INTERSECTION THEORY ON SPHERICAL VARIETIES

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1. Introduction

The aim of this note is to give a presentation for the Chow homology groups for an arbitrary variety with an action of a connected solvable linear algebraic group, and to compute the operational Chow cohomology ring of a complete variety on which such a group acts with finitely many orbits. These results apply to any spherical variety, which is a variety with an action of a reductive group that contains a Borel subgroup with a dense orbit in the variety.[1] They also apply to the closures of the orbits of the Borel subgroup in any such variety. These varieties include Grassmannians, flag manifolds, and homogeneous spaces G/P and their Schubert subvarieties, toric varieties, varieties of complete quadrics and other compactifications of symmetric spaces, and varieties of complexes. For the smooth varieties on this list, their cohomology rings, with their rich combinatorial structure, have been vital in many areas of mathematics, dating back to the Schubert calculus of classical enumerative geometry. Our primary goal here is to extend this as far as possible to the singular case.

The Chow group A_kX of an arbitrary variety or scheme[2] X is defined to be Z_kX/R_kX, where Z_kX is the free abelian group generated by all k-dimensional closed subvarieties of X, and R_kX is the subgroup generated by divisors $[\mathrm{div}(f)]$ of nonzero rational functions f on $(k+1)$-dimensional subvarieties W of X (see [6, §1]). When an algebraic group Γ acts on X, one can form a group $A_k^\Gamma X = Z_k^\Gamma X/R_k^\Gamma X$, with $Z_k^\Gamma X$ the free abelian group generated by all Γ-stable closed subvarieties of X, and $R_k^\Gamma X$ is the subgroup generated by all divisors of eigenfunctions on Γ-stable $(k+1)$-dimensional subvarieties; here a function f in $R(W)^*$ is an eigenfunction if $g \cdot f = \chi(g)f$ for all g in Γ, for some character $\chi = \chi_f$ on Γ.

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[1]In the literature, spherical varieties are taken to be normal, but this condition is not needed here. The general theory of spherical varieties has been developed primarily by Luna and Vust, Brion, Vinberg, Pauer, and Knop, cf. [1], [15], [17].

[2]See the end of the introduction for conventions about schemes and varieties.

Theorem 1. *If a connected solvable linear algebraic group Γ acts on a scheme X, then the canonical homomorphism $A_k^\Gamma X \to A_k X$ is an isomorphism.*

Theorem 1 was proved by for projective varieties by Hirschowitz [11]. Brion [3, §1.3] pointed out that Vust had proved the surjectivity for complete varieties, and asked if the isomorphism holds without the assumption of projectivity.

When Γ acts on X with only finitely many orbits, this gives a finite presentation of the Chow groups of X. This follows from two facts: (i) when Γ has a dense orbit in W, each eigenspace $R(W)_\chi$ can have dimension at most one; (ii) the group of characters is finitely generated (see [12, §16, Ex. 12]). It is not always easy to make this presentation explicit. However, for toric varieties, the invariant subvarieties correspond to cones in the defining fan, and eigenfunctions on a corresponding variety come from points in the lattice dual to the cone; this gives a finite presentation of the Chow groups in terms of the combinatorics of the fan (see [10]).

If Γ is unipotent, then the groups $R_k^\Gamma X$ are trivial, and the theorem says that $A_k X$ is the free abelian group on the orbit closures. For example, if X is a Schubert variety, one can take Γ to be a unipotent group, so the Chow group is the free abelian group on the Schubert subvarieties of X.

Corollary. *Suppose a connected solvable linear algebraic group acts on a complete scheme X with only finitely many orbits. Then*

(i) *The cone of effective cycles in $A_k(X) \otimes \mathbb{Q}$ is a polyhedral cone, generated by the classes of the closures of the orbits.*
(ii) *Rational equivalence and algebraic equivalence coincide on X.*

Let $A_*X = \bigoplus_k A_k X$. For any schemes X and Y one has a "Künneth map" $A_*X \otimes A_*Y \to A_*(X \times Y)$, taking $[V] \otimes [W]$ to $[V \times W]$, where V and W are subvarieties of X and Y. This is an isomorphism only for very special algebraic varieties, but when it is, there are strong consequences.

Theorem 2. *If a connected solvable linear algebraic group acts on a scheme X with only finitely many orbits, then for any scheme Y the Künneth map $A_*X \otimes A_*Y \to A_*(X \times Y)$ is an isomorphism.*

Corollary. *If, in addition, X is nonsingular and complete, then the cycle map $A_*X \to H_*X$ is an isomorphism.*

The Chow groups $A_k X$ are called Chow "homology" groups, since they are covariant for arbitrary proper maps. Historically, it has not been easy to construct directly Chow "cohomology" groups out of geometric "cocycles", to pair with these Chow homology groups. In [9, §9], cf. [6, §17], *operational* Chow cohomology groups $A^k X$ were defined, to have the expected functorial properties:

(i) "cup products" $A^p X \otimes A^q X \to A^{p+q} X$, $a \otimes b \mapsto a \cup b$, making $A^*X = \bigoplus_k A^k X$ into a graded associative ring (commutative when resolution of singularities is known);
(ii) contravariant graded ring maps $f^*\colon A^k X \to A^k Y$ for arbitrary morphisms $f\colon Y \to X$;
(iii) "cap products" $A^k X \otimes A_m X \to A_{m-k} X$, $c \otimes z \mapsto c \cap z$, making A_*X into an A^*X-module and satisfying the usual projection formula;
(iv) when X is a nonsingular n-dimensional variety, the natural "Poincaré duality" map from $A^k X$ to $A_{n-k} X$ taking c to $c \cap [X]$ is an isomorphism, and

the ring structure on A^*X is that determined by intersection products of cycles on X;

(v) vector bundles on X have Chern classes in A^*X.

(There are other functorial properties, such as Gysin push-forward maps for proper maps that are local complete intersection morphisms.) A class c in A^kX determines a collection of homomorphisms $A_mY \to A_{m-k}Y$, written $z \mapsto f^*c \cap z$, for all morphisms $f\colon Y \to X$ and all integers $m \geq k$; in fact, elements of A^kX are *defined* to be collections of such homomorphisms that are compatible with standard intersection-theoretic constructions; the ring structure is defined by composition of such homomorphisms. We point out that the Chow cohomology rings A^*X used here are not the ideal Chow cohomology rings one hopes will someday be constructed; rather, they form the coarsest possible Chow cohomology theory with the properties listed. For a discussion of this, see [7, §10.3]. From work of Kimura [14], however, they are more computable, and closer to the ideal, than one might have expected.

An element in A^kX determines, by the cap product, a homomorphism from A_kX to A_0X; if X is complete, composing with the degree homomorphism from A_0X to $\mathbb{Z}$, one has a natural "Kronecker duality" homomorphism

$$A^kX \to \mathrm{Hom}(A_kX, \mathbb{Z})\,, \quad c \mapsto [a \mapsto \deg(c \cap a)].$$

For a general complete variety this is far from being an isomorphism; for example, when X is a nonsingular curve and $k = 1$, the kernel is the Jacobian of the curve. This is the map analogous to the Kronecker map $H^kX \to \mathrm{Hom}(H_kX, \mathbb{Z})$ in topology, which is always an isomorphism, at least up to torsion. If X is nonsingular and the cycle map from A_*X to H_*X is an isomorphism, then it follows from topology that the Kronecker duality is an isomorphism for the Chow groups. For singular spherical varieties, however, the cycle map need not be an isomorphism; in fact, singular toric varieties can have non-trivial odd-dimensional homology which is rather hard to calculate [18], and the cycle map need not be surjective in even dimensions. This makes the following result somewhat unexpected:

Theorem 3. *If a connected solvable linear algebraic group acts on a complete scheme X with only finitely many orbits, then the Kronecker duality map $A^kX \to \mathrm{Hom}(A_kX, \mathbb{Z})$ is an isomorphism.*

For example, let X be the closure of a generic torus orbit in the Grassmannian $G(2,4)$; X is the toric variety constructed from the fan over the faces of a cube, with lattice generated by the vertices of the cube. The Chow homology groups are $\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}^{\oplus 5}$, and $\mathbb{Z}$ in dimensions 0, 1, 2, and 3, while the Chow cohomology groups are $\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}^{\oplus 5}$, and $\mathbb{Z}$ in codimensions 0, 1, 2, and 3. Note that the Poincaré duality maps $A^kX \to A_{n-k}X$ are not isomorphisms. (In this example, the maps $A_kX \to H_{2k}X$ are isomorphisms, but $H_3X = \mathbb{Z}^{\oplus 2}$.) In general, even for toric varieties, the groups A_kX can have large torsion subgroups. By Theorem 3, the groups A^kX are torsion free. More generally, if one considers Chow groups $A_k(X;R)$ and $A^k(X;R)$ with coefficients in a commutative ring R, the same theorems are valid, and, under the hypothesis of Theorem 3, the canonical map

$$A^k(X;R) \to \mathrm{Hom}(A_kX, R) = \mathrm{Hom}_R(A_k(X;R), R)$$

is an isomorphism.

The orbits, so their closures—the Γ-stable subvarieties—can be indexed by a finite set Σ, with V_σ denoting the corresponding orbit closure for σ in Σ. We write $\tau \prec \sigma$ to mean that $V_\tau \subset V_\sigma$ (the "Bruhat" order), and we set $d(\sigma) = \dim(V_\sigma)$, and denote by $\Sigma^{(k)}$ the set of σ in Σ with $d(\sigma) = k$. Theorem 3 means that a Chow cohomology class c in $A^k X$ can be identified with a function $c\colon \Sigma^{(k)} \to \mathbb{Z}$, $\sigma \mapsto c(\sigma)$, where we write $c(\sigma)$ for the degree of $c \cap [V_\sigma]$; these functions must satisfy the condition that for any τ in $\Sigma^{(k+1)}$, and any eigenfunction f in $R(V_\tau)$, $\sum \operatorname{ord}_\sigma(f)\, c(\sigma) = 0$, the sum over those $\sigma \prec \tau$ with $d(\sigma) = k$, where $\operatorname{ord}_\sigma(f)$ is the order of f along V_σ. Our last goal in this note is to describe the ring structure of A^*X in terms of such combinatorial data. It follows from Theorem 2 that, for any γ, the class of the diagonal in $V_\gamma \times V_\gamma$ can be written in the form

$$(*) \qquad [\delta(V_\gamma)] \;=\; \sum m^\gamma_{\sigma,\tau}\, [V_\sigma] \otimes [V_\tau] \quad \text{in} \quad A_{d(\gamma)}(V_\gamma \times V_\gamma),$$

the sum over pairs σ, τ with $\sigma \prec \gamma$, $\tau \prec \gamma$, and $d(\sigma) + d(\tau) = d(\gamma)$, and some integers $m^\gamma_{\sigma,\tau}$. When Γ is unipotent, e.g., in the case of Schubert varieties, these coefficients are uniquely defined; on the Grassmannian they are the classical Littlewood-Richardson coefficients. In general these coefficients $m^\gamma_{\sigma,\tau}$ are not unique; at least in the toric case they can be found explicitly by choosing appropriate one-parameter subgroups to deform the diagonal [10]. These coefficients determine the cap and cup products:

Theorem 4.

(a) *For $c \in A^pX$ and $d(\gamma) = p + q$, the cap product $c \cap [V_\gamma]$ in A_qX is $\sum m^\gamma_{\sigma,\tau}\, c(\sigma)\, [V_\tau]$, the sum over $(\sigma,\tau) \in \Sigma^{(p)} \times \Sigma^{(q)}$.*

(b) *For $c \in A^pX$ and $c' \in A^qX$, the cup product $c \cup c'$ in $A^{p+q}X$ is given by the formula $(c \cup c')(\gamma) = \sum m^\gamma_{\sigma,\tau}\, c(\sigma)\, c'(\tau)$, the sum over $(\sigma,\tau) \in \Sigma^{(p)} \times \Sigma^{(q)}$.*

If H is a closed subgroup of a reductive group G, with a Borel subgroup B such that $B \cdot H$ is dense in G, then G/H is a spherical variety, and all G-equivariant normal and complete compactifications $G/H \hookrightarrow X$ are spherical varieties. One can form the direct limit $\varinjlim A^*X$ of the Chow cohomology rings of these compactifications. Each of the rings A^*X is embedded as a subring of this direct limit. This limit ring can be regarded as the natural intersection ring for solving enumerative geometry problems arising on G/H, at least those compatible with the group action. When G/H is a symmetric variety, these rings have been studied by De Concini and Procesi [4], and the general spherical case is discussed by Brion [2]. For toric varieties, they are in fact the polytope algebras introduced by McMullen [19], Morelli [20], and Khovanskii and Pukhlikov [13], as shown in [10].

For an arbitrary scheme X the first Chern class determines a functorial homomorphism from $Pic(X)$ to A^1X. One defect of the operational definition of A^1X is that this need not always be an isomorphism [6, Ex. 17.4.9]. The work of Brion [2], together with Theorem 3, shows that this map is an isomorphism when X is a projective spherical variety; Brion has informed us that his result extends to arbitrary complete spherical varieties. A direct proof for complete toric varieties is given in [10].

If X is an arbitrary surface with only rational singularities, it is easy to see that the canonical map from $Pic(X)$ to A^1X is an isomorphism. It follows from [16, Prop. 12.1.4] that, for complex varieties of arbitrary dimension with rational singularities, this map becomes an isomorphism after tensoring with $\mathbb{Q}$. However, A. Corti has constructed a threefold with rational singularities for which the map from $Pic(X)$ to A^1X is not an isomorphism. In this sense the singularities of spherical varieties are better behaved than those of general rational singularities.

The rings A^*X exhibit a rich combinatorial structure, which should be interesting to investigate. It is not always easy to compute the coefficients $m^{\gamma}_{\sigma,\tau}$ needed to describe the ring structure. In a sequel [10] devoted to toric varieties, we will identify the functions c with Minkowski weights on the corresponding fan, calculate the integers $m^{\gamma}_{\sigma,\tau}$ in terms of the fan, and describe the ring A^*X as a subalgebra of the polytope algebra.

Any orbit of a connected solvable linear algebraic group is isomorphic to a product of an affine space and an algebraic torus. B. Totaro [22] has answered a natural question left open by this paper, by showing that Theorems 2 and 3 are valid whenever X can be decomposed into a finite number of pieces isomorphic to $\mathbb{A}^a \times {\mathbb{G}_m}^*$. Moreoever, he shows that for any such variety over $\mathbb{C}$, the canonical map from $A_i(X)$ to the Borell-Moore homology $H^{BM}_{2i}(X)$ determines an isomorphism of $A_i(X) \otimes \mathbb{Q}$ with the smallest weight space $W_{2i}H^{BM}_{2i}(X, \mathbb{Q})$. This identifies $A^i(X) \otimes \mathbb{Q}$ with a direct factor of $H^{2i}(X, \mathbb{Q})$.

Conventions. We work in the category of algebraic schemes over an arbitrary algebraically closed field. A variety or subvariety is understood to be reduced and irreducible. When discussing homology, we take the ground field to be the complex numbers; the results extend without difficulty to ℓ-adic homology, using corresponding coefficients for the cycles. Throughout, Γ denotes a connected solvable linear algebraic group. A Γ-scheme is a scheme with an algebraic action of Γ; a subvariety is Γ-stable if it is mapped to itself by each element of Γ.

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2. Proof of Theorem 1

The case when X is projective was proved by Hirschowitz [11]. The essential point there is to use the Chow variety of effective cycles of given degree X, which is a projective variety with an action of Γ. The fact that a solvable group always has a fixed point on such a variety (Borel's fixed point theorem) implies that any effective cycle is rationally equivalent to a cycle fixed by Γ, which in turn implies the surjectivity of the map from A^{Γ}_*X to A_*X. An additional argument is needed to show that a chain of rational curves connecting two fixed points in the Chow variety can be deformed to a chain of Γ-stable rational curves, which implies the injectivity of the map.

To deduce the general case from Hirschowitz's theorem, we argue by induction on the dimension of X, assuming the result for all Γ-schemes of smaller dimension. The reduction requires two simple lemmas:

Lemma 1. *If Z is a closed Γ-stable subscheme of a Γ-scheme Y, and $U = Y \setminus Z$,* *there is a commutative diagram with exact rows:*

$$\begin{array}{ccccccc} A_*^\Gamma Z & \longrightarrow & A_*^\Gamma Y & \longrightarrow & A_*^\Gamma U & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A_* Z & \longrightarrow & A_* Y & \longrightarrow & A_* U & \longrightarrow & 0 \end{array}$$

with horizontal maps determined by the inclusion of Z in Y and the restriction from Y to U.

Proof. The exactness of the bottom row is proved in [6, §1.9]; the proof for the top row is the same, but using only invariant subvarieties and eigenfunctions. □

Lemma 2. *Let $p\colon X' \to X$ be a proper equivariant morphism of Γ-schemes, such that every closed (resp. Γ-stable closed) subvariety of X is the birational image of a closed (resp. Γ-stable closed) subvariety of X'. Suppose S is a Γ-stable closed subscheme of X such that p maps $X' \smallsetminus p^{-1}(S)$ isomorphically onto $X \smallsetminus S$. Let $E = p^{-1}(S)$. Then there is a commutative diagram*

$$\begin{array}{ccccccc} A_*^\Gamma E & \longrightarrow & A_*^\Gamma S \oplus A_*^\Gamma X' & \longrightarrow & A_*^\Gamma X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A_* E & \longrightarrow & A_* S \oplus A_* X' & \longrightarrow & A_* X & \longrightarrow & 0 \end{array}$$

with exact rows, and vertical maps the canonical maps of Theorem 1.

Proof. Form and label the fibre square

$$\begin{array}{ccc} E & \xrightarrow{\ j\ } & X' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ S & \xrightarrow[\ i\ ]{} & X \end{array}$$

The left horizontal maps in the diagram send a cycle class γ on E to $(q_*\gamma, -j_*\gamma)$, and the right horizontal maps send a pair (α, β) to $i_*\alpha + p_*\beta$. The surjectivity of the maps $p_*\colon A_*^\Gamma(X') \to A_*^\Gamma(X)$ and $q_*\colon A_*^\Gamma(E) \to A_*^\Gamma(S)$ are obvious from the assumptions. The exactness of the bottom row was mentioned in [8, §1], and that of the top is proved in the same way. For completeness, we include a proof. Since the cokernels of i_* and j_* are isomorphic (Lemma 1), p_* maps $\mathrm{Im}(j_*)$ onto $\mathrm{Im}(i_*)$. By a diagram chase, it suffices to show that q_* maps $\mathrm{Ker}(j_*)$ onto $\mathrm{Ker}(i_*)$. An element in the kernel of i_* is represented by a cycle α on S which has the form $\sum[\mathrm{div}(f_i)]$ on X, for eigenfunctions f_i on Γ-stable subvarieties W_i of X. Let W_i' be a closed Γ-stable subvariety of X' such that p maps W_i' birationally onto W_i. Then f_i defines a rational eigenfunction f_i' on W_i', and the cycle $\alpha' = \sum[\mathrm{div}(f_i')]$ is supported on E (since its restriction to $X' \smallsetminus E = X \smallsetminus S$ vanishes); α' is equivalent to zero in $Z_*^\Gamma(X')$, so α' represents a class in the kernel of j_*, and, by construction, $q_*(\alpha') = \alpha$. □

We claim next that, given any Γ-scheme X, there is a morphism $p\colon X' \to X$ satisfying the conditions of Lemma 2, such that X' is a disjoint union of Γ-stable open subvarieties of normal projective Γ-varieties, and such that the dimensions of S and $E = p^{-1}(S)$ are strictly smaller than the dimension of X. Theorem 1 will then follow by induction on the dimension of X. Indeed, the isomorphism of the theorem will be known for X' by Hirschowitz's theorem, Lemma 1, and the induction on the dimension; the isomorphism will be known for S and E by induction on the dimension, and then Lemma 2 implies the isomorphism for X. The essential point for proving the claim is the theorem of Sumihiro [21] that any Γ-variety V has an equivariant Chow cover, i.e., there is a proper equivariant morphism $\widetilde{V} \to V$, with $\widetilde{V}$ a Γ-stable open subvariety of a normal projective Γ-variety, and a Γ-stable open subvariety U of V over which this morphism is an isomorphism. Given an arbitrary X, start by constructing $X_0 \to X$, with X_0 a disjoint union of equivariant Chow covers of the irreducible components of X. Let Y_1 be a Γ-stable closed subscheme of X of dimension smaller than that of X, so that $X_0 \to X$ is an isomorphism on the complement of Y_1. Then repeat the process on Y_1, constructing an equivariant Chow cover $X_1 \to Y_1$; take $Y_2 \subset Y_1$ so that this map is an isomorphism off Y_2, and take an equivariant Chow cover $X_2 \to Y_2$. This process stops after at most $\dim(X)$ steps, and the variety X' can be taken to be the disjoint union of these X_i, with the induced morphism to X, and one can take $S = Y_1$. The conditions of Lemma 2 are clear, since every subvariety of X intersects in an open subset with one of the relatively open sets over which some $X_i \to Y_i$ is an isomorphism.

The deduction of the corollary to Theorem 1 from the theorem is exactly as in [3, §1.3]: for (i) one uses the existence of a Chow cover to reduce to the case where X is projective; then moving in a Chow variety of cycles to a point (cycle) fixed by the solvable group; (ii) follows from the fact that the group of cycles algebraically equivalent to zero modulo those rationally equivalent to zero is divisible.

3. Proof of Theorem 2

Now we assume that Γ acts on X with only finitely many orbits. Let Y be an arbitrary scheme, and take the trivial action of Γ on Y. Theorem 2 will be an easy consequence of the following lemma (for which the solvability of Γ is irrelevant).

Lemma 3.

(a) *Every Γ-stable closed subvariety of $X \times Y$ has the form $V \times W$, where V is a Γ-stable closed subvariety of X and W is a closed subvariety of Y.*

(b) *For such V and W, every eigenfunction for Γ in the field of rational functions on $V \times W$ has the form $f \cdot g$, where f is an eigenfunction for Γ in $R(V)$ and g is in $R(W)^*$.*

Proof. To prove (a), let Z be a Γ-stable closed subvariety of $X \times Y$. For each orbit closure V of X, let Z_V be the set of y in Y such that $V \times \{y\}$ is an irreducible component of $Z \cap (X \times \{y\})$. The projection of Z into Y is the union of the sets Z_V, each of which is a constructible subset of Y. One of the Z_V must be dense in the projection, since there are only finitely many Γ-orbits on X. This implies that $V \times Z_V$ is dense in Z, and it follows that $Z = V \times W$, where W is the closure of Z_V in Y.

For (b), let h be an eigenfunction for Γ in $R(V \times W)$, with χ its corresponding character. There is an open set $U \subset W$ of points w such that the restriction of h to $V \times \{w\}$ is a nonzero rational function h_w on V; this h_w is an eigenfunction in $R(V)$ with the same character χ. Fix $w_0 \in U$, and set $f = h_{w_0}$. For any $w \in U$,

h_w/f is an eigenfunction with trivial character. Since Γ has a dense orbit in V, such an eigenfunction is a constant $g(w)$. This function g is a rational function on W, and $h = f \cdot g$, as required. $\square$

Consider the commutative diagram

$$\begin{array}{ccc} A_*^\Gamma X \otimes A_*^\Gamma Y & \longrightarrow & A_*^\Gamma(X \times Y) \\ \downarrow & & \downarrow \\ A_* X \otimes A_* Y & \longrightarrow & A_*(X \times Y) \end{array}$$

It follows immediately from Lemma 3 that the top horizontal map is an isomorphism. Theorem 1 implies that the vertical maps are isomorphisms, so it follows that the bottom map is an isomorphism.

The Corollary to Theorem 2 follows from a result of Ellingsrud and Strømme [5]: whenever X is a complete nonsingular variety such that the class of the diagonal in $X \times X$ is in the image of $A_*X \otimes A_*X$, then the cycle map from A_*X to H_*X is an isomorphism. The point is that if $[\delta(X)] = \sum u_i \otimes v_i$, then any z in A_*X has the form $\sum (u_i \cdot z) v_i$, which implies that the v_i generate A_*X and H_*X, and that numerical equivalence implies rational equivalence.

4. Proofs of Theorems 3 and 4

These theorems follow quite formally from the Künneth property of Theorem 2. Theorem 3 is a special case of the following proposition.

Proposition. *Suppose X is a complete scheme such that the Künneth map $A_*X \otimes A_*Y \to A_*(X \times Y)$ is an isomorphism for all schemes Y. Then the Kronecker duality maps $A^kX \to \mathrm{Hom}(A_kX, \mathbb{Z})$ are isomorphisms.*

Proof. We construct the inverse map from $\mathrm{Hom}(A_kX, \mathbb{Z})$ to A^kX. Given a homomorphism $\phi\colon A_kX \to \mathbb{Z}$, we must construct an element c_ϕ in A^kX. For every morphism $f\colon Y \to X$ and $m \geq k$, we must construct a homomorphism from $A_m(Y)$ to $A_{m-k}(Y)$. This homomorphism is defined to be the composite

$$\begin{aligned} A_mY \;\longrightarrow\; A_m(X \times Y) \;&=\; \sum \bigl(A_pX \otimes A_{m-p}Y\bigr) \\ &\longrightarrow A_kX \otimes A_{m-k}Y \;\longrightarrow\; \mathbb{Z} \otimes A_{m-k}Y \;=\; A_{m-k}Y. \end{aligned}$$

The first map in this sequence is induced by the inclusion of Y in $X \times Y$ by the graph of f; the second is the isomorphism of Theorem 2; the third is the projection to the factor of the direct sum; the fourth is the tensor product of ϕ with the identity on $A_{m-k}Y$; and the last is the usual identification of $\mathbb{Z} \otimes M$ with M for any abelian group M. To be an element of A^kX, these homomorphisms must satisfy three conditions of compatibility: with proper push-forward (resp. flat pull-back, resp. intersection with a divisor) for maps $Y' \to Y \to X$, with $Y' \to Y$ proper (resp. flat, resp. determined by intersection with a divisor); see [6, §17] for precise statements. All of these follow readily from the definitions, since each of the maps in the above sequence is clearly compatible with such operations.

We next verify that the composite $A^kX \to \mathrm{Hom}(A_kX, \mathbb{Z}) \to A^kX$ is the identity. Given $c \in A^kX$, $f\colon Y \to X$, and $z \in A_mY$, let $\gamma_f\colon Y \to X \times Y$ be the graph of f, and let π_1 and π_2 be the two projections from $X \times Y$ to X and Y. Since

$\pi_1 \circ \gamma_f = f$ and $\pi_2 \circ \gamma_f = id_Y$, the fact that operational classes commute with proper push-forward implies that

$$f^*c \cap z \quad = \quad (\pi_2)_*(\gamma_f)_*\big(\gamma_f{}^*\pi_1{}^*c \cap z\big) \quad = \quad (\pi_2)_*\big(\pi_1{}^*c \cap (\gamma_f)_*(z)\big).$$

Now write $(\gamma_f)_*(z) = \sum u_i \otimes v_i$ with $u_i \in A_{p(i)}X$ and $v_i \in A_{m-p(i)}Y$. Since c commutes with flat pull- back, $\pi_1{}^*c \cap (u_i \otimes v_i) = (c \cap u_i) \otimes v_i$. The projection $(\pi_2)_*$ maps such a class to 0 unless $p(i) = k$, and in this case, $(\pi_2)_*\big((c \cap u_i) \otimes v_i\big) = \deg(c \cap u_i)\, v_i$. Therefore $f^*c \cap z$ is the sum of the terms $\deg(c \cap u_i)\, v_i$ for which $p(i) = k$. This shows that c can be recovered from the functional $\deg(c \cap \cdot)$ on A_kX by applying the above sequence of three maps, which is the required assertion. The same calculation shows that the other composite $\mathrm{Hom}(A_kX, \mathbb{Z}) \to A^kX \to \mathrm{Hom}(A_kX, \mathbb{Z})$ is the identity as well. □

For $c \in A^kX$, $z \in A_kX$, we write $c(z)$ for $\deg(c \cap z)$. The first corollary was proved in the proof of the proposition.

Corollary 1. *Let $f\colon Y \to X$, $c \in A^kX$, $z \in A_mY$. Suppose $(\gamma_f)_*(z) = \sum u_i \otimes v_i$ with $u_i \in A_{p(i)}X$, $v_i \in A_{m-p(i)}Y$. Then*

$$f^*c \cap z \quad = \quad \sum_{p(i)=k} c(u_i)\, v_i\,.$$

Corollary 2. *Let $c \in A^kX$, $c' \in A^lX$, $z \in A^{k+l}X$, and write $\delta_*(z) = \sum u_i \otimes v_i$ with $u_i \in A_{p(i)}X$, $v_i \in A_{k+l-p(i)}X$. Then*

$$(c \cup c')(z) \quad = \quad \sum_{p(i)=k} c(u_i)\, c'(v_i)\,.$$

Proof. We apply the first corollary with f the identity on X, $\gamma_f = \delta$ the diagonal embedding. Since the cup product in A^*X is defined by composition of the corresponding operators, and since $\delta_*(z) = \sum v_i \otimes u_i$, (by permuting the two factors), one has

$$(c \cup c') \cap z \;=\; c \cap (c' \cap z) \;=\; c \cap \left(\sum_{p(i)=k} c'(v_i) u_i\right) \;=\; \sum_{p(i)=k} c'(v_i)(c \cap u_i),$$

and the corollary follows by taking the degrees of both sides. □

Parts (a) and (b) of Theorem 4 are special cases of Corollaries 1 and 2, respectively.

It may be remarked that the general proof that the rings A^*X are commutative relies on resolution of singularities. For the varieties considered in Theorem 4, however, the commutativity is obvious from the formula in (b), given the symmetry of the diagonal embedding.

For spherical varieties, at least in characteristic zero, another proof of Theorem 3 can be given by constructing $X' \to X$ as in Lemma 2, but using equivariant resolution of singularities, and then using the cohomology version of the exact sequence of that lemma:

$$0 \longrightarrow A^*X \longrightarrow A^*S \oplus A^*X' \longrightarrow A^*E$$

which has been proved by Kimura ([14, Thm. 2.3]); a similar induction reduces the theorem to the case when X is smooth and projective, where one knows that the Chow groups and topological homology groups are the same.

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