$. Let $\prec$ be the degree reverse lexicographic term order on $k[\mathcal{C}^q_{p,m}]$ induced by any linear extension of the poset $\mathcal{C}^q_{p,m}$. This term order on $k[\mathcal{C}^q_{p,m}]$ and the previous term order on $k[X]$ are fixed throughout this paper.

Theorem 2. *The reduced Gröbner basis of the kernel of φ consists of quadratic polynomials in $k[\mathcal{C}^q_{p,m}]$ which are indexed by pairs of incomparable variables $\gamma^{(c)}, \delta^{(d)}$ in the poset $\mathcal{C}^q_{p,m}$,*

$$S(\gamma^{(c)}, \delta^{(d)}) \quad = \quad \gamma^{(c)} \cdot \delta^{(d)} \; - \; (\gamma^{(c)} \vee \delta^{(d)}) \cdot (\gamma^{(c)} \wedge \delta^{(d)}) \; + \; \text{lower terms in } \prec,$$

and all lower terms $\lambda \beta^{(b)} \alpha^{(a)}$ in $S(\gamma^{(c)}, \delta^{(d)})$ satisfy $\beta^{(b)} < \gamma^{(c)} \wedge \delta^{(d)}$ and $\gamma^{(c)} \vee \delta^{(d)} < \alpha^{(a)}$.

The join $\vee$ and meet $\wedge$ appearing in the above formula are the lattice operations in $\mathcal{C}^q_{p,m}$. The combinatorial structure of this distributive lattice will become clear in Section 2, when we introduce the toric variety and Hibi ring associated with $\mathcal{C}^q_{p,m}$. In Section 3 we interpret the subalgebra image(φ) of $k[X]$ as the coordinate ring of the quantum Grassmannian. Section 4 contains the proofs of Theorems 1 and 2. These results generalize the classical sagbi basis property of maximal minors [20, Theorem 3.2.9] and its geometric interpretation as a toric deformation [21, Proposition 11.10] from the case of the Grassmannian to the quantum Grassmannian. In Section 5 we discuss corollaries, applications and some open problems. One such application is that the row-consecutive $p \times p$-minors of any matrix of indeterminates form a sagbi basis.

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2. The toric variety of the distributive lattice

Theorem 1 asserts that the initial algebra of our subalgebra image(φ) is generated by the initial monomials of its generators $\varphi(\alpha^{(a)})$. Our first step is to identify the initial monomials. Here are two examples. The first one is the underlined monomial right before Theorem 1:

$$\mathrm{in}_\prec\big(\varphi(456^{(2)})\big) \;=\; x^{(0)}_{3,6} x^{(1)}_{1,5} x^{(1)}_{2,4} \qquad \text{and} \qquad \mathrm{in}_\prec\big(\varphi(2457^{(5)})\big) \;=\; x^{(1)}_{2,7} x^{(1)}_{3,5} x^{(1)}_{4,4} x^{(2)}_{1,2}.$$

In general, the initial monomial of $\varphi(\alpha^{(a)})$ is given by the following lemma:

Lemma 3. *Let $\alpha \in \binom{[m+p]}{p}$ and $a = pl + r$ with integers $p > r \geq 0$. Then*

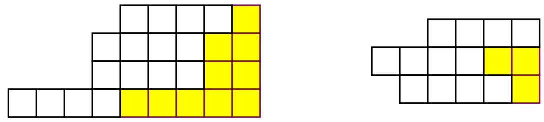
$$\mathrm{in}_\prec\big(\varphi(\alpha^{(a)})\big) \quad = \quad x^{(l)}_{r+1,\alpha_p} x^{(l)}_{r+2,\alpha_{p-1}} \cdots x^{(l)}_{p,\alpha_{r+1}} x^{(l+1)}_{1,\alpha_r} x^{(l+1)}_{2,\alpha_{r-1}} \cdots x^{(l+1)}_{r,\alpha_1}.$$

Proof. Let $x^{(l_1)}_{i_1,j_1} x^{(l_2)}_{i_2,j_2} \cdots x^{(l_p)}_{i_p,j_p}$ be a monomial which appears in $\varphi(\alpha^{(a)})$. We claim that

$$(1) \qquad x^{(l_1)}_{i_1,j_1} x^{(l_2)}_{i_2,j_2} \cdots x^{(l_p)}_{i_p,j_p} \quad \preceq \quad x^{(l)}_{r+1,\alpha_p} x^{(l)}_{r+2,\alpha_{p-1}} \cdots x^{(l)}_{p,\alpha_{r+1}} x^{(l+1)}_{1,\alpha_r} \cdots x^{(l+1)}_{r,\alpha_1}.$$

We may assume $x^{(l_1)}_{i_1,j_1} \prec x^{(l_2)}_{i_2,j_2} \prec \cdots \prec x^{(l_p)}_{i_p,j_p}$ and hence $l_1 \leq \cdots \leq l_p$. Since $l_1 + \cdots + l_p = a$, either $l_1 < l$, from which (1) follows, or else $l_1 = \cdots = l_{p-r} = l$ and $l_{p+1-r} = \cdots = l_p = l+1$. In the second case, as $\{i_1, \ldots, i_p\} = \{1, \ldots, p\}$ and the monomial is in order, we must have $i_1 < \cdots < i_{p-r}$ and $i_{p+1-r} < \cdots < i_p$. If $i_1 \leq r$, then (1) follows, and if $i_1 = r+1$, then the ordered sequence $i_1, i_2, \ldots, i_p$ equals $r+1, r+2, \ldots, p, 1, \ldots, r$. Among all monomials satisfying this new second case, the largest in the degree reverse lexicographic order $\prec$ has the second lower index appearing in reverse order. This completes the proof. ∎

We next introduce some combinatorics to help understand the poset $\mathcal{C}_{p,m}$. A *row* with shift a consists of p consecutive empty unit boxes shifted a units to the left of a given vertical line. A *skew shape* is an array of such rows whose shifts are weakly increasing read from top to bottom. For example, the unshaded boxes in the figure on the left form a skew shape (with shifts 1,2,2, and 5) while those in the other figure do not:



A *(skew) tableau* T is a filling of a skew shape with integers that increase across each row. When the entries lie in $[m+p]$, the ith row of a tableau is a sequence $\alpha_{(i)} \in \binom{[m+p]}{p}$. If a_i is the shift of the ith row and T has j rows, then T corresponds to a monomial $\alpha_{(1)}^{(a_1)}\alpha_{(2)}^{(a_2)}\cdots\alpha_{(j)}^{(a_j)}$. Conversely, any monomial in the variables $\alpha^{(a)}$ corresponds to a tableau. A tableau T is *standard* if the entries are weakly increasing in each column, read top to bottom. Equivalently, T is standard if we have $\alpha_{(1)}^{(a_1)} \leq \alpha_{(2)}^{(a_2)} \leq \cdots \leq \alpha_{(j)}^{(a_j)}$ in $\mathcal{C}_{p,m}$. For example, the following two tableaux correspond to the monomials $345^{(0)}123^{(1)}245^{(3)}$ and $135^{(0)}123^{(1)}257^{(3)}$. The first tableau is not standard and the second tableau is standard.

The elements of the poset $\mathcal{C}_{p,m}^q$ are represented by one-row tableaux with entries in $[m+p]$ and shift at most q. Two elements satisfy $\alpha^{(a)} \leq \beta^{(b)}$ if and only if the two-rowed tableau $T = \alpha^{(a)}\beta^{(b)}$ is standard. This representation implies that $\mathcal{C}_{p,m}^q$ is a distributive lattice. Indeed, the two lattice operations $\wedge$ and $\vee$ are described as follows. If a two-rowed tableau $T = \alpha^{(a)}\beta^{(b)}$ is non-standard then interchanging the entries in every column in which a violation ($\alpha_{a-b+i} < \beta_i$) occurs yields a standard tableau. The first row of this new tableau is the *meet* $\alpha^{(a)} \wedge \beta^{(b)}$ of $\alpha^{(a)}$ and $\beta^{(b)}$ in $\mathcal{C}_{p,m}^q$ and the second row is their *join* $\alpha^{(a)} \vee \beta^{(b)}$.

Let $\psi : k[\mathcal{C}_{p,m}^q] \to k[X]$ denote the k-algebra homomorphism which sends the variable $\alpha^{(a)}$ to the monomial $\text{in}_{\prec}\big(\varphi(\alpha^{(a)})\big)$. Its kernel is a *toric ideal* (i.e. binomial prime) in $k[\mathcal{C}_{p,m}^q]$.

Proposition 4. *The reduced Gröbner basis for the kernel of* ψ *consists of the binomials*

$$\underline{\alpha^{(a)} \cdot \beta^{(b)}} \;-\; (\alpha^{(a)} \vee \beta^{(b)}) \cdot (\alpha^{(a)} \wedge \beta^{(b)}),$$

where $\alpha^{(a)}, \beta^{(b)}$ *runs over all incomparable pairs of* $\mathcal{C}_{p,m}^q$. *The initial monomial is underlined.*

Proof. This follows from Hibi's Theorem [9] since $\mathcal{C}_{p,m}^q$ is a distributive lattice. ∎

Thus $\text{image}(\psi) \simeq k[\mathcal{C}_{p,m}^q]/\text{kernel}(\psi)$ is the Hibi ring of the distributive lattice $\mathcal{C}_{p,m}^q$.

Corollary 5. *The set of standard tableaux is a* k*-basis for* $\text{image}(\psi)$.

Here is a typical element in the reduced Gröbner basis of $\text{kernel}(\psi)$ for $p = 5, m = 4, q = 9$:

$$\underline{45789^{(1)} \cdot 12356^{(3)}} \;-\; 35689^{(1)} \cdot 12457^{(3)}.$$

Note that the second monomial corresponds to a standard tableau while the first does not.

We write $T_{p,m}^q$ for the projective toric variety cut out by the binomials in Proposition 4. Its coordinate ring is the Hibi ring $\text{image}(\psi)$. The geometry of toric varieties associated with distributive lattices is discussed in [23]. The analogue to Corollary 5 always holds, i.e., multichains in the poset correspond to basis monomials in the Hibi ring.

Corollary 6. *The degree of the toric variety* $T_{p,m}^q$ *is the number of maximal chains in* $\mathcal{C}_{p,m}^q$.

The closed intervals of the poset $\mathcal{C}^q_{p,m}$ are also distributive lattices. They are denoted

$$[\beta^{(b)},\alpha^{(a)}] \quad := \quad \{\gamma^{(c)}\in\mathcal{C}^q_{p,m} \,:\, \beta^{(b)}\leq\gamma^{(c)}\leq\alpha^{(a)}\}.$$

Proposition 4 and Corollaries 5 and 6 hold essentially verbatim for the distributive sublattice $[\beta^{(b)},\alpha^{(a)}]$ as well. The projective toric variety associated with $[\beta^{(b)},\alpha^{(a)}]$ is gotten from the toric variety of $\mathcal{C}^q_{p,m}$ by setting $\gamma^{(c)}=0$ for all $\gamma^{(c)}\notin[\beta^{(b)},\alpha^{(a)}]$. The degree of that variety is the number of saturated chains in $\mathcal{C}^q_{p,m}$ which start at $\beta^{(b)}$ and end at $\alpha^{(a)}$.

We close this section with an alternative proof, to be used in Section 4, for the fact that $\mathcal{C}_{p,m}$ is a distributive lattice. We claim that $\mathcal{C}_{p,m}$ is a sublattice of *Young's lattice.* Given $\alpha^{(a)}\in\mathcal{C}_{p,m}$, write $a=pl+r$ with integers $p>r\geq 0$, and define a sequence $J(\alpha^{(a)})$ by

$$(2)\qquad J(\alpha^{(a)})_i \quad := \quad \begin{cases} l(m+p)+\alpha_{r+i} & \text{if } 1\leq i\leq p-r \\ (l+1)(m+p)+\alpha_{i-p+r} & \text{if } p-r<i\leq p \end{cases}.$$

This gives an order-preserving bijection between the poset $\mathcal{C}_{p,m}$ and the poset of sequences $J:=j_1<j_2<\cdots<j_p$ of positive integers with $j_p-(m+p)<j_1$, and it preserves meet and join. This bijection preserves the rank function in the two distributive lattices:

$$(3)\qquad |\alpha^{(a)}| \quad := \quad a(m+p)+\sum_{j=1}^{p}(\alpha_j-j) \quad = \quad \sum_{i=1}^{p}\big(J(\alpha^{(a)})_i-i\big) \quad =: \quad |J(\alpha^{(a)})|.$$

3. The quantum Grassmannian

Let $Grass_p k^{m+p}$ denote the Grassmannian of p-planes in the vector space k^{m+p}. This is a smooth projective variety of dimension mp. Consider the space $S^q_{p,m}$ of maps $\mathbb{P}^1\to Grass_p k^{m+p}$ of degree q. Such a map may be (non-uniquely) represented as the row space of a $p\times(m+p)$-matrix of polynomials in t whose maximal minors have degree q. Results in [4] imply that it suffices to consider the matrices $\mathcal{M}(t)$ in the introduction. The coefficients of these maximal minors define the *Plücker embedding* of $S^q_{p,m}$ into $\mathbb{P}(\wedge^p k^{m+p}\otimes k^{q+1})$; see [19, 17]. The *quantum Grassmannian* $K^q_{p,m}$ is the Zariski closure of $S^q_{p,m}$ in this Plücker embedding. It is an irreducible projective variety of dimension $mp+q(m+p)$. Its prime ideal is $\text{kernel}(\varphi)\subset k[\mathcal{C}^q_{p,m}]$ and its coordinate ring is our subalgebra $\text{image}(\varphi)\subset k[X]$.

The quantum Grassmannian $K^q_{p,m}$ is singular and it differs from other spaces used to study rational curves in Grassmann varieties (the quot scheme [19], the Kontsevich space of stable maps [13], or the set of autoregressive systems [15]). Nevertheless, $K^q_{p,m}$ has been crucial in two important advances: in computing the intersection number $\text{degree}(K^q_{p,m})$ in quantum cohomology [16], and in showing that this intersection problem can be fully solved over the real numbers [18]. Our result will give a new derivation of this intersection number.

Corollary 7. [16] *The degree of $K^q_{p,m}$ is the number of maximal chains in $\mathcal{C}^q_{p,m}$.*

Proof. This follows immediately from Theorem 2 and Corollary 6. ∎

This degree can also be computed in the small quantum cohomology ring of the Grassmannian [2]. Note that $\text{degree}(K^1_{2,3})=55$, by counting maximal chains in Figure 1. Ravi, Rosenthal, and Wang [16] were motivated by a problem in applied mathematics. The degree of $K^q_{p,m}$ is the number of dynamic feedback compensators that stabilize a certain linear system, in the sense of systems theory. This number can be described in classical projective geometry as follows. The Schubert subvariety of $Grass_p k^{m+p}$ consisting of p-planes meeting a fixed m-plane L is a hyperplane section in the Plücker embedding of $Grass_p k^{m+p}$. Thus the set of maps $M \in S^q_{p,m}$ such that $M(t)$ meets L non-trivially is a hyperplane section of $S^q_{p,m}$ in its Plücker embedding. Since $GL_{m+p}(k)$ acts transitively on $Grass_p k^{m+p}$, Kleiman's Theorem on generic transversality [12] implies the following statement when k is algebraically closed of characteristic zero. Set $N := mp + q(m+p)$ and suppose $t_1, \ldots, t_N \in \mathbb{P}^1$ are general points and $L_1, \ldots, L_N$ are general m-planes in k^{m+p}, then the degree of $K^q_{p,m}$ counts those maps M for which $M(t_i)$ meets L_i non-trivially for each $i = 1, \ldots, N$. As to computing the desired maps M numerically, we note that the sagbi basis in Theorem 1 and the Gröbner basis in Theorem 2 each lead to an *optimal homotopy algorithm* for finding these degree$(K^q_{p,m})$ maps. These algorithms generalize the ones in [11].

Remark 8. *The sagbi basis in Theorem 1 defines a flat deformation from the quantum Grassmannian $K^q_{p,m}$ to the projective toric variety $T^q_{p,m}$ associated with the poset $\mathcal{C}^q_{p,m}$.*

See [5] for a precise algebraic discussion of such deformations, and see [21, Equation (11.9)] for the simplest example relevant to us, namely, $K^0_{2,3} = Grass_2 k^5$. The flat deformation is given algebraically by deleting all but the first two terms in the Gröbner basis elements $S(\gamma^{(c)}, \delta^{(d)})$ given in Theorem 2. Consider the deformation from $K^3_{3,3}$ to $T^3_{3,3}$. The incomparable pair $156^{(1)}$ and $234^{(2)}$ in $\mathcal{C}^3_{3,3}$ indexes the quadratic polynomial $S(156^{(1)}, 234^{(2)}) =$

$$\begin{array}{c} \underline{156^{(1)}234^{(2)} - 146^{(1)}235^{(2)}} + 145^{(1)}236^{(2)} + 136^{(1)}245^{(2)} - 135^{(1)}246^{(2)} \\ +134^{(1)}256^{(2)} - 126^{(1)}345^{(2)} + 125^{(1)}346^{(2)} - 124^{(1)}356^{(2)} + 123^{(1)}456^{(2)} \\ -456^{(0)}123^{(3)} + 356^{(0)}124^{(3)} - 346^{(0)}125^{(3)} + 345^{(0)}126^{(3)} - 256^{(0)}134^{(3)} \\ +246^{(0)}135^{(3)} - 245^{(0)}136^{(3)} - 236^{(0)}145^{(3)} + 235^{(0)}146^{(3)} - 234^{(0)}156^{(3)} \\ +2{\cdot}156^{(0)}234^{(3)} - 2{\cdot}146^{(0)}235^{(3)} + 2{\cdot}145^{(0)}236^{(3)} + 2{\cdot}136^{(0)}245^{(3)} - 2{\cdot}135^{(0)}246^{(3)} \\ +2{\cdot}134^{(0)}256^{(3)} - 2{\cdot}126^{(0)}345^{(3)} + 2{\cdot}125^{(0)}346^{(3)} - 2{\cdot}124^{(0)}356^{(3)} + 2{\cdot}123^{(0)}456^{(3)}, \end{array} \tag{4}$$

which vanishes on the quantum Grassmannian $K^3_{3,3}$. The underlined leading binomial vanishes on the toric variety $T^3_{3,3}$, by Proposition 4. Our main technical problem, to be solved in the next section, is the reconstruction of quadrics such as (4) from their leading binomial.

A key tool in proving Theorems 1 and 2 is the Schubert decomposition of the quantum Grassmannian $K^q_{p,m}$ indexed by $\mathcal{C}^q_{p,m}$. For $\alpha^{(a)} \in \mathcal{C}^q_{p,m}$, the *quantum Schubert variety* is

$$Z_{\alpha^{(a)}} \quad := \quad \{ (\gamma^{(c)}) \in K^q_{p,m} \mid \gamma^{(c)} = 0 \text{ if } \gamma^{(c)} \not\leq \alpha^{(a)} \} .$$

More generally, for $\beta^{(b)} \leq \alpha^{(a)}$ in $\mathcal{C}^q_{p,m}$, we define the *skew quantum Schubert variety*

$$Z_{\alpha^{(a)}/\beta^{(b)}} \quad := \quad \{ (\gamma^{(c)}) \in K^q_{p,m} \mid \gamma^{(c)} = 0 \text{ if } \gamma^{(c)} \notin [\beta^{(b)}, \alpha^{(a)}] \} .$$

Among the quantum Schubert varieties of $K^q_{p,m}$ are the $K^d_{p,m}$ for $d < q$; namely, if $\delta^{(d)}$ is the supremum of $\mathcal{C}^d_{p,m}$, then $K^d_{p,m} = Z_{\delta^{(d)}}$. This allows us to deduce assertions about the general quantum Grassmannian $K^q_{p,m}$ from results about quantum Schubert varieties of $K^{np}_{p,m}$.

The quantum Schubert varieties and skew quantum Schubert varieties have rational parameterizations which are constructed as follows. Let $\alpha^{(a)} \in \mathcal{C}^{np}_{p,m}$ and write $a = ps + r$ with integers $p > r \geq 0$. We define the matrix $\mathcal{M}_{\alpha^{(a)}}(t)$ to be the specialization of $\mathcal{M}(t)$ where

$$x^{(l)}_{i,j} \;=\; 0 \quad \text{if} \quad \left\{ \begin{array}{lll} (l > s+1 \;\text{ and }\; i \leq r) & \text{or} & (l = s+1 \text{ and } j > \alpha_{r+1-i}) \;\text{ or} \\ (l > s \;\text{ and }\; i > r) & \text{or} & (l = s \text{ and } j > \alpha_{p+r+1-i}) \,. \end{array} \right.$$

Here we use the conventions $\alpha_\nu = 0$ if $\nu \le 0$ and $\alpha_\nu = +\infty$ if $\nu > p$. For example,

$$\mathcal{M}_{235^{(2)}}(t) \quad = \quad \begin{bmatrix} x^{(0)}_{1,1} + x^{(1)}_{1,1}\cdot t & x^{(0)}_{1,2} + x^{(1)}_{1,2}\cdot t & x^{(0)}_{1,3} + x^{(1)}_{1,3}\cdot t & x^{(0)}_{1,4} & x^{(0)}_{1,5} & x^{(0)}_{1,6} \\ x^{(0)}_{2,1} + x^{(1)}_{2,1}\cdot t & x^{(0)}_{2,2} + x^{(1)}_{2,2}\cdot t & x^{(0)}_{2,3} & x^{(0)}_{2,4} & x^{(0)}_{2,5} & x^{(0)}_{2,6} \\ x^{(0)}_{3,1} & x^{(0)}_{3,2} & x^{(0)}_{3,3} & x^{(0)}_{3,4} & x^{(0)}_{3,5} & 0 \end{bmatrix}.$$

If we specialize the variables $x^{(l)}_{i,j}$ in $\mathcal{M}_{\alpha^{(a)}}(t)$ to field elements in k in such a way that the resulting matrix over $k(t)$ has maximal row rank, then that matrix defines a map from k to $Grass_p k^{m+p}$. If we extend this to $\mathbb{P}^1$, we obtain a map in $Z_{\alpha^{(a)}}$. Proposition 9 below implies that such maps constitute a dense subset of $Z_{\alpha^{(a)}}$. This means that the coefficients with respect to t of the maximal minors of $\mathcal{M}_{\alpha^{(a)}}(t)$ give a rational parameterization of $Z_{\alpha^{(a)}}$.

This construction extends to skew quantum Schubert varieties as follows. Given $\beta^{(b)} \le \alpha^{(a)}$, write $b = ps + r$ with integers $p > r \ge 0$ and define the matrix $\mathcal{M}_{\alpha^{(a)}/\beta^{(b)}}(t)$ to be the specialization of $\mathcal{M}_{\alpha^{(a)}}(t)$ where

$$x^{(l)}_{i,j} \;=\; 0 \quad \text{if} \quad \begin{cases} (l < s+1 \;\text{ and }\; i \le r) & \text{or} \quad (l = s+1 \text{ and } j < \beta_{r+1-i}) \quad \text{or} \\ (l < s \;\text{ and }\; i > r) & \text{or} \quad (l = s \text{ and } j < \beta_{p+r+1-i}) \end{cases}.$$

The matrix $\mathcal{M}_{\alpha^{(a)}/\beta^{(b)}}(t)$ gives a rational map into $Z_{\alpha^{(a)}/\beta^{(b)}}$, which is described algebraically as follows. We define $\varphi_{\alpha^{(a)}}$ and $\varphi_{\alpha^{(a)}/\beta^{(b)}}$ to be the composition of the map $\varphi : k[\mathcal{C}^{np}_{p,m}] \to k[X]$ with the specializations to $\mathcal{M}_{\alpha^{(a)}}(t)$ and $\mathcal{M}_{\alpha^{(a)}/\beta^{(b)}}(t)$ respectively. We claim that these matrices parameterize dense subsets of the (skew) quantum Schubert varieties.

Proposition 9. *The kernel of $\varphi_{\alpha^{(a)}}$ is the homogeneous ideal of the quantum Schubert variety $Z_{\alpha^{(a)}}$. Likewise, the kernel of $\varphi_{\alpha^{(a)}/\beta^{(b)}}$ is the homogeneous ideal of the skew quantum Schubert variety $Z_{\alpha^{(a)}/\beta^{(b)}}$. In particular, the varieties $Z_{\alpha^{(a)}}$ and $Z_{\alpha^{(a)}/\beta^{(b)}}$ are irreducible.*

We postpone the proof of this proposition until the next section. Here is an example which illustrates the parameterization of skew quantum Schubert varieties for $p = m = 3$:

$$\mathcal{M}_{235^{(2)}/146^{(1)}}(t) \quad = \quad \begin{bmatrix} x^{(1)}_{1,1}\cdot t & x^{(1)}_{1,2}\cdot t & x^{(1)}_{1,3}\cdot t & 0 & 0 & 0 \\ x^{(1)}_{2,1}\cdot t & x^{(1)}_{2,2}\cdot t & 0 & 0 & 0 & x^{(0)}_{2,6} \\ 0 & 0 & 0 & x^{(0)}_{3,4} & x^{(0)}_{3,5} & 0 \end{bmatrix}.$$

We evaluate the 3×3-minors of this matrix to find the k-algebra homomorphism $\varphi_{235^{(2)}/146^{(1)}}$. It takes polynomials in 12 variables $\gamma^{(c)}$ to polynomials in 8 variables $x^{(l)}_{i,j}$ as follows:

$$\begin{gathered} 146^{(1)} \mapsto -x^{(1)}_{1,1}x^{(0)}_{2,6}x^{(0)}_{3,4}\,, \quad 156^{(1)} \mapsto -x^{(1)}_{1,1}x^{(0)}_{2,6}x^{(0)}_{3,5}\,, \quad 246^{(1)} \mapsto -x^{(1)}_{1,2}x^{(0)}_{2,6}x^{(0)}_{3,4}\,, \\ 256^{(1)} \mapsto -x^{(1)}_{1,2}x^{(0)}_{2,6}x^{(0)}_{3,5}\,, \quad 346^{(1)} \mapsto -x^{(1)}_{1,3}x^{(0)}_{2,6}x^{(0)}_{3,4}\,, \quad 356^{(1)} \mapsto -x^{(1)}_{1,3}x^{(0)}_{2,6}x^{(0)}_{3,5}\,, \\ 124^{(2)} \mapsto x^{(1)}_{1,1}x^{(1)}_{2,2}x^{(0)}_{3,4} - x^{(1)}_{2,1}x^{(1)}_{1,2}x^{(0)}_{3,4}\,, \quad 125^{(2)} \mapsto x^{(1)}_{1,1}x^{(1)}_{2,2}x^{(0)}_{3,5} - x^{(1)}_{2,1}x^{(1)}_{1,2}x^{(0)}_{3,5}\,, \\ 134^{(2)} \mapsto -x^{(1)}_{2,1}x^{(1)}_{1,3}x^{(0)}_{3,4}\,, \qquad 135^{(2)} \mapsto -x^{(1)}_{2,1}x^{(1)}_{1,3}x^{(0)}_{3,5}\,, \\ 234^{(2)} \mapsto -x^{(1)}_{2,2}x^{(1)}_{1,3}x^{(0)}_{3,4}\,, \qquad 235^{(2)} \mapsto -x^{(1)}_{2,2}x^{(1)}_{1,3}x^{(0)}_{3,5}. \end{gathered}$$

The 12 variables $\gamma^{(c)}$ appearing on the left sides above are precisely the elements in the interval $[146^{(1)}, 235^{(2)}]$ of the distributive lattice $\mathcal{C}_{3,3}$. There are 18 incomparable pairs in this interval, each giving a quadratic generator for the kernel of $\varphi_{235^{(2)}/146^{(1)}}$. This set of 18 quadrics consists of 14 binomials and four trinomials, and it equals the reduced Gröbner basis with respect to $\prec$. For example, one of the 14 binomials in this Gröbner basis is the underlined leading binomial of $S(156^{(1)}, 234^{(2)})$ in (4), and one of the four trinomials is

$$\underline{346^{(1)} \cdot 125^{(2)}} \; - \; 246^{(1)} \cdot 135^{(2)} \; + \; 146^{(1)} \cdot 235^{(2)}.$$

The underlined term is an incomparable pair in $\big[146^{(1)}, 235^{(2)}\big]$, while the other two monomials are comparable pairs. Erasing the third term gives a binomial as in Proposition 4.

4. Construction of Straightening Syzygies

The following theorem is the technical heart of this paper. All three of Theorem 1, Theorem 2, and Proposition 9 will be derived from Theorem 10 in the end of this section.

Theorem 10. *Let $\gamma^{(c)}, \delta^{(d)}$ be a pair of incomparable variables in the poset $\mathcal{C}^{np}_{p,m}$. There is a quadric $S(\gamma^{(c)}, \delta^{(d)})$ in the kernel of $\varphi : k[\mathcal{C}^{np}_{p,m}] \to k[X]$ whose first two monomials are*

$$\gamma^{(c)} \cdot \delta^{(d)} \; - \; (\gamma^{(c)} \vee \delta^{(d)}) \cdot (\gamma^{(c)} \wedge \delta^{(d)}).$$

Moreover, if $\lambda\beta^{(b)}\alpha^{(a)}$ is any non-initial monomial in $S(\gamma^{(c)}, \delta^{(d)})$, then $\gamma^{(c)}, \delta^{(d)} \in [\beta^{(b)}, \alpha^{(a)}]$.

The pair $\beta^{(b)}\alpha^{(a)}$ in the second assertion is necessarily standard, i.e. $\beta^{(b)} < \alpha^{(a)}$. The quadrics $S(\gamma^{(c)}, \delta^{(d)})$ are not constructed explicitly, but rather through an iterative procedure modeled on the *subduction algorithm* in image(φ). A main idea is to utilize the well-known subduction process [20, Algorithm 3.2.6] modulo the $p \times p$-minors of a generic $p \times N$-matrix.

Set $N := (n+1)(m+p)$. Let $\mathcal{N}$ be the $p \times N$-matrix whose i,jth entry is $x^{(l)}_{i,r}$, where $j = (m+p)l + r$ with $1 \le r \le m+p$. If $\mathcal{N}_l$ is the submatrix of $\mathcal{N}$ consisting of the entries $x^{(l)}_{i,j}$, then $\mathcal{N}$ is the concatenation of $\mathcal{N}_0, \mathcal{N}_1, \ldots, \mathcal{N}_n$ and $\mathcal{M}(t) = \mathcal{N}_0 + t\mathcal{N}_1 + \cdots + t^n\mathcal{N}_n$. Sequences $J : j_1 < \cdots < j_p \in \binom{[N]}{p}$ are regarded as variables. We write $\phi(J)$ for the Jth maximal minor of $\mathcal{N}$. *Young's poset* on sequences J is given by componentwise comparison and is graded via $|J| := \sum_i (j_i - i)$. The coefficient $\varphi(\alpha^{(a)})$ of t^a in the αth maximal minor of $\mathcal{M}(t)$ is an alternating sum of maximal minors of $\mathcal{N}$. The exact formula is

$$\varphi(\alpha^{(a)}) \quad = \sum_{\substack{|J| = |\alpha^{(a)}| \\ J \equiv \alpha \bmod (m+p)}} \epsilon_J \cdot \phi(J), \tag{5}$$

where ϵ_J is the sign of the permutation that orders the following sequence:

$$j_1 \bmod (m+p),\; j_2 \bmod (m+p),\; \ldots,\; j_p \bmod (m+p).$$

The polynomial rings $k[\mathcal{C}^{np}_{p,m}]$ and $k[\binom{[N]}{p}]$ are graded with $\deg \alpha^{(a)} = |\alpha^{(a)}|$ and $\deg J = |J|$. Consider the degree-preserving k-algebra homomorphism $\pi : k[\mathcal{C}^{np}_{p,m}] \to k[\binom{[N]}{p}]$ defined by

$$\pi(\alpha^{(a)}) \quad = \sum_{\substack{|J| = |\alpha^{(a)}| \\ J \equiv \alpha \bmod (m+p)}} \epsilon_J \cdot J\,. \tag{6}$$

Lexicographic order on the sequences $J \in \binom{[N]}{p}$ gives a linear extension of Young's poset. In this ordering, the initial term of (6) is the sequence $J(\alpha^{(a)})$ defined in (2). This sequence is characterized by $\mathrm{in}_\prec\, \varphi(\alpha^{(a)}) = \mathrm{in}_\prec\, \phi(J(\alpha^{(a)}))$. It can be checked that all other terms $\epsilon_J \cdot J$ appearing in (6) satisfy $J_1 < J(\alpha^{(a)})_1$ and $J_p - J_1 > m+p$. For example, for $m = 4$,

$$\pi(235^{(2)}) \quad = \quad \underline{(5,9,10)} - (3,9,12) + (3,5,16) + (2,10,12) - (2,5,17) + (2,3,19),$$

$$\text{and} \qquad \mathrm{in}_{\prec}\big(\varphi(\,235^{(2)}\,)\big) \;=\; \mathrm{in}_{\prec}\big(\phi(5,9,10)\big) \;=\; x^{(0)}_{3,5}x^{(1)}_{1,3}x^{(1)}_{2,2}.$$

For $J \in \binom{[N]}{p}$, let $\mathcal{N}_J$ be the specialization of $\mathcal{N}$ where in each row i, all entries in columns greater than j_i are set to zero. Under the identification of $\mathcal{N}$ with $\mathcal{M}(t)$, we have $\mathcal{N}_{J(\alpha^{(a)})} = \mathcal{M}_{\alpha^{(a)}}$. Let $\phi_J : k[\binom{[N]}{p}] \to k[X]$ denote the k-algebra homomorphism which maps the formal variable I to the Ith maximal minor of $\mathcal{N}_J$. Then $\phi_J(I)$ vanishes unless $I \leq J$. In particular, if $|I| = |J|$, then $\phi_J(I)$ vanishes unless $I = J$, and in that case, it is just the product of the last non-zero variables in each row of $\mathcal{N}_J$. From this it follows that

$$\begin{aligned} \varphi_{\alpha^{(a)}} &= \phi_{J(\alpha^{(a)})} \circ \pi \\ \varphi_{\alpha^{(a)}}(\alpha^{(a)}) &= \phi_{J(\alpha^{(a)})}(J(\alpha^{(a)})) \;=\; \mathrm{in}_{\prec}\, \varphi(\alpha^{(a)}) \;=\; \psi(\alpha^{(a)}). \end{aligned} \tag{7}$$

In the Plücker embedding of $Grass_p k^N$ into $\mathbb{P}(\wedge^p k^N)$, the Schubert variety indexed by J is

$$\Omega_J \;:=\; \{y = (y_I) \in Grass_p k^N \mid y_I = 0 \text{ if } I \not\leq J\}\ .$$

The homogeneous ideal $\mathcal{I}(\Omega_J)$ which defines this Schubert variety is precisely the kernel of ϕ_J. The following identity of ideals in $k[\binom{[N]}{p}]$ follows from the classical Plücker relations.

Proposition 11. *For any $J \in \binom{[N]}{p}$ we have*

$$\bigcap_{I<J} \mathcal{I}(\Omega_I) \;=\; \mathcal{I}(\Omega_J) + \langle\, J \,\rangle\ .$$

The map $\pi : k[\mathcal{C}^{np}_{p,m}] \to k[\binom{[N]}{p}]$ induces a birational isomorphism $\pi^* : Grass_p k^N \dashrightarrow K^{np}_{p,m}$. From the identification of $\mathcal{M}_{\alpha^{(a)}}(t)$ with $\mathcal{N}_{J(\alpha^{(a)})}$ and Proposition 9, we will see that $\pi^*(\Omega_{J(\alpha^{(a)})})$ is a dense subset of $Z_{\alpha^{(a)}}$. We also consider the image under π^* of the Schubert varieties Ω_J for $J < J(\alpha^{(a)})$.

Proposition 12 (Ravi-Rosenthal-Wang [16])**.** *If $J < J(\alpha^{(a)})$, then*

$$\pi^*(\Omega_J) \;\subset\; \bigcup_{\beta^{(b)}<\alpha^{(a)}} Z_{\beta^{(b)}}\ .$$

Proof. The inclusion $\Omega_J \subset \Omega_{J(\alpha^{(a)})}$ implies $\pi^*(\Omega_J) \subset Z_{\alpha^{(a)}}$. Since $\varphi_{\alpha^{(a)}}(\alpha^{(a)})$ is the product of leading entries in the rows of $\mathcal{N}_{J(\alpha^{(a)})}$, it follows that $\varphi_{\alpha^{(a)}}(\alpha^{(a)})$ vanishes under the specialization to $\mathcal{N}_J$, and hence $\pi(\alpha^{(a)})$ vanishes on Ω_J. This implies our claim because $\bigcup_{\beta^{(b)}<\alpha^{(a)}} Z_{\beta^{(b)}}$ is defined as a subvariety of $Z_{\beta^{(b)}}$ by the vanishing of $\alpha^{(a)}$. ∎

For $L < J$ in Young's poset, define $\mathcal{N}_{J/L}$ to be the specialization of $\mathcal{N}$ where in the ith row, only the entries in columns $l_i, l_i+1, \ldots, j_i$ are non-zero. Then $\mathcal{M}_{\alpha^{(a)}/\beta^{(b)}}(t)$ is the specialization of $\mathcal{M}(t)$ corresponding to $\mathcal{N}_{J(\alpha^{(a)})/J(\beta^{(b)})}(t)$. Define the k-algebra homomorphism $\phi_{J/I} : k[\binom{[N]}{p}] \to k[X]$ by evaluating the appropriate minors on $\mathcal{N}_{J/L}$. We observe that

$$\begin{aligned} \varphi_{\alpha^{(a)}/\beta^{(b)}}(\alpha^{(a)}) &= \mathrm{in}_{\prec}\, \varphi(\alpha^{(a)}) \;=\; \phi_{J(\alpha^{(a)})/J(\beta^{(b)})}\big(J(\alpha^{(a)})\big), \\ \varphi_{\alpha^{(a)}/\beta^{(b)}}(\beta^{(b)}) &= \mathrm{in}_{\prec}\, \varphi(\beta^{(b)}) \;=\; \phi_{J(\alpha^{(a)})/J(\beta^{(b)})}\big(J(\beta^{(b)})\big). \end{aligned} \tag{8}$$

The following lemma is very useful in our proof of Theorem 10.

Lemma 13. *Fix $\alpha^{(a)} \in \mathcal{C}^{np}_{p,m}$ and let $f \in k[\mathcal{C}^{np}_{p,m}]$ be a quadratic form of degree d.*

(1) *Suppose that* $\varphi_{\beta^{(b)}}(f)=0$ *for all* $\beta^{(b)}<\alpha^{(a)}$. *Then there exist constants* $\lambda_J\in k$ *with*

$$\varphi_{\alpha^{(a)}}(f) \quad = \quad \varphi_{\alpha^{(a)}}(\alpha^{(a)}) \cdot \sum_{\substack{J\in\binom{[N]}{p}\\ |J|+|\alpha^{(a)}|=d}} \lambda_J\cdot\phi_{J(\alpha^{(a)})}(J)\,.$$

(2) *Suppose* $\beta^{(b)}<\alpha^{(a)}$ *and* $\varphi_{\alpha^{(a)}/\gamma^{(c)}}(f)=0$ *for all* $\beta^{(b)}<\gamma^{(c)}\le\alpha^{(a)}$. *For some* $\lambda_J\in k$,

$$\varphi_{\alpha^{(a)}/\beta^{(b)}}(f) \quad = \quad \varphi_{\alpha^{(a)}/\beta^{(b)}}(\beta^{(b)}) \cdot \sum_{\substack{J\in\binom{[N]}{p}\\ |J|+|\beta^{(b)}|=d}} \lambda_J\cdot\phi_{J(\alpha^{(a)})/J(\beta^{(b)})}(J)\,.$$

Proof. We only prove part 1. The hypothesis states that $\phi_{J(\alpha^{(a)})}(\pi(f)) = \varphi_{\alpha^{(a)}}(f)$ vanishes on all matrices $\mathcal{N}_{J(\beta^{(b)})}$ for $\beta^{(b)}<\alpha^{(a)}$. Proposition 12 implies that $\pi(f)$ vanishes on all Schubert varieties Ω_J with $J<J(\alpha^{(a)})$. But then, using Proposition 11,

$$\pi(f) \quad \in \quad \bigcap_{J<J(\alpha^{(a)})}\mathcal{I}(\Omega_J) \quad = \quad \mathcal{I}(\Omega_{J(\alpha^{(a)})}) \,+\, \big\langle\, J(\alpha^{(a)})\,\big\rangle\,.$$

This means $\pi(f)=g+J(\alpha^{(a)})\cdot h$, where $g\in\mathcal{I}(\Omega_{J(\alpha^{(a)})})=\ker\big(\phi_{J(\alpha^{(a)})}\big)$ and $h\in k[\binom{[N]}{p}]$ is a linear form of degree $d-|\alpha^{(a)}|$. Such a linear form can be written as follows

$$h \quad = \quad \sum_{|J|+|\alpha^{(a)}|=d} \lambda_J\, J\,.$$

By applying the map $\phi_{J(\alpha^{(a)})}$ to both sides of the equation $\pi(f)=g+J(\alpha^{(a)})\cdot h$, we obtain the first assertion of Lemma 13. Part 2 is proved by similar arguments. ∎

Our proof of Theorem 10 will show that the sums in Lemma 13 are actually sums of terms of the form $\lambda_{J(\delta^{(d)})}\cdot\varphi_{\alpha^{(a)}}(\delta^{(d)})$ and $\lambda_{J(\delta^{(d)})}\cdot\varphi_{\alpha^{(a)}/\beta^{(b)}}(\delta^{(d)})$ respectively. The next lemma provides the initial step in our inductive proof of Theorem 10.

Lemma 14. *Let* $\gamma^{(c)}$ *and* $\delta^{(d)}$ *be incomparable variables in the poset* $\mathcal{C}^{np}_{p,m}$ *and set* $\alpha^{(a)}:=\gamma^{(c)}\vee\delta^{(d)}$. *Then* $\varphi_{\alpha^{(a)}}(\gamma^{(c)}\cdot\delta^{(d)} \;-\; \gamma^{(c)}\vee\delta^{(d)}\cdot\gamma^{(c)}\wedge\delta^{(d)})=0$.

Proof. We prove the lemma by inductively showing that, for each $\beta^{(b)}\le\alpha^{(a)}$,

$$\varphi_{\alpha^{(a)}/\beta^{(b)}}\big(\,\gamma^{(c)}\cdot\delta^{(d)} \;-\; \gamma^{(c)}\vee\delta^{(d)}\cdot\gamma^{(c)}\wedge\delta^{(d)}\,\big) \quad = \quad 0. \tag{9}$$

If $\beta^{(b)}\not\le\gamma^{(c)}\wedge\delta^{(d)}$, then $\varphi_{\alpha^{(a)}/\beta^{(b)}}(\gamma^{(c)}\wedge\delta^{(d)})$ vanishes, and either $\varphi_{\alpha^{(a)}/\beta^{(b)}}(\gamma^{(c)})$ vanishes or $\varphi_{\alpha^{(a)}/\beta^{(b)}}(\delta^{(d)})$ vanishes. This implies that (9) holds.

Next suppose $\beta^{(b)}=\gamma^{(c)}\wedge\delta^{(d)}$. We claim that $\varphi_{\alpha^{(a)}/\beta^{(b)}}$ maps each variable appearing in (9) to its initial term in $k[X]$. In view of Proposition 4, this claim implies (9). To establish this claim, we need only show that $\varphi_{\alpha^{(a)}/\beta^{(b)}}(\gamma^{(c)})=\mathrm{in}_{\prec}\varphi(\gamma^{(c)})$, as the case for $\delta^{(d)}$ is similar and that of the other terms follow from (8). Consider the expansion of $\varphi_{\alpha^{(a)}/\beta^{(b)}}(\gamma^{(c)})$ in terms of the minors $\phi(J)$ of $\mathcal{N}_{J(\alpha^{(a)})/J(\beta^{(b)})}$. First observe that the submatrix given by the columns from $J(\gamma^{(c)})$ is block anti-diagonal, with each block either upper or lower triangular along its anti-diagonal. This is because for each i, $J(\gamma^{(c)})_i$ is either $J(\beta^{(b)})_i$ or $J(\alpha^{(a)})_i$, and the non-zero entries in the ith row of $\mathcal{N}_{J(\alpha^{(a)})/J(\beta^{(b)})}$ lie between these two numbers. Thus the contribution of term $J(\gamma^{(c)})$ to $\varphi_{\alpha^{(a)}/\beta^{(b)}}(\gamma^{(c)})$ is simply $\mathrm{in}_{\prec}\,\varphi(\gamma^{(c)})$.

We claim there are no other terms. If there were another term indexed by L, then the Lth maximal minor of $\mathcal{N}_{J(\alpha^{(a)})/J(\beta^{(b)})}$ would be non-zero, and so $J(\beta^{(b)}) \leq L \leq J(\alpha^{(a)})$. Thus

$$J(\beta^{(b)}) \quad = \quad J(\gamma^{(c)}) \wedge J(\delta^{(d)}) \quad \leq \quad L \wedge J(\delta^{(d)}).$$

Comparing the first components of these sequences gives $\min\{J(\gamma^{(c)})_1, J(\delta^{(d)})_1\} \leq L_1$. Since $L_1 < J(\gamma^{(c)})_1$, this implies $J(\delta^{(d)})_1 \leq L_1$. Similarly, using $J(\alpha^{(a)}) \geq L \vee J(\delta^{(d)})$, we see that $J(\delta^{(d)})_p \geq L_p$. Lastly, as L is a summand in $\pi(\gamma^{(c)})$ and $L \neq J(\gamma^{(c)})$, we have $L_p - L_1 > m+p$ and thus

$$m+p \;\geq\; J(\delta^{(d)})_p - J(\delta^{(d)})_1 \;\geq\; L_p - L_1 \;>\; m+p,$$

a contradiction, which proves the claim. Thus (9) holds for $\beta^{(b)} = \gamma^{(c)} \wedge \delta^{(d)}$.

Finally, let $\zeta^{(z)} < \gamma^{(c)} \wedge \delta^{(d)}$ and suppose that (9) holds for all $\beta^{(b)}$ with $\zeta^{(z)} < \beta^{(b)} \leq \alpha^{(a)}$. Then by Lemma 13,

$$\varphi_{\alpha^{(a)}/\zeta^{(z)}}\left(\gamma^{(c)} \cdot \delta^{(d)} \;-\; \gamma^{(c)} \vee \delta^{(d)} \cdot \gamma^{(c)} \wedge \delta^{(d)}\right) \quad = \quad \varphi_{\alpha^{(a)}/\zeta^{(z)}}(\zeta^{(z)}) \cdot \sum_J \lambda_J \cdot \phi_{J(\alpha^{(a)})/J(\zeta^{(z)})}(J)\,,$$

the sum over sequences J of rank $|J| = |\gamma^{(c)}| + |\delta^{(d)}| - |\zeta^{(z)}|$. But this exceeds the rank of $\alpha^{(a)}$, since $\zeta^{(z)} < \gamma^{(c)} \wedge \delta^{(d)}$ and $|\alpha^{(a)}| + |\gamma^{(c)} \wedge \delta^{(d)}| = |\gamma^{(c)}| + |\delta^{(d)}|$. Thus the sum vanishes and so (9) holds for all $\beta^{(b)} \leq \alpha^{(a)}$, which proves the lemma. ∎

Proof of Theorem 10. Let $\gamma^{(c)}$ and $\delta^{(d)}$ be incomparable variables in the poset $\mathcal{C}^{np}_{p,m}$. For each $\alpha^{(a)} \in \mathcal{C}^{np}_{p,m}$ we inductively construct quadratic polynomials $S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)}) \in k[\beta^{(s)} \mid \beta^{(s)} \leq \alpha^{(a)}]$, and then show $S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)})$ is in the kernel of the map $\varphi_{\alpha^{(a)}}$. The case when $\alpha^{(a)}$ is the top element in the poset $\mathcal{C}^{np}_{p,m}$ proves the theorem. These polynomials have the following restriction property: If $\beta^{(b)} < \alpha^{(a)}$, then $S_{\beta^{(b)}}(\gamma^{(c)}, \delta^{(d)})$ is the image of $S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)})$ under the map which sets variables $\zeta^{(z)} \not\leq \beta^{(b)}$ to zero. They also have the further homogeneity properties that each non-zero term $\lambda\zeta^{(z)}\beta^{(b)}$ must have $z + b = c + d$ and satisfy the multiset equality $\beta \cup \zeta \;=\; \gamma \cup \delta$, and if it is not the initial term, then $\gamma^{(c)}, \delta^{(d)} \in [\zeta^{(z)}, \beta^{(b)}]$.

For $\alpha^{(a)} \not\geq \gamma^{(c)} \vee \delta^{(d)}$, set $S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)}) := 0$ and if $\alpha^{(a)} = \gamma^{(c)} \vee \delta^{(d)}$, then set

$$S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)}) \quad := \quad \gamma^{(c)} \cdot \delta^{(d)} \;-\; \gamma^{(c)} \vee \delta^{(d)} \cdot \gamma^{(c)} \wedge \delta^{(d)}.$$

These polynomials have the restriction and homogeneity properties, and, for $\alpha^{(a)} \not> \gamma^{(c)} \vee \delta^{(d)}$, we have $\varphi_{\alpha^{(a)}}(S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)})) = 0$, by Lemma 14.

Let $\alpha^{(a)} > \gamma^{(c)} \vee \delta^{(d)}$ and suppose we have constructed $S_{\beta^{(b)}}(\gamma^{(c)}, \delta^{(d)})$ for each $\beta^{(b)} < \alpha^{(a)}$. By the restriction property, there is a polynomial $S' \in k[\beta^{(b)} \mid \beta^{(b)} < \alpha^{(a)}]$ which restricts to $S_{\beta^{(b)}}(\gamma^{(c)}, \delta^{(d)})$ for each $\beta^{(b)} < \alpha^{(a)}$. Thus $\varphi_{\beta^{(b)}}(S') = 0$ for all $\beta^{(b)} < \alpha^{(a)}$.

Set $e := |\gamma^{(c)}| + |\delta^{(d)}|$, the degree of S'. By Lemma 13,

$$\varphi_{\alpha^{(a)}}(S') \quad = \quad \varphi_{\alpha^{(a)}}(\alpha^{(a)}) \cdot \sum_{|J|+|\alpha^{(a)}|=e} \lambda_J \cdot \phi_{J(\alpha^{(a)})}(J)\,. \tag{10}$$

If we consider the columns of $\mathcal{M}_{\alpha^{(a)}}(t)$ involved in $\varphi_{\alpha^{(a)}}(S')$, we see that this sum is further restricted to those J which satisfy the multiset equality $(\gamma \cup \delta) \setminus \alpha \equiv J \mod (m+p)$, with $J \bmod (m+p)$ consisting of distinct integers, and with $J \leq J(\alpha^{(a)})$. If there are no such J, then $\varphi_{\alpha^{(a)}}(S') = 0$ and we set $S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)}) = S'$.

Otherwise, let $z := c+d-a$ and $\zeta := (\gamma \cup \delta) \setminus \alpha$. Then the summands in (10) are among those J which appear in $\pi(\zeta^{(z)})$ so we have $J(\zeta^{(z)}) < J(\alpha^{(a)})$ and hence $\zeta^{(z)} < \alpha^{(a)}$. Observe that $\varphi_{\alpha^{(a)}/\zeta^{(z)}}(S') = \lambda_{J(\zeta^{(z)})} \cdot \varphi_{\alpha^{(a)}/\zeta^{(z)}}(\alpha^{(a)}\zeta^{(z)})$. Define

$$S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)}) \quad := \quad S' - \lambda_{J(\zeta^{(z)})}\alpha^{(a)}\zeta^{(z)}.$$

We claim that if $\lambda_{J(\zeta^{(z)})} \neq 0$, then $\zeta^{(z)} \leq \gamma^{(c)}, \delta^{(d)}$. If not, then every term of S' contains a variable $\xi^{(x)}$ with $\zeta^{(z)} \not\leq \xi^{(x)}$, and so we must have $\varphi_{\alpha^{(a)}/\zeta^{(z)}}(S') = 0$, a contradiction.

We complete the proof of Theorem 10 by showing that for $\beta^{(b)} \leq \alpha^{(a)}$,

$$\varphi_{\alpha^{(a)}/\beta^{(b)}}(S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)})) \quad = \quad 0. \tag{11}$$

If $\beta^{(b)} \not\leq \zeta^{(z)}$, then $\varphi_{\alpha^{(a)}/\beta^{(b)}}(\zeta^{(z)}) = 0$ and so $\varphi_{\alpha^{(a)}/\beta^{(b)}}(S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)})) = \varphi_{\alpha^{(a)}/\beta^{(b)}}(S')$, which is zero as $\phi_{J(\alpha^{(a)})/J(\beta^{(b)})}(J) = 0$ for all J which appear in $\pi(\zeta^{(z)})$. By the construction of $S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)}))$, we also have $\varphi_{\alpha^{(a)}/\zeta^{(z)}}(S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)})) = 0$.

Let $\xi^{(x)} < \zeta^{(z)}$ and suppose (11) holds for all $\beta^{(b)}$ with $\xi^{(x)} < \beta^{(b)}$. Then by Lemma 13,

$$\varphi_{\alpha^{(a)}/\xi^{(x)}}(S_{\alpha^{(a)}}(\gamma^{(c)}, \delta^{(d)})) \quad = \quad \varphi_{\alpha^{(a)}/\xi^{(x)}}(\xi^{(x)}) \cdot \sum_{|J|+|\xi^{(x)}|=e} \lambda_J \cdot \phi_{J(\alpha^{(a)})/J(\xi^{(x)})}(J).$$

Since $|J| = e - |\xi^{(x)}| > e - |\zeta^{(z)}| = |\alpha^{(a)}|$, each term in the right hand sum is zero. ∎

It is now straightforward to derive all our assertions that were left unproven so far.

Proof of Theorem 1. Theorem 10 together with Proposition 4 shows that the subduction criterion for sagbi bases (see e.g. [5, Proposition 1.1] or [21, Theorem 11.4]) is satisfied. ∎

Proof of Theorem 2. A standard fact on sagbi bases, proved in [5, Corollary 2.2] or in [21, Corollary 11.6 (1)], states that the reduced Gröbner basis for the binomial ideal kernel(ψ) lifts to a reduced Gröbner basis for the non-binomial ideal kernel(φ). ∎

Proof of Proposition 9. If $\beta^{(b)}$ is the minimal element in the poset $\mathcal{C}_{p,m}$, then $\varphi_{\alpha^{(a)}}$ and $\varphi_{\alpha^{(a)}/\beta^{(b)}}$ have the same kernel, as the varieties $Z_{\alpha^{(a)}/\beta^{(b)}}$ and $Z_{\alpha^{(a)}}$ are equal. Hence it suffices to prove the second statement about $\varphi_{\alpha^{(a)}/\beta^{(b)}}$. Clearly, the kernel of $\varphi_{\alpha^{(a)}/\beta^{(b)}}$ contains the homogeneous ideal of the skew quantum Schubert variety $Z_{\alpha^{(a)}/\beta^{(b)}}$. If this containment were proper, then we would also get proper containment at the level of initial ideals with respect to the induced partial term order, which was denoted by $\mathcal{A}^T\omega$ in [21, Chapter 11]. But that is impossible since every binomial relation on the monomials $\text{in}_\prec \varphi_{\alpha^{(a)}/\beta^{(b)}}(\gamma^{(c)}) = \text{in}_\prec \varphi(\gamma^{(c)})$ lifts to a polynomial which vanishes on $Z_{\alpha^{(a)}/\beta^{(b)}}$, as shown in the proof Theorem 10. ∎

5. Applications and future directions

We first summarize some algebraic consequences of our main results.

Corollary 15. *The coordinate ring of the quantum Grassmannian $K^q_{p,m}$ is an algebra with straightening law on the distributive lattice $\mathcal{C}^q_{p,m}$. It has a presentation by a non-commutative Gröbner basis consisting of quadratic elements.*

Proof. The first statement follows from Theorem 1 and the form of the syzygies $S(\gamma^{(c)}, \delta^{(d)})$ of Theorem 2. For the second statement, consider the coordinate ring of $K^q_{p,m}$ as the quotient of the free associative algebra on $\mathcal{C}^q_{p,m}$ modulo a two-sided ideal. By [7, Proposition 3.2] that two-sided ideal has a quadratic Gröbner basis, obtained from lifting the Gröbner basis in Theorem 2. For the classical Grassmannian ($n = 0$) this result appeared in [8]. ∎

Corollary 16. *The coordinate ring of* $K^q_{p,m}$ *is a Koszul algebra.*

Proof. This is a well-known consequence of the existence of a quadratic Gröbner basis; see e.g. [8, Theorem 3]. ∎

Corollary 17. *The coordinate ring of* $K^q_{p,m}$ *is a normal Cohen-Macaulay domain. It has rational singularities if* $\mathrm{char}(k) = 0$ *and it is* F*-rational if* $\mathrm{char}(k) > 0$.

Proof. By Theorem 2 and the results of [5], these properties of $K^q_{p,m}$ follow from the corresponding properties of the toric variety $T^q_{p,m}$. But these were established in [9], as $T^q_{p,m}$ is the toric variety associated to the distributive lattice $\mathcal{C}^q_{p,m}$. ∎

We remark that both Corollary 17 and the analog of Corollary 15 (with the poset $\mathcal{C}^q_{p,m}$ replaced by the appropriate interval) hold for the skew quantum Schubert varieties.

Our next application is the sagbi property of the row-consecutive $p \times p$-minors of a matrix of indeterminates. This result is non-trivial since the set of *all* $p \times p$-minors is not a sagbi basis in general [21, Example 11.3]. A finite sagbi basis for the algebra of all $p \times p$-minors was found by Bruns and Conca [3]. Let $\mathcal{L}$ be the $p(n+1) \times (m+p)$-matrix whose i, jth entry is $x^{(l)}_{r,j}$, where $i = pl + r$. This matrix is obtained from $\mathcal{N}$ by stacking the matrices $\mathcal{N}_0, \ldots, \mathcal{N}_n$. Let $\chi : k[\mathcal{C}^{np}_{p,m}] \to k[X]$ denote the k-algebra homomorphism which sends the variable $\alpha^{(a)}$ to the αth maximal minor of the submatrix of $\mathcal{L}$ consisting of rows $a+1, a+2, \ldots, a+p$. Thus the collection of polynomials $\chi(\alpha^{(a)})$ are the row-consecutive $p \times p$-minors of $\mathcal{L}$.

Theorem 18. *The set* $\{\chi(\alpha^{(a)}) : \alpha^{(a)} \in \mathcal{C}^{np}_{p,m}\}$ *of row-consecutive* $p \times p$*-minors of a generic matrix is a sagbi basis with respect to the degree reverse lexicographic term order* $\prec$ *on* $k[X]$.

This may also be deduced from Theorem 7.5 in [22].

Proof. Let ω be the weight on the variables in $k[X]$ defined by $\omega(x^{(l)}_{i,j}) := -(pl+i)^2$. Then $\chi(\alpha^{(a)}) = \mathrm{in}_\omega\big(\varphi(\alpha^{(a)})\big)$, the initial form of $\varphi(\alpha^{(a)})$, and we have $\mathrm{in}_\prec\big(\chi(\alpha^{(a)})\big) = \mathrm{in}_\prec\big(\varphi(\alpha^{(a)})\big)$ for all $\alpha^{(a)} \in \mathcal{C}^{np}_{p,m}$. Thus image($\varphi$) and image($\chi$) have the same initial algebra, and so we deduce the sagbi property for the polynomials $\chi(\alpha^{(a)})$ from Theorem 1. ∎

Let $\mathbf{w}$ denote the weight on the variables $\mathcal{C}^{np}_{p,m}$ defined by $\mathbf{w}(\alpha^{(a)}) := -a^2$. For every incomparable pair $\gamma^{(c)}, \delta^{(d)}$ in the poset $\mathcal{C}^{np}_{p,m}$, we define the quadratic polynomial

$$R(\gamma^{(c)}, \delta^{(d)}) \quad := \quad \mathrm{in}_{\mathbf{w}}\big(S(\gamma^{(c)}, \delta^{(d)})\big),$$

where $S(\gamma^{(c)}, \delta^{(d)})$ is the element of the reduced Gröbner basis for the kernel of φ. For example, $R(156^{(1)}, 234^{(2)})$ equals the sum of the first ten terms in (4). The weight $\mathbf{w}$ is equivalent, modulo the homogeneities of kernel(φ), to the induced weight which was denoted by $\mathcal{A}^T\omega$ in [21, Chapter 11]. The only-if direction in [21, Theorem 11.4] implies

Corollary 19. *The reduced Gröbner basis of the kernel of* χ *consists of the quadratic polynomials* $R(\gamma^{(c)}, \delta^{(d)})$ *as* $\gamma^{(c)}, \delta^{(d)}$ *run over the set of incomparable pairs in the poset* $\mathcal{C}^{np}_{p,m}$.

For the Plücker ideal defining the classical Grassmannian ($n = 0$), an explicit (but non-reduced) quadratic Gröbner basis is known. It appears in the work of Hodge-Pedoe [10] and Doubilet-Rota-Stein [6], and it consists of the van der Waerden syzygies. They are discussed in Gröbner basis language in [20, Section 3.1]. Our next aim is to introduce an analogous non-reduced Gröbner basis for the ideal kernel(φ) of the quantum Grassmannian.

We begin by defining the skew van der Waerden syzygies for its initial ideal

$$\text{kernel}(\chi) \quad = \quad \text{in}_{\mathbf{w}}(\text{kernel}(\varphi)), \tag{12}$$

which consists of the algebraic relations among the row-consecutive minors. Given a sequence of integers $D : 1 \leq d_1, \cdots, d_p \leq m+p$ and any integer $0 \leq a \leq np$, let $D^{(a)}$ denote $\pm\alpha^{(a)}$, where α is the reordering of the sequence D and $\pm$ is the sign of the permutation which sorts the sequence D. Let $T = \alpha^{(a)}\beta^{(b)}$ with $a < b$ be a non-standard tableau and i the smallest index of a violation $\beta_i < \alpha_{i-b+a}$. Define increasing sequences

$$A \quad := \quad \alpha_1, \ldots, \alpha_{i-b+a-1} \qquad B \quad := \quad \beta_{i+1}, \ldots, \beta_p$$
$$C \quad := \quad \beta_1, \ldots, \beta_i, \alpha_{i-b+a}, \ldots, \alpha_p.$$

For a subset $I \in \binom{[p+b-a+1]}{i}$, let C_I be the corresponding numbers from C (in order) and C_{I^c} be the other numbers from C, also in order. Define the *skew van der Waerden syzygy*

$$W(T) \quad := \quad \sum_{I \in \binom{[p+b-a+1]}{i}} (A, C_{I^c})^{(a)} \cdot (C_I, B)^{(b)} . \tag{13}$$

Proposition 20. *The syzygies $W(T)$ form a Gröbner basis for the kernel of χ.*

Proof. Our choice of term order implies $\text{in}_{\prec}\big(W(T)\big) = T = \alpha^{(a)}\beta^{(b)}$. Therefore it suffices to show that $\chi(W(T)) = 0$. Let $Y_1, \ldots, Y_{m+p}$ be the columns of the submatrix of $\mathcal{L}$ given by its rows $a+1, \ldots, b+p$. The skew van der Waerden syzygy $\chi(W(T))$ is an anti-symmetric, multilinear form in the $p+b-a+1$ vectors $Y_{b_1}, \ldots, Y_{b_{p+b-a+1}}$ in $(p+b-a)$-space. ∎

The non-reduced Gröbner basis in Proposition 20 can be lifted to the quantum Grassmannian as follows. We define the *quantum van der Waerden syzygy* of the non-standard tableau T to be the unique quadratic polynomial $V(T)$ in kernel(φ) which satisfies

$$\text{in}_{\mathbf{w}}\big(V(T)\big) \quad = \quad W(T),$$

and is a sum of syzygies $S(\gamma^{(c)}, \delta^{(d)})$ with $\mathbf{w}(\gamma^{(c)}\delta^{(d)}) = \mathbf{w}(T)$. This syzygy exists by (12) and it is unique because the quadratic generators of the initial ideal are k-linearly independent, and any two such quadratic lifts of $W(T)$ in kernel(φ) differ by terms whose weights are strictly less than $\mathbf{w}(T)$. For instance, the quantum van der Waerden syzygy $V(156^{(1)}234^{(2)})$ is the polynomial with 30 terms given in (4). It would be desirable to find an explicit formula, perhaps in terms of the combinatorial formalism in [6], for all of the skew van der Waerden syzygies $V(T)$, but at present we have no clue how to do this.

The ideal of the quantum Grassmannian $K^q_{p,m}$ contains certain *obvious relations* which are derived from the Grassmannian $Grass_p k^{m+p}$. For each $\alpha \in \binom{[m+p]}{p}$ consider the polynomial

$$g_\alpha(t) \quad = \quad \alpha^{(q)} \cdot t^q + \cdots + \alpha^{(1)} \cdot t + \alpha^{(0)} .$$

Given any quadratic form $F(\alpha)$ in the Plücker ideal defining $Grass_p k^{m+p}$ and any $0 \leq r \leq 2q$, let F_r be the coefficient of t^r in polynomial $F(g_\alpha(t))$. Since $F(g_\alpha(t))$ is a polynomial in t which vanishes identically on $K^q_{p,m}$, each of its coefficients F_r must also vanish on $K^q_{p,m}$. We call the collection of quadratic polynomials F_r as F ranges over a generating set for the Plücker ideal of $Grass_p k^{m+p}$ the *obvious relations*. Rosenthal [17] showed the following.

Proposition 21. *The obvious relations define $K^q_{p,m}$ set-theoretically, provided k is infinite.*

When p or $m \leq 2$, the obvious relations coincide with the reduced Gröbner basis of Theorem 2, in particular, they generate the ideal of the quantum Grassmannian $K^q_{2,m}$. This is no longer true for $m = p = 3$. There are 35 incomparable pairs in $\mathcal{C}^0_{3,3}$, and hence 35 linearly independent quadrics in the Plücker ideal of $Grass_3 k^6$. These give rise to $35(2q+1)$ linearly independent obvious relations but when $q > 0$ there are $35(2q+1)+2q-1$ incomparable pairs in $\mathcal{C}^q_{3,3}$. Thus the obvious relations do not generate the homogeneous ideal of $K^q_{3,3}$.

When $q = 1$ or $q = 2$ then the obvious relations generate the homogeneous ideal of $K^q_{3,3}$ together with an embedded component supported on the irrelevant ideal. Thus the obvious relations define $K^q_{3,3}$ scheme-theoretically, but not ideal-theoretically. It remains an open problem whether the the obvious relations define $K^q_{p,m}$ scheme-theoretically.

The varieties $K^q_{p,m}$ are in general singular when $q > 0$. For instance $K^1_{2,2}$ is singular along the classical Grassmannian $K^0_{2,2}$. The coordinate rings of classical Schubert varieties are unique factorization domains. We conjecture that this also holds for the coordinate rings of quantum Schubert varieties.

The coordinate rings of the classical Grassmannian varieties are unique factorization domains. We conjecture that this also holds for the coordinate rings of quantum Grassmannian varieties. In particular we conjecture that the quantum Grassmannian varieties are Gorenstein.

Batyrev et. al. [1] applied the familiar sagbi property for the Grassmannian in the construction of certain pairs of mirror 3-folds from Calabi-Yau complete intersections in Grassmannians. We are optimistic that the results in this paper will be similarly useful for researchers in the fascinating interplay of algebraic geometry and theoretical physics.

The classical straightening law for the Grassmannian and its Schubert varieties were the starting point for the general *standard monomial theory* for flag varieties. For details and references we refer to the recent work on sagbi bases by Gonciulea and Lakshmibai [14]. Our results suggest that standard monomial theory might be extended to certain spaces of rational curves in flag varieties generalizing the quantum Grassmannian $K^q_{p,m}$.

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