Hopf Structures on Planar Binary Trees

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Work with Marcelo Aguiar,
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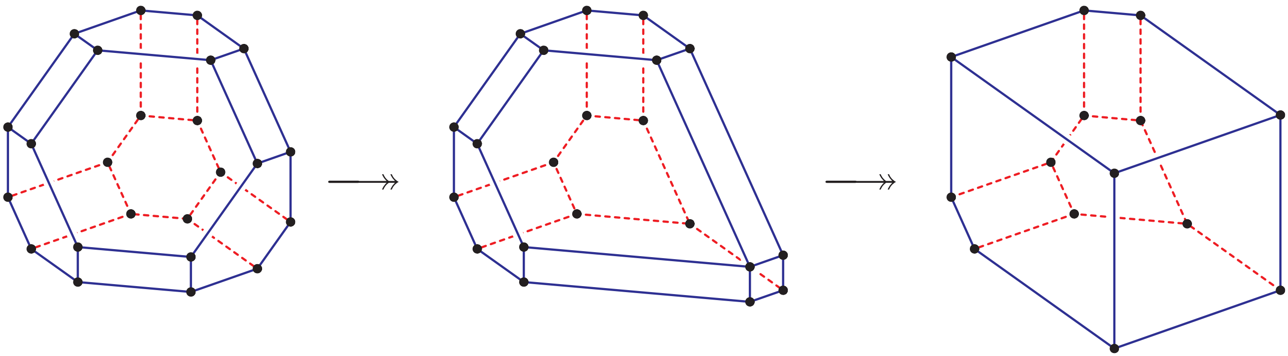


Hopf algebras in combinatorics

A *Hopf algebra* H is an algebra whose linear dual is also an algebra, with some compatibility between the two structures.

This means that H has a coassociative *coproduct*, $\Delta\colon H \longrightarrow H \otimes H$, which is an algebra homomorphism.

Joni and Rota ('79): coproducts are natural in combinatorics; they encode the disassembly of combinatorial objects.

Today, I'll discuss some old and new Hopf structures based on trees.

See also the poster by Aaron Lauve.

(Hopf) algebra of symmetric functions

$Sym := \mathbb{Q}[h_1, h_2, \ldots\,]$ is the (Hopf) algebra of symmetric functions. (Newton, Jacobi (1841))

- Graded with bases indexed by partitions.
- Hopf structures described combinatorially.
- Self-dual.
- $\Delta h_k = \displaystyle\sum_{i+j=k} h_i \otimes h_j$.

$\longrightarrow$ Perhaps the most fundamental object in algebraic combinatorics.

Sym

Quasi-symmetric functions

$QSym$: Quasi-symmetric functions. (Gessel '83)

- Introduced for combinatorial enumeration.
- Aguiar-Bergeron-S.: Universal target for combinatorial generating functions.
- Bases F_α, M_α for α a composition.

$$\Delta M_{(a_1,\dots,a_p)} = \sum_{i=0}^{p} M_{(a_1,\dots,a_i)} \otimes M_{(a_{i+1},\dots,a_p)}$$

$$\begin{tikzcd} & QSym \\ Sym \arrow[ur] & \end{tikzcd}$$

$\Rightarrow$ Primitives are $\{M_{(n)} \mid n > 0\}$ and $QSym$ is cofree.

Non-commutative symmetric functions

$NSym \ := \ \mathbb{Q}\langle h_1, h_2, \ldots \rangle$. (Gel'fand, Krob, Lascoux, Leclerc, Retakh, Thibon '95)
(also Malvenuto-Reutenauer '95)

- Related to Solomon's descent algebra.
- Graded dual to $QSym$,
  (product and coproduct are dual).

$$\Delta h_k \ = \ \sum_{i+j=k} h_i \otimes h_j \ .$$

$$\begin{tikzcd}
NSym \arrow[dr] & & QSym \\
& Sym \arrow[ur] &
\end{tikzcd}$$

Malvenuto-Reutenauer Hopf algebra

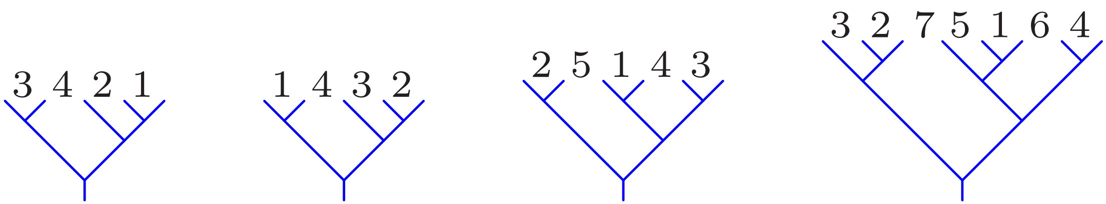
$\mathfrak{S}Sym$: Malvenuto-Reutenauer Hopf algebra of permutations ('95).

- Self-dual Hopf algebra.
- Basis $\{F_w \mid w \in \mathfrak{S}_n,\ n \geq 0\}$ indexed by permutations.

$$\begin{tikzcd} & \mathfrak{S}Sym \arrow[dr] & \\ NSym \arrow[ur] \arrow[dr] & & QSym \\ & Sym \arrow[ur] & \end{tikzcd}$$

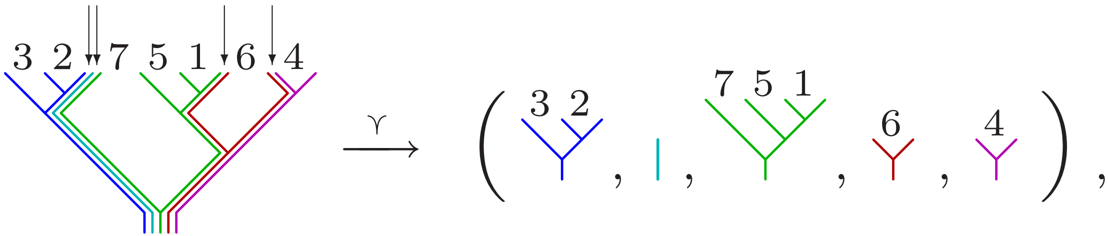
permutations = ordered trees

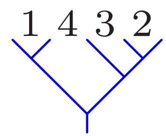
Ordered tree : linear extension of node poset of a planar binary tree. ($w \in \mathfrak{S}_n$ has n nodes.)



Splitting and grafting trees

Split ordered tree w to get an ordered forest, $w \xrightarrow{\curlyvee} (w_0, \ldots, w_p)$,



graft it onto $v \in \mathfrak{S}_p$, to get $(w_0, \ldots, w_p)/v$. If v is 

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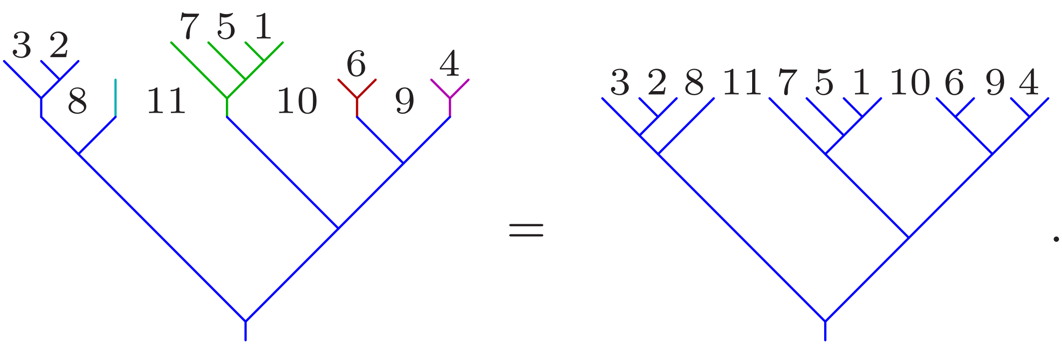
then $(w_0, \ldots, w_p)/v$ is



Hopf structure on $\mathfrak{S}Sym$

For $w \in \mathfrak{S}_n$ and $v \in \mathfrak{S}_p$,

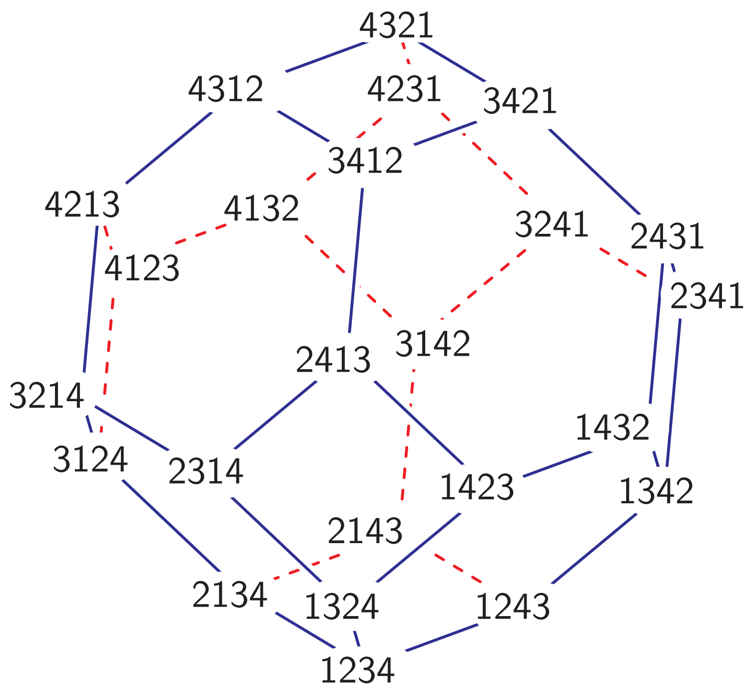
$$
\begin{aligned}
F_w \cdot F_v &= \sum_{w \xrightarrow{\curlyvee} (w_0,\ldots,w_p)} F_{(w_0,\ldots,w_p)/v}\ , \\
1 &= F_{|}\ , \quad \text{and} \\
\Delta F_w &= \sum_{w \xrightarrow{\curlyvee} (w_0,w_1)} F_{w_0} \otimes F_{w_1}\ .
\end{aligned}
$$

Second basis for $\mathfrak{S}Sym$

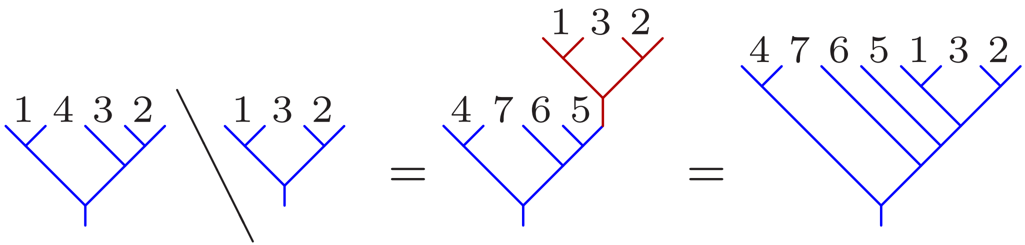
Weak order on $\mathfrak{S}_n$ has covers $w \lessdot (i, i{+}1)w$ if i is before $i{+}1$ in w.

Use Möbuis function $\mu(\cdot,\cdot)$ to define a second basis,

$$M_w \;:=\; \sum_v \mu(w,v) F_v.$$



Primitives and indecomposable trees



Prune w along its rightmost branch with all nodes above the cut smaller than all those below to get $w = u \backslash v$.

w is *indecomposable* if only trivial prunings are possible.

$w \in \mathfrak{S}_n$ is uniquely pruned $w = u_1 \backslash \cdots \backslash u_p$ into indecomposables.

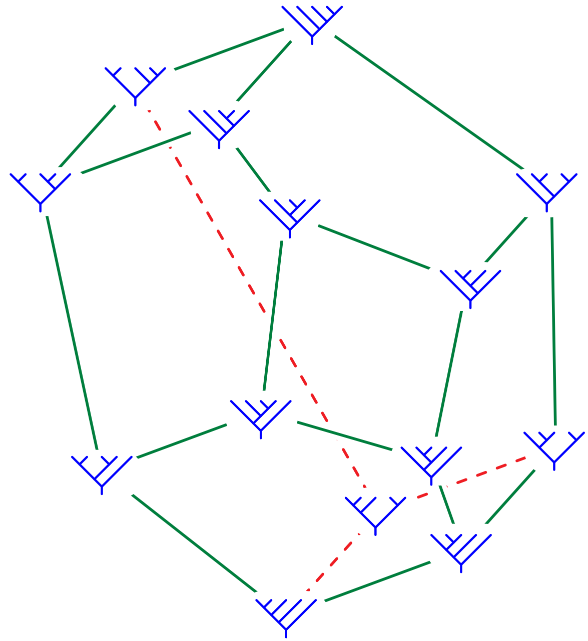
Theorem. (Aguiar-S.) $\displaystyle \Delta M_w \;=\; \sum_{w=u\backslash v} M_u \otimes M_v\ .$

$\Rightarrow$ Primitives are $\{M_w \mid w \text{ is indecomposable}\}$ and $\mathfrak{S}Sym$ is cofree (known previously, but not so explicitly).

Planar binary trees

$\mathcal{Y}_n$:= planar binary trees with n nodes.

Forgetful map $\tau\colon \mathfrak{S}_n \to \mathcal{Y}_n$ induces *Tamari order*, (child nodes move from left to right across their parent), with 1-skeleton the associahedron.



τ induces constructions on trees:

splitting $t \xrightarrow{\curlyvee} (t_0, \ldots, t_p)$,

grafting $(t_0, \ldots, t_p)/s$, and

pruning $t = r \backslash s$ (cut along rightmost branch).

Loday-Ronco Hopf algebra

$\mathcal{Y}Sym$: Loday-Ronco Hopf algebra of trees
Defined in 1998, and related to
Connes-Kreimer Hopf algebra.

$$\begin{tikzcd}
 & \mathfrak{S}Sym \arrow[d] & \\
 & \mathcal{Y}Sym \arrow[dr] & \\
NSym \arrow[ur] \arrow[dr] & & QSym \\
 & Sym \arrow[ur] &
\end{tikzcd}$$

- Self-dual Hopf algebra.
- Basis $\{F_t \mid t \in \mathcal{Y}_n,\ n \geq 0\}$ of trees.
- $F_w \mapsto F_{\tau(w)}$ defines a map $\boldsymbol{\tau} : \mathfrak{S}Sym \to \mathcal{Y}Sym$, which induces structure of Hopf algebra on $\mathcal{Y}Sym$:

For $s \in \mathcal{Y}_p$,
$$F_t \cdot F_s \;=\; \sum_{t \xrightarrow{\curlyvee} (t_0,\ldots,t_p)} F_{(t_0,\ldots,t_p)/s}\ ,$$

$$1 = F_{|}\,, \quad \text{and} \quad \Delta F_t \;=\; \sum_{t \xrightarrow{\curlyvee} (r,s)} F_r \otimes F_s\ .$$

Möbius inversion and primitives

$\mu(\cdot,\cdot)$ = Möbius function of Tamari order.

Define $M_t := \sum_s \mu(t,s) F_s$, a second basis for $\mathcal{Y}Sym$.

Theorem. (Aguiar-S.)

$$\boldsymbol{\tau}(M_w) = \begin{cases} M_{\tau(w)} & \text{if } w \text{ is 132-avoiding} \\ 0 & \text{otherwise} \end{cases}, \quad \text{and}$$

$$\Delta M_t = \sum_{t=r\backslash s} M_r \otimes M_s \,.$$

$\Rightarrow$ Primitives are $\{M_t \mid t \text{ is indecomposable}\}$ and $\mathcal{Y}Sym$ is cofree.
(known previously, but not so explicitly).

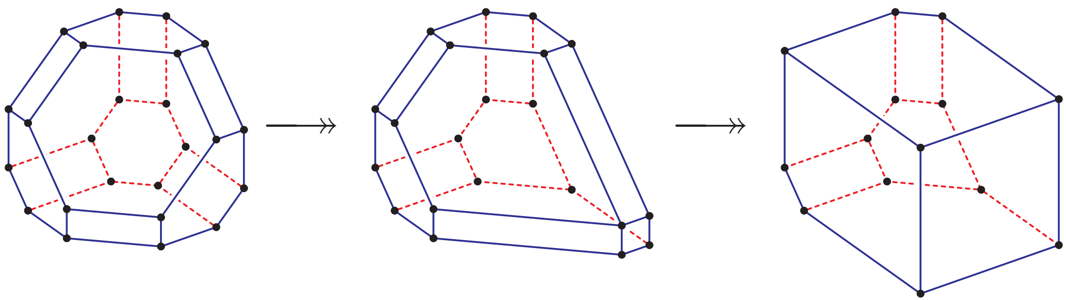
Stasheff's multiplihedron

Stasheff, who introduced the associahedron to encode higher homotopy associativity of H-spaces ('63), introduced the *multiplihedron* to encode higher homomotopy associativity for maps of H-spaces ('70).

Saneblidze and Umble ('04) described maps of cell complexes

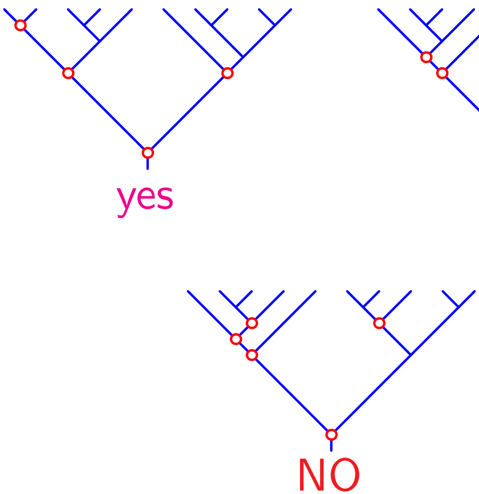
$$\text{permutahedra} \longrightarrow \text{multiplihedra} \longrightarrow \text{associahedra}$$

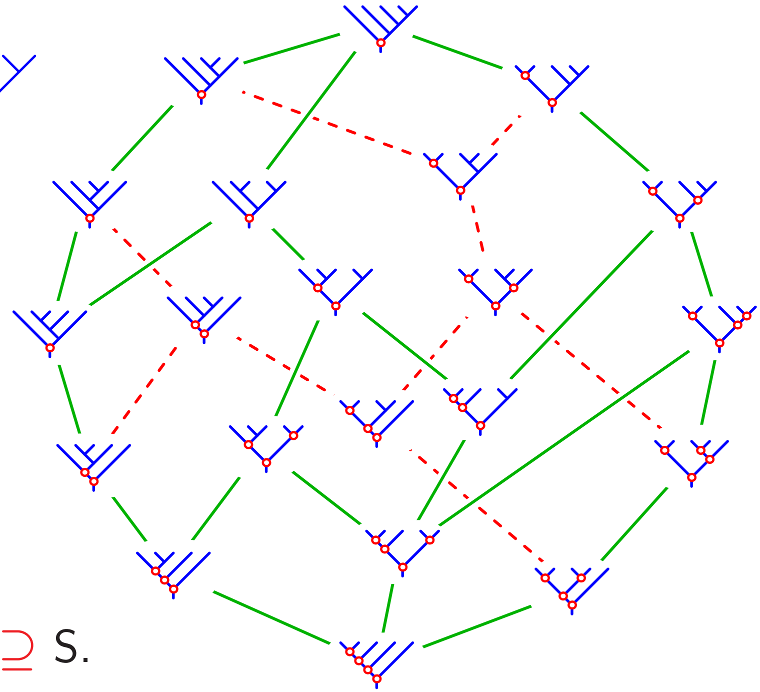
Forcey ('08) gave a polytopal realization of the multiplihedra.



Bi-leveled trees

A *bi-leveled tree* (t, T) in $\mathcal{M}_n$ is a tree $t \in \mathcal{Y}_n$ with an upper order ideal T of its node poset having leftmost node as a minimal element.





$\mathcal{M}_4$

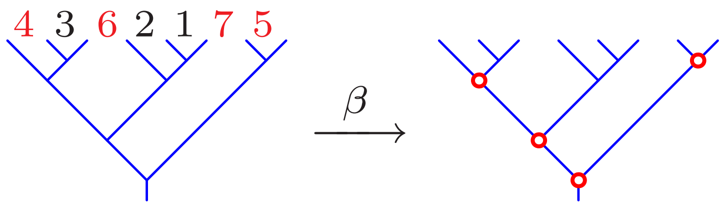
$(t, \mathsf{T}) \leq (s, \mathsf{S})$ if $t \leq s$ and $\mathsf{T} \supseteq \mathsf{S}$.

Poset maps

The map $\tau\colon \mathfrak{S}_n \to \mathcal{Y}_n$ factors through $\mathcal{M}_n$.

Define $\beta\colon \mathfrak{S}_n \to \mathcal{M}_n$ by

$$\beta(w) \ := \ \big(\tau(w),\ w^{-1}\{w(1), 1+w(1), \ldots, n-1, n\}\big).$$



$(t, \mathsf{T}) \mapsto t$ gives poset map $\phi\colon \mathcal{M}_n \to \mathcal{Y}_n$ and the composition

$$\mathfrak{S}_n \xrightarrow{\beta} \mathcal{M}_n \xrightarrow{\phi} \mathcal{Y}_n$$

is the map $\tau\colon \mathfrak{S}_n \to \mathcal{Y}_n$.

The $\mathfrak{S}Sym$-module $\mathcal{M}Sym$

$\mathcal{M}Sym$: graded vector space with basis $\{F_b \mid b \in \mathcal{M}_n,\ n \geq 0\}$.

$F_w \mapsto F_{\beta(w)}$ and $F_b \mapsto F_{\phi(b)}$ for $w \in \mathfrak{S}_n$ and $b \in \mathcal{M}_n$ induce linear surjections $\mathfrak{S}Sym \xrightarrow{\boldsymbol{\beta}} \mathcal{M}Sym \xrightarrow{\boldsymbol{\phi}} \mathcal{Y}Sym$.

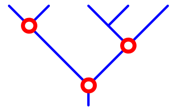
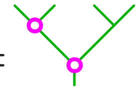
For $b = \beta(u) \in \mathcal{M}_n$ and $c = \beta(v) \in \mathcal{M}_m$, set
$F_b \cdot F_c := \beta(F_w \cdot F_u)$.

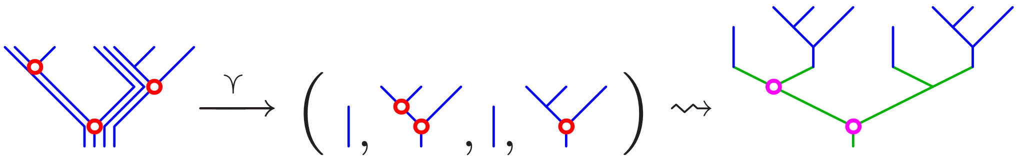
Theorem. This is well-defined and gives an associative product, so that $\mathcal{M}Sym$ is a graded algebra and $\boldsymbol{\beta}$ is an algebra homomorphism. Furthermore, $\mathcal{M}Sym$ is an $\mathfrak{S}Sym$-$\mathfrak{S}Sym$ bimodule,

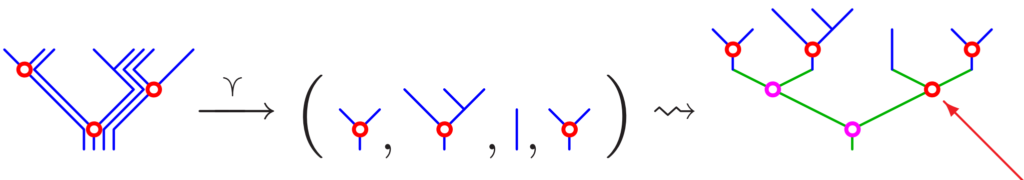
$$F_w.F_b.F_u \;=\; F_{\beta(w)} \cdot F_b \cdot F_{\beta(u)}\,.$$

The product, combinatorially

$$F_b \cdot F_c \;=\; \sum_{b \xrightarrow{\curlyvee} (b_0,\ldots,b_p)} F_{\overline{(b_0,\ldots,b_p)/c}}\,.$$

Here are two different graftings for $b =$  , $c =$  .



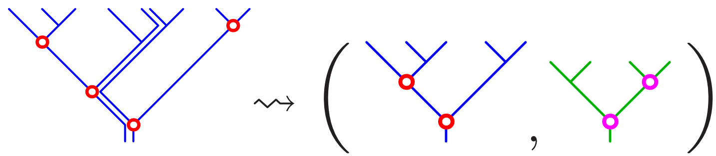


If $b_0 = |$, the order ideal is that of c.

If $b_0 \neq |$, the order ideal is all nodes of c and the order ideal of b.

$\mathcal{Y}Sym$-comodule $\mathcal{M}Sym$

Splitting a bi-leveled tree does not give a pair of bi-leveled trees



The first tree is bi-leveled, but subsequent trees need not be.

Ignoring the order ideal in the second component gives a splitting $b \xrightarrow{\curlyvee} (b_0, t_1)$, where b_0 is bi-leveled and t_1 is an ordinary tree.

Theorem. $F_b \;\longmapsto\; \displaystyle\sum_{b \xrightarrow{\curlyvee} (c,t)} F_c \otimes F_t$ gives a coaction,

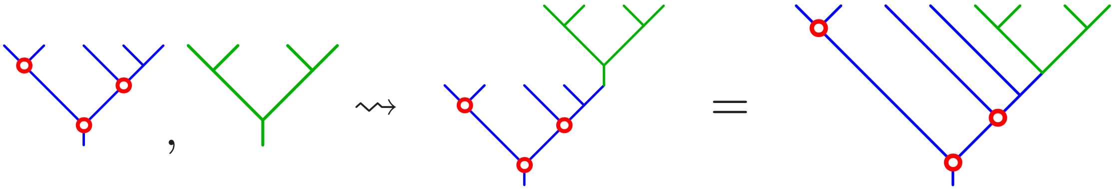
$\boldsymbol{\rho}\colon \mathcal{M}Sym \longrightarrow \mathcal{M}Sym \otimes \mathcal{Y}Sym$, endowing $\mathcal{M}Sym$ with the structure of a $\mathcal{Y}Sym$-comodule. $\boldsymbol{\phi}$ is a comodule map.

(restricts to $\mathcal{M}Sym_+ = \text{Span}\{F_b \mid b \neq |\}$, with structure map $\boldsymbol{\rho}_+$.)

Coinvariants and a second basis

Set $M_b := \sum_c \mu(b,c) F_c$, a second basis.

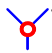
For $c \in \mathcal{M}_n$ and $t \in \mathcal{Y}_m$, we have $c \backslash t \in \mathcal{M}_{n+m}$:



If $b = (t, \mathsf{T})$ we can write $b = c \backslash s$ only when $\mathsf{T} \subset c$.

Theorem. Let $b \in \mathcal{M}_n$ with $n > 0$. In $\mathcal{M}Sym_+$ we have,

$$\boldsymbol{\rho}(M_b) \;=\; \sum_{b = c \backslash t} M_c \otimes M_t \,.$$

$\Rightarrow \{M_{(t,\mathsf{T})} \mid \mathsf{T} \ni \text{rightmost node of } t\}$ spans coinvariants of $\mathcal{M}Sym_+$.
& its subset $\{M_{(t,\mathsf{T})} \mid t \neq$  $\backslash s\}$ spans coinvariants of $\mathcal{M}Sym$.

Covariant consequences

We have the generating series

$$M(q) := \sum_{n\geq 0} |\mathcal{M}_n| q^n\,, \qquad M_+(q) := \sum_{n>0} |\mathcal{M}_n| q^n\,, \quad \text{and}$$

$$Y(q) := \sum_{n\geq 0} |\mathcal{Y}_n| q^n\,, \qquad \text{the Catalan generating series.}$$

$$\mathcal{B}_n := \{(t, \mathsf{T}) \in \mathcal{M}_n \mid \mathsf{T} \ni \text{ rightmost node of } t\} \quad n > 0$$

$$\mathcal{B}'_n := \{(t, \mathsf{T}) \in \mathcal{B}_n \mid t \neq \setminus s\} \cup \{|\}.$$

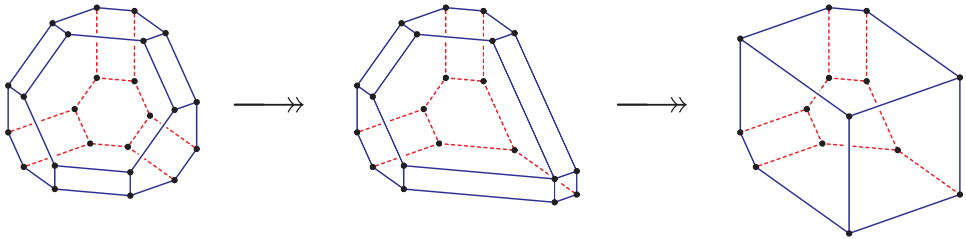
Corollary. $M_+(q)/Y(q) \;=\; qY\big(qY(q)\big) \;=\; \sum_{n>0} |\mathcal{B}_n| q^n$

$$M(q)/Y(q) \;=\; \sum_{n\geq 0} |\mathcal{B}'_n| q^n \qquad \text{(Both are positive!)}$$

Existence of coinvariants $\Rightarrow$ $\mathcal{M}Sym$ must be a $\mathcal{Y}Sym$ Hopf module algebra, which can be understood combinatorially.

Conclusion

The middle polytope of the cellular surjections



corresponds to a type of tree nestled between ordered trees and planar binary trees and gives maps

$$\mathfrak{S}Sym \twoheadrightarrow \mathcal{M}Sym \twoheadrightarrow \mathcal{Y}Sym$$

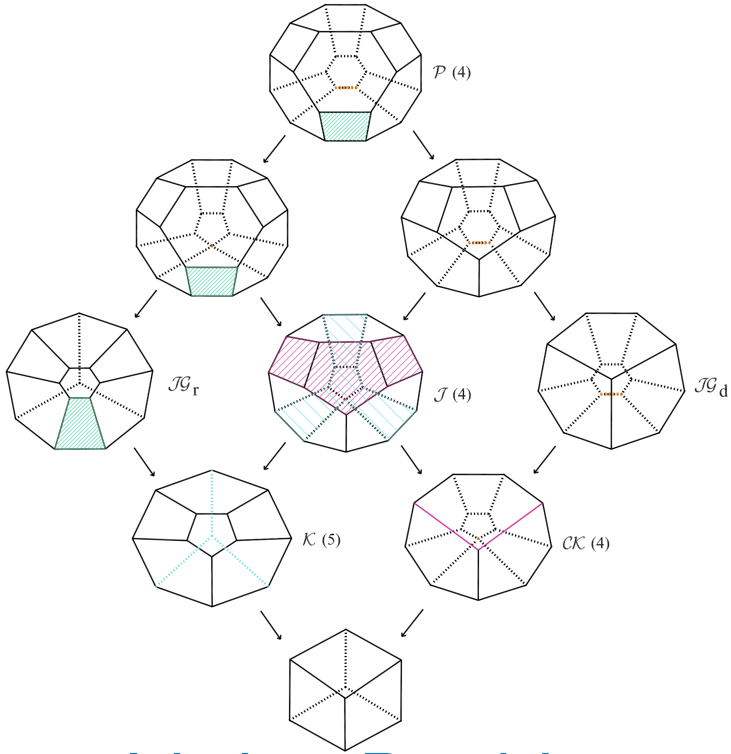
factoring the Hopf algebra map $\mathfrak{S}Sym \twoheadrightarrow \mathcal{Y}Sym$.

The Hopf structures weaken, but do not vanish, for $\mathcal{M}Sym$:

$\mathcal{M}Sym$ is an algebra, a $\mathfrak{S}Sym$-module, and a $\mathcal{Y}Sym$-comodule.

Beyond $\mathcal{M}Sym$

There are many other polytopes/trees to be studied in this way:



Vielen Dank!