Frontiers of Reality in Schubert Calculus

AMS Current Events Bulletin

7 January 2009, AMS National Meeting

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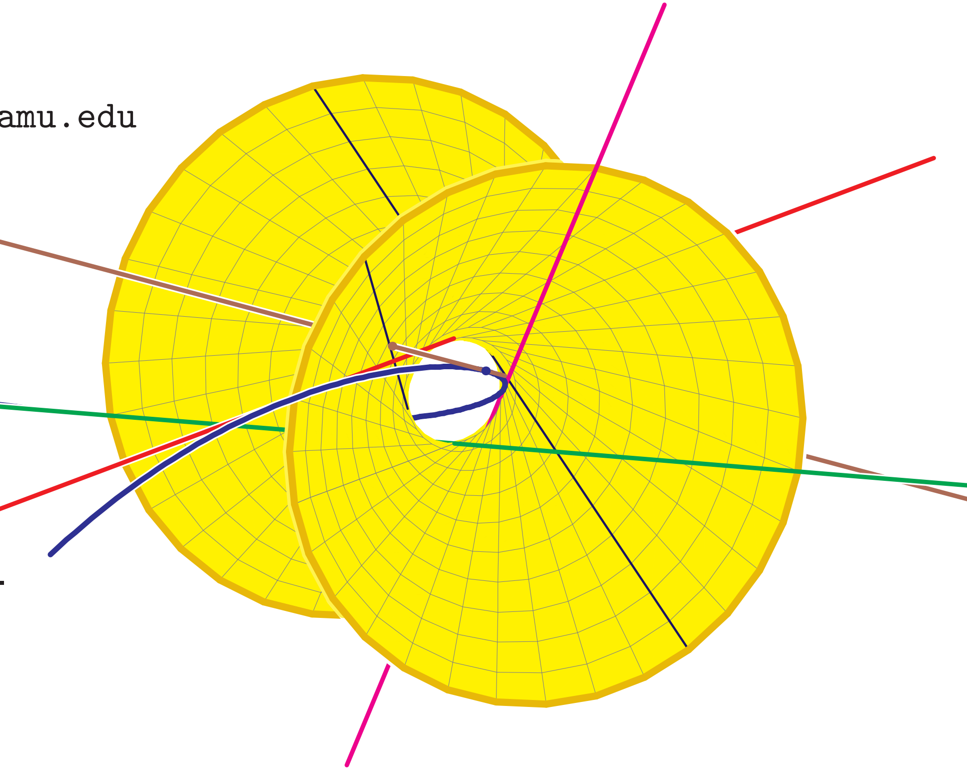
In which the theory
of integrable systems
resolves a compelling
conjecture
in the real Schubert
calculus by reducing it
to an elementary
fact from linear algebra.



Polynomial systems with only real roots

Among the roots of a real univariate polynomial f, some are real and the rest occur in complex-conjugate pairs.

Rarely are all roots of f real.

A primary example that comes to mind is when f is the characteristic polynomial of a real symmetric matrix, which only has real eigenvalues.

Similarly, a first example of a system of multivariate polynomials with only real solutions is the system for the eigenvalues/eigenvectors of a symmetric matrix.

It will turn out that this is the elementary fact from linear algebra behind this reality in Schubert calculus.

Wronski map and MTV Theorem

The Wronskian of degree-d polynomials $f_0, \ldots, f_n \in \mathbb{C}[t]$ is

$$Wr := \det \begin{pmatrix} f_0(t) & f_1(t) & \cdots & f_n(t) \\ f_0'(t) & f_1'(t) & \cdots & f_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)}(t) & f_1^{(n)}(t) & \cdots & f_n^{(n)}(t) \end{pmatrix} .$$

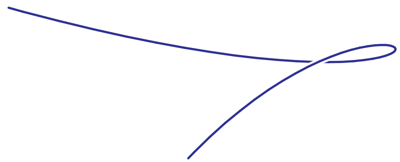
Up to scalar, Wr depends only on the linear span P of the f_i, and only finitely many spans P have a given Wronskian.

Theorem. (Mukhin, Tarasov, Varchenko) *If $Wr(P)$ has only real roots, then P has a basis of real polynomials.*

More generally, "Geometric problems in Schubert calculus on a Grassmannian involving osculating flags have only real solutions."

MTV Theorem in 3–space

Let $\gamma(t) = (t, t^2, t^3)$ be the rational normal curve



A cubic polynomial $f(t)$	$\Longleftrightarrow$	affine function applied to γ
	$\Longleftrightarrow$	affine hyperplane, $f^\perp$
Two polynomials f, g	$\Longleftrightarrow$	two affine hyperplanes
	$\Longleftrightarrow$	a line $f^\perp \cap g^\perp$
$Wr(f,g)(s) = 0$	$\Longleftrightarrow$	the line $f^\perp \cap g^\perp$ meets tangent line to γ at $\gamma(s)$.
$Wr(f,g)$ is a given quartic F	$\Longleftrightarrow$	$f^\perp \cap g^\perp$ meets tangents to γ at the four roots of F

View Animation

Numerical accident ?

The proof begins with a numerical accident.

In 1884, Schubert (essentially) determined that

$$\# Wr^{-1}(F(t)) \;=\; [(n+1)(d-n)]!\,\frac{1!2!\cdots n!}{(d-n)!(d-n+1)!\cdots d!}\,.$$

Call this number $\deg(n,d)$.

$\deg(n,d)$ is also the dimension of the space of invariants

$$\left(\left(\mathbb{C}^{n+1}\right)^{\otimes(n+1)(d-n)}\right)^{\mathfrak{sl}_{n+1}\mathbb{C}}\,.$$

Strengthening this coincidence is at the heart of our story.

A remarkable function

For $i = 0, \ldots, n$ let $y_i(t)$ be a polynomial of degree $(i+1)(d-n)$.
Define the master function

$$\Phi \ := \ \prod_{i=0}^{n} \mathrm{Discr}(y_i) \Big/ \prod_{i=1}^{n} \mathrm{Res}(y_{i-1}, y_i) \ ,$$

where Discr and Res are the classical discriminant and resultant.

Writing Φ in terms of the roots $s_{i,j}$ of y_i gives,

$$\Phi(s) \ = \ \prod_{i=0}^{n} \prod_{j \neq k} (s_{i,j} - s_{i,k})^2 \cdot \prod_{i=1}^{n} \prod_{j,k} (s_{i-1,j} - s_{i,k})^{-1} \ .$$

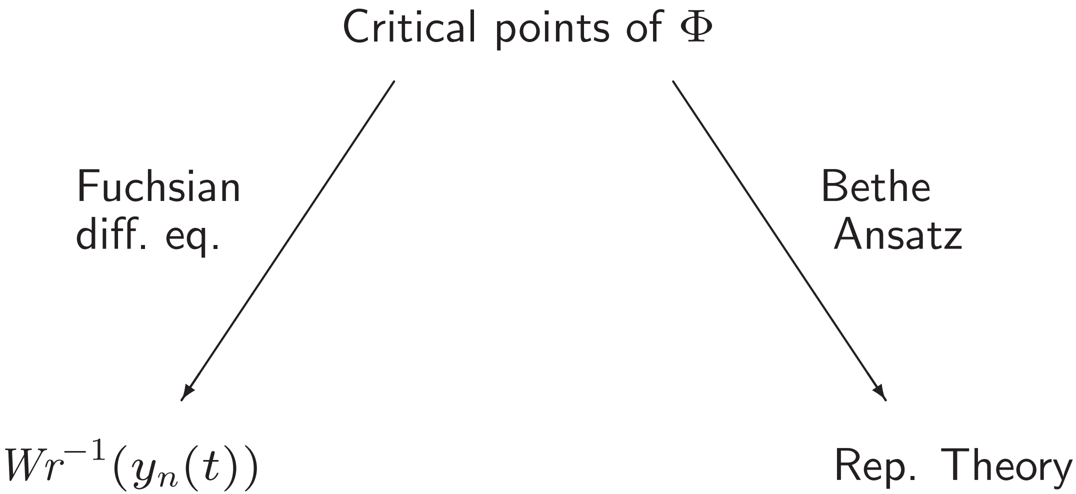
(The exponents come from the Cartan matrix of type A.)

Remarkably, $\deg(n,d)$ counts the (orbits of) critical points of $\Phi(s)$.

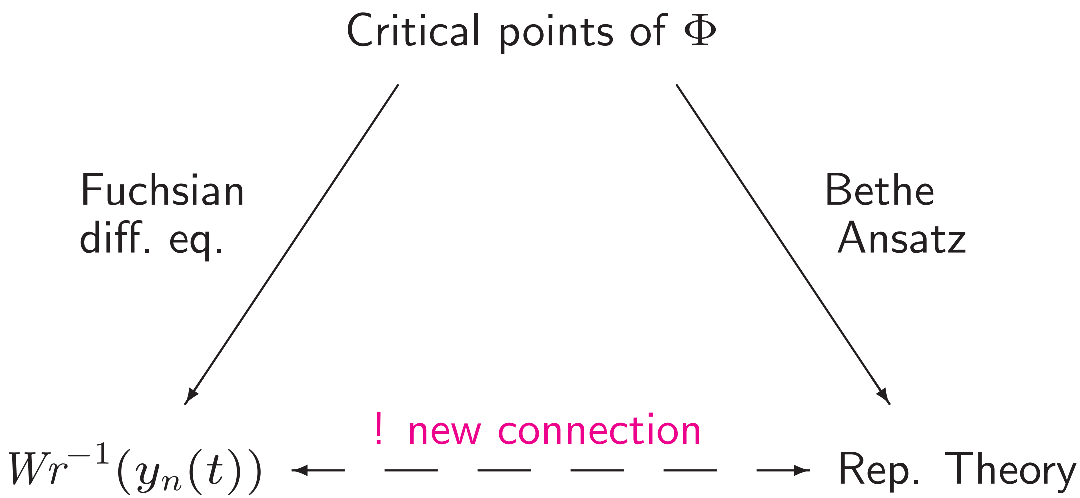
Schematic of proof

Critical points of Φ

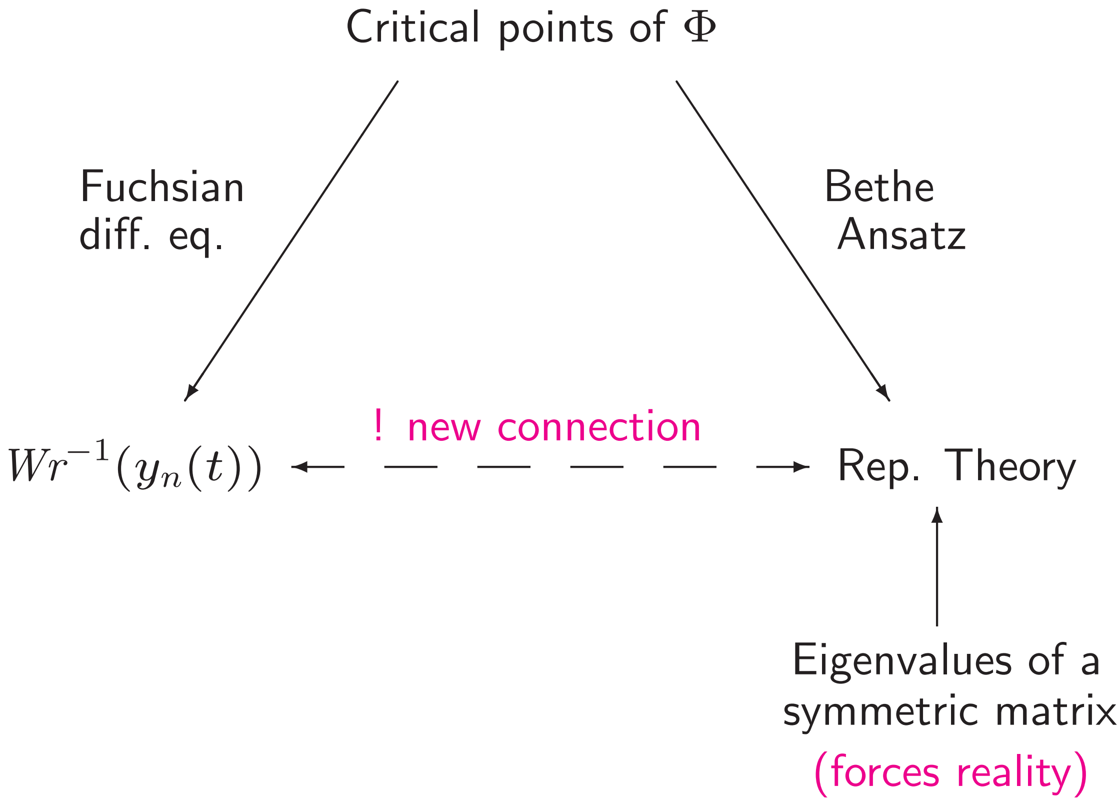
Schematic of proof



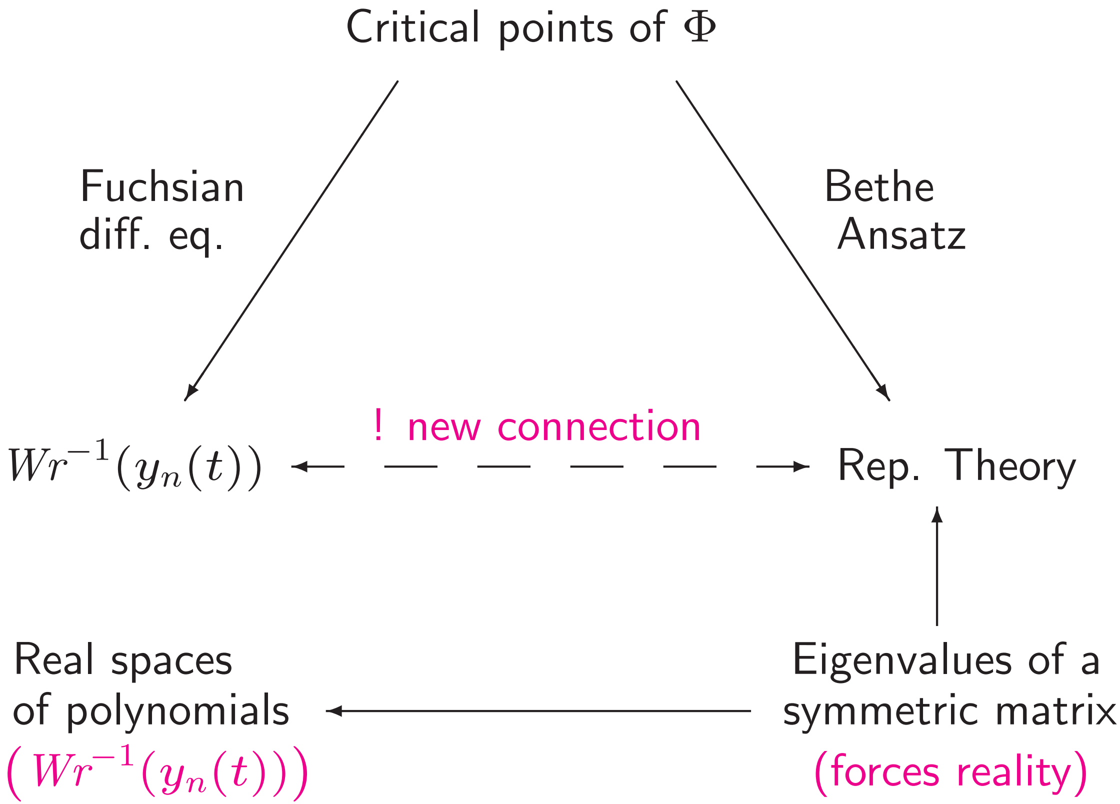
Schematic of proof



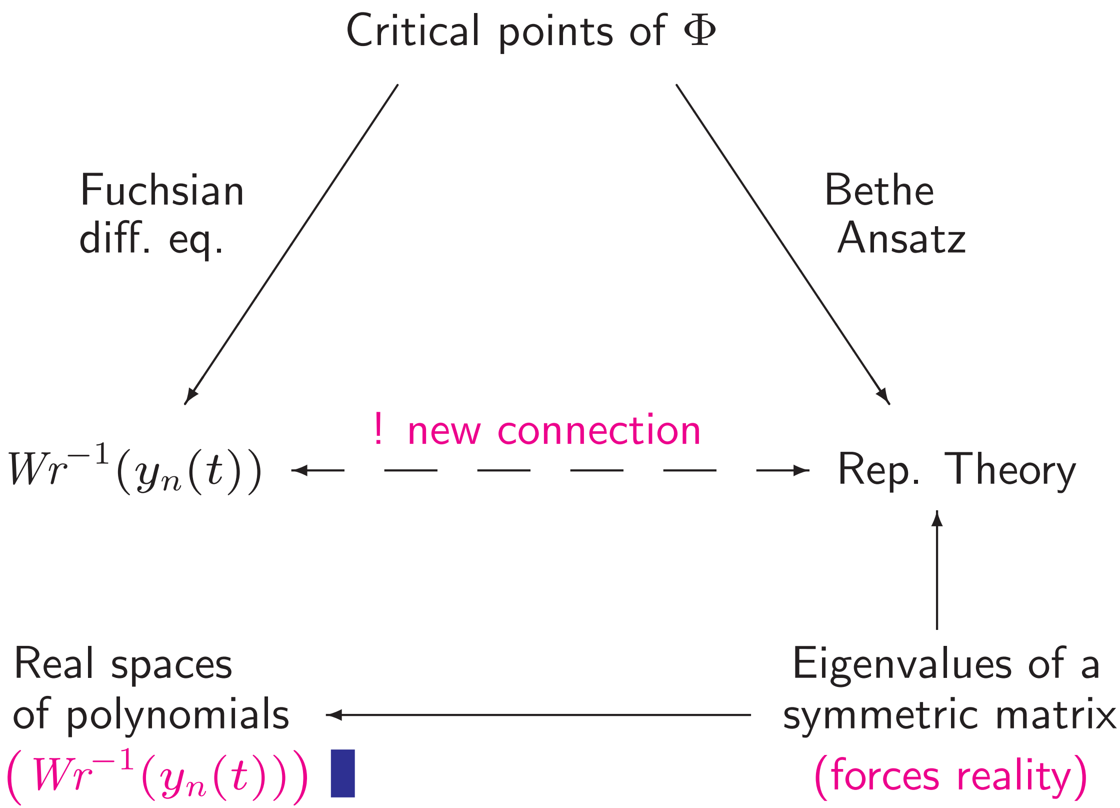
Schematic of proof



Schematic of proof

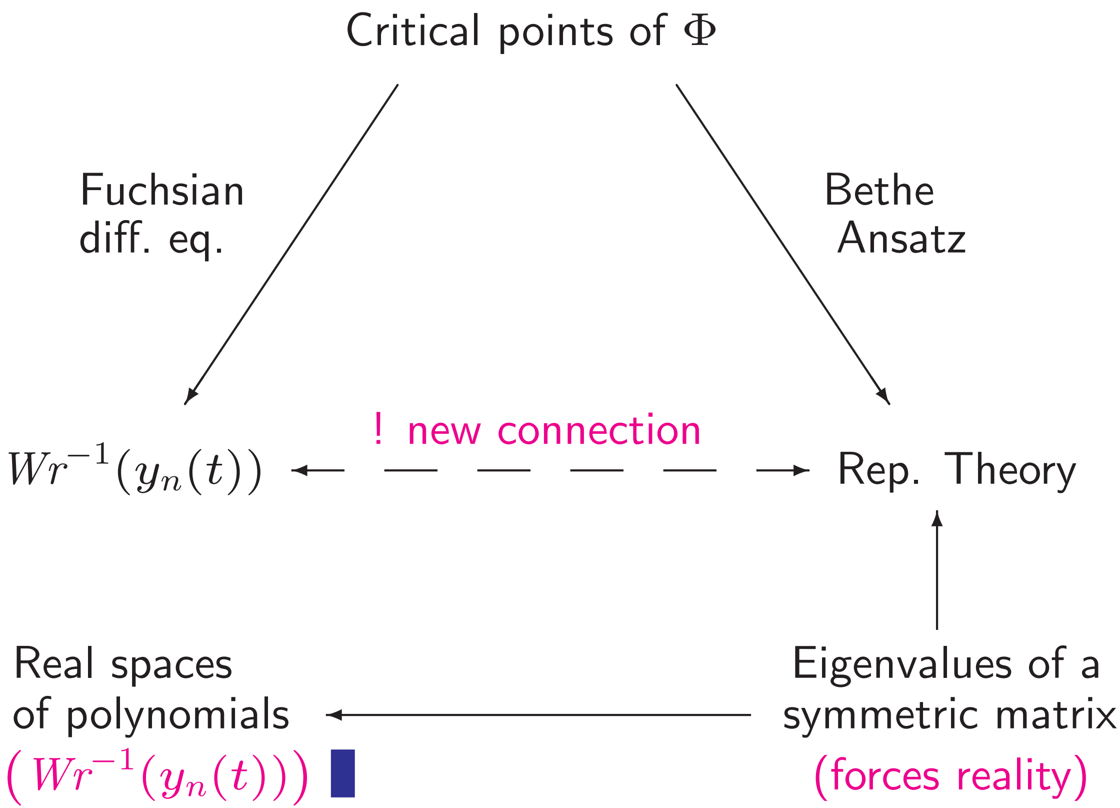


Schematic of proof



Intermission

Reprise: Schematic of proof



Critical points of master function

For $i = 0, \ldots, n$ let $y_i(t)$ be a polynomial of degree $(i{+}1)(d{-}n)$. Recall the master function

$$\Phi \;:=\; \prod_{i=0}^{n} \mathrm{Discr}(y_i) \Big/ \prod_{i=1}^{n} \mathrm{Res}(y_{i-1}, y_i)\ .$$

Fix $y_n(t)$ to be a polynomial of degree $(n{+}1)(d{-}n)$ with roots $\mathbf{s} = (s_1, \ldots, s_{(n+1)(d-n)})$. This will be our Wronski polynomial.

The master function $\Phi_{\mathbf{s}}(\mathbf{x})$ depends on the roots $\mathbf{x}$ of the other y_i.

Let $\mathbf{x}$ be a critical point of $\Phi_{\mathbf{s}}(\mathbf{x})$, and $\mathbf{y} := (y_0, \ldots, y_{n-1})$ the corresponding polynomials whose roots are $\mathbf{x}$.

Theorem. (MV) *There are* $\deg(n, d)$ *such critical points* $\mathbf{y}$.

Spaces of polynomials from $\mathbf{y}$

For polynomials $\mathbf{y} = (y_0, \ldots, y_{n-1})$, with $\deg y_i = (i{+}1)(d-n)$, the *fundamental differential operator* $D_{\mathbf{y}}$ is

$$\left(\tfrac{d}{dt} - \ln'\left(\tfrac{y_n}{y_{n-1}}\right)\right) \cdots \left(\tfrac{d}{dt} - \ln'\left(\tfrac{y_1}{y_0}\right)\right)\left(\tfrac{d}{dt} - \ln'(y_0)\right) .$$

Let $V_{\mathbf{y}}$ be the kernel of $D_{\mathbf{y}}$.

Theorem. (MV)

1. *$V_{\mathbf{y}}$ is a space of polynomials iff $\mathbf{y}$ is a critical point of $\Phi_{\mathbf{s}}$.*
2. *If $f_0, \ldots, f_n$ span $V_{\mathbf{y}}$ with $\deg f_i = d{-}n{+}i$, then*

$$\begin{aligned} y_0 &= f_0\,, \\ y_1 &= Wr(f_0, f_1)\,, \\ &\vdots \\ y_n &= Wr(f_0, f_1 \ldots, f_n)\,. \end{aligned}$$

Periodic Gaudin model

V := dual of vector representation of $\mathfrak{sl}_{n+1}\mathbb{C}$ (also $\mathfrak{gl}_{n+1}\mathbb{C}$). Let $e_{i,j} \in \mathfrak{gl}_{n+1}\mathbb{C}$ be the elementary matrix with 1 in i, j position.

Define an operator $X_{i,j}(t) := \delta_{i,j}\dfrac{d}{dt} - \displaystyle\sum_{k=1}^{(n+1)(d-n)} \frac{e_{i,j}^{(k)}}{t - s_k}$,

where $e_{i,j}^{(k)}$ acts on the kth factor in $V^{\otimes(n+1)(d-n)}$.

Formal conjugate of the expansion of the row determinant of $(X_{i,j})$ is

$$\frac{d^{n+1}}{dt^{n+1}} + K_1(t)\frac{d^n}{dt^n} + \cdots + K_n(t)\frac{d}{dt} + K_{n+1}(t)\,.$$

$K_1(t), \ldots, K_{n+1}(t)$ are the Gaudin Hamiltonians. They form a family of commuting operators on $V^{\otimes(n+1)(d-n)}$, centralizing $\mathfrak{gl}_{n+1}\mathbb{C}$.

Bethe Ansatz for Gaudin model

In the theory of integrable systems, Bethe Ansätze are conjectural methods to find the joint eigenvectors and spectra of families of commuting operators.

As the Gaudin Hamiltonians centralize the action of $\mathfrak{sl}_{n+1}\mathbb{C}$, the Bethe Ansatz also gives a precise way to understand $\left(V^{\otimes(n+1)(d-n)}\right)^{\mathfrak{sl}_{n+1}\mathbb{C}}$.

The idea is to define a (rational) universal weight function

$$\beta \;:\; \begin{array}{c}\text{spaces of roots of}\\ \underbrace{y_0,\,\ldots,\,y_{n-1}}_{\mathbf{x}},\ \underbrace{y_n}_{\mathbf{s}}\end{array} \longrightarrow \text{0-weight space of } V^{\otimes(n+1)(d-n)}\,,$$

and then address for which values $(\mathbf{x},\mathbf{s})$ is $\beta(\mathbf{x},\mathbf{s})$ $\mathfrak{sl}_{n+1}\mathbb{C}$-invariant.

Completeness of the Bethe Ansatz

Theorem. (MTV) *Let* $\mathbf{x}$ *be a critical point of the master function* $\Phi_{\mathbf{s}}$.

1. $\beta(\mathbf{x},\mathbf{s})$ *is well-defined, non-zero, and a joint eigenvector of the Gaudin Hamiltonians.*
2. $\beta(\mathbf{x},\mathbf{s})$ *is a* $\mathfrak{sl}_{n+1}\mathbb{C}$*-invariant.*
3. *When* $\mathbf{s}$ *is general, the vectors* $\beta(\mathbf{x},\mathbf{s})$ *for* $\mathbf{x}$ *a critical point form a basis of* $\left(V^{\otimes(n+1)(d-n)}\right)^{\mathfrak{sl}_{n+1}\mathbb{C}}$.
4. *When* $\mathbf{s}$ *is general, the Gaudin Hamiltonians have simple spectrum.*
5. *The eigenvalues* $\lambda_i(t)$ *of* $K_i(t)$ *on* $\beta(\mathbf{x},\mathbf{s})$ *satisfy*

$$\frac{d^{n+1}}{dt^{n+1}} + \lambda_1(t)\frac{d^n}{dt^n} + \cdots + \lambda_n(t)\frac{d}{dt} + \lambda_{n+1}(t) \;=\; \left(\tfrac{d}{dt} - \ln'\left(\tfrac{y_n}{y_{n-1}}\right)\right)\cdots\left(\tfrac{d}{dt} - \ln'\left(\tfrac{y_1}{y_0}\right)\right)\left(\tfrac{d}{dt} - \ln'(y_0)\right),$$

the fundamental differential operator of the critical point $\mathbf{x}$.

Proof of the Shapiro Conjecture

Usual Euclidean inner product on V induces the Shapovalov form on $V^{\otimes(n+1)(d-n)}$, which is $\mathfrak{sl}_{n+1}\mathbb{C}$-invariant.

Gaudin Hamiltonians are symmetric w.r.t the Shapovalov form.

Therefore, when $\mathbf{s}$ and t are real, their eigenvalues $\lambda_i(t)$ on a vector $\beta(\mathbf{x}, \mathbf{s})$ for a critical point $\mathbf{x}$ are real.

Then the fundamental differential operator is real, and thus its kernel $V_{\mathbf{x}}$ is also real.

This implies the Shapiro Conjecture, as spaces $V_{\mathbf{x}}$ for $\mathbf{x}$ a critical point give all spaces of polynomials whose Wronskian has roots $\mathbf{s}$.

Thank you for your attention!