Experimentation at the Frontier of Reality in Schubert Calculus

Special Session on Experimental Mathematics

5 January 2009, AMS National Meeting

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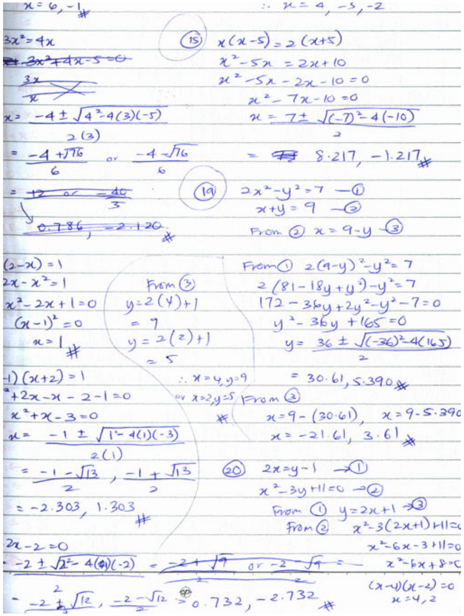
With Luis-David Garcia-Puente, Christopher Hillar, James Ruffo, Zach Teitler, Nickolas Hein, Corey Irving, and Abraham Martin del Campo-Sanchez. Also thanks to Yuval Sivan and Evgenia Soprunova.

Computer-based experimentation

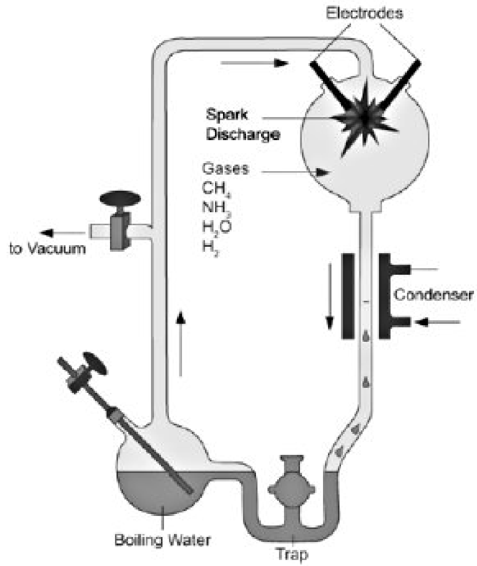
Computer-based experimentation in mathematics takes many forms.



Sometimes, a computer is high-powered scratch paper.

Computer-based experimentation

Computer-based experimentation in mathematics takes many forms.



Sometimes, a computer is used to find an important example or to prove a concept.

A third paradigm

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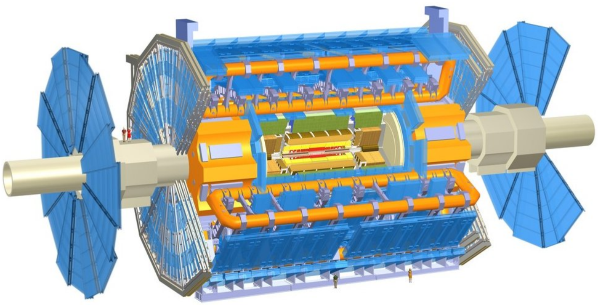

Sometimes, computer expermentation is at a completely different level.

I'll discuss an instance of this third paradigm in which we search for, discover, refine, and test an interesting mathematical conjecture on a very large scale.

Wronski map and MTV Theorem

The Wronskian of degree-d polynomials $f_0, \ldots, f_n \in \mathbb{C}[t]$ is

$$Wr \;:=\; \det \begin{pmatrix} f_0(t) & f_1(t) & \cdots & f_n(t) \\ f_0'(t) & f_1'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)}(t) & f_1^{(n)}(t) & \cdots & f_n^{(n)}(t) \end{pmatrix} .$$

Up to scalar, Wr depends only on the linear span P of the f_i.

Theorem. (Mukhin, Tarasov, Varchenko) *If $Wr(P)$ has only real roots, then P has a basis of real polynomials.*

Equivalently, "Geometric problems in Schubert calculus on a Grassmannian involving tangent flags have only real solutions."

(The form of the statement is important, not its details.)

Shapiro Conjecture

In 1994 Boris and Michael Shapiro conjectured: "Geometric problems in Schubert calculus involving tangent flags have only real solutions."

Originally considered too strong to be true, but became well-known & studied due to overwhelming experimental/computational evidence, conducted on Grassmannians.

S., *Real Schubert Calculus*, Exp. Math., 9, (2000), 161–182.

The story of this conjecture and its proof are the subject of my Current Events Bulletin talk on Wednesday, 7 January at 4:00 PM.

S., *Frontiers of Reality in Schubert Calculus*, AMS Current Events Bulletin, 2009.

Monotone Conjecture

The Shapiro Conjecture is false for flag varieties—computer experimentation revealed a shockingly simple counterexample. Further experimentation suggested a refinement, the Monotone Conjecture.

Sivan, *et. al*, *Experimentation and conjectures in the real Schubert calculus for flag manifolds*, Exp. Math., 15, (2006), 199–221.

We studied this with 15.76 GHz-years of computing, solving 520,420,135 polynomial systems, representing 1126 Schubert problems on 29 different flag manifolds.

Eremenko, et. al proved a special case of the Monotone Conjecture. Interpreting it for Grassmannians leads to a new Secant Conjecture involving *disjoint secant flags*. The Shapiro Conjecture is a limiting case of this Secant Conjecture.

Secant Conjecture

With 4 Ph.D. students and 3 postdocs, we wrote software to study this Secant Conjecture on personal computers and a 1.1 teraflop Beowulf cluster whose day job is Calculus instruction.

The automated computation is organized from a MySQL database.

Processors query the database for problems to solve, record what is computing, and update the results after successful computation.

The computation is robust—it automatically recovers from failures, and is repeatable using fixed random seeds and a standard pseudorandom number generator.

Its progress is monitored from the web and fine-tuned with MySQL browsers and other software tools we have written.

Status of experiment

To date, we have

— used over 105 GHz-years of computation.

— solved over 460 million polynomial systems

Much more is planned.

This includes using a second supercomputer to verify our results.

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