Khovanskii-Rolle Continuation for Real Solutions

Bounds for real solutions to equations from geometry,

Lecture 2

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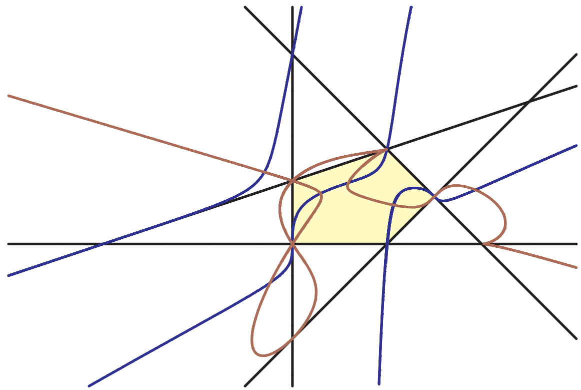


Work with

Dan Bates, Colorado State U

Khovanskii-Rolle Continuation

Khovanskii-Rolle continuation is a new numerical method to compute real solutions.

— Based on proof of fewnomial bounds for real solutions.

— Uses 2 symbolic steps:

1) Gale duality reduces a (potentially high-degree) polynomial system to a system of rational functions on a different space.

2) Reducing this to solving some systems of low-degree polynomials & some path-continuation.

— Complexity is essentially the fewnomial bound.

Gale duality, via example

Suppose we have the system of polynomials,

$$\begin{aligned} v^2w^3 &= 1 - u^2v - uv^2w\,, \\ v^2w &= \tfrac{1}{2} - u^2v + uv^2w\,, \\ uvw^3 &= \tfrac{10}{11}(1 + u^2v - 3uv^2w)\,. \end{aligned} \tag{1}$$

Observe that

$$\begin{aligned} (u^2v)^2 \cdot (v^2w^3)^3 &= (uv^2w)^2 \cdot (v^2w) \cdot (uvw^3)^2 \quad \text{and} \\ (uv^2w)^3 \cdot (v^2w^3) &= (u^2v) \cdot (v^2w)^3 \cdot (uvw^3)\,. \end{aligned}$$

Substituting (1) into this, writing x for u^2v and y for uv^2w, and solving for 0, gives the Gale system of master functions

$$f := x^2(1-x-y)^3 - y^2(\tfrac{1}{2}-x+y)(\tfrac{10}{11}(1+x-3y))^2 = 0\,,$$

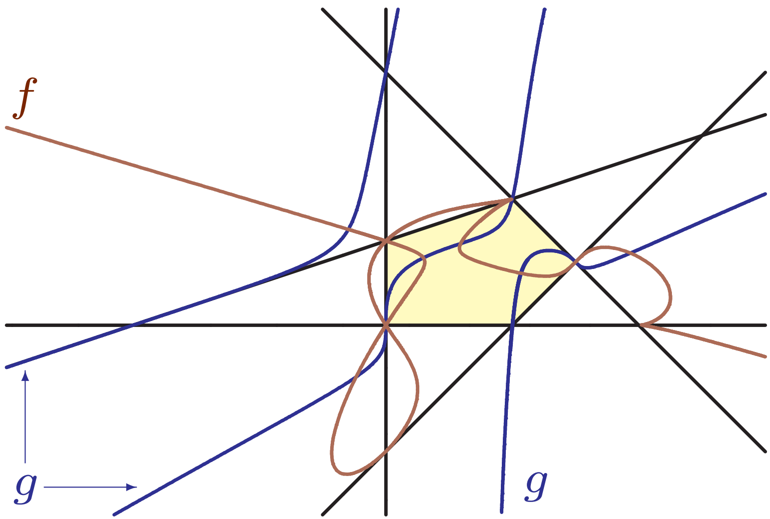
$$g := y^3(1-x-y) \;-\; x(\tfrac{1}{2}-x+y)^3\tfrac{10}{11}(1+x-3y) = 0\,.$$

Gale duality, continued

The original system is equivalent to the Gale system

$$f := x^2(1-x-y)^3 - y^2(\tfrac{1}{2}-x+y)(\tfrac{10}{11}(1+x-3y))^2 = 0\,,$$
$$g := y^3(1-x-y) \ - \ x(\tfrac{1}{2}-x+y)^3\tfrac{10}{11}(1+x-3y) = 0\,,$$

in the complement of the lines given by the linear factors.



Khovanskii-Rolle continuation

Given a system of master functions

$$\prod_{i=1}^{\ell+n} p_i(x)^{a_{i,j}} \;=\; 1 \qquad j = 1, \ldots, \ell\,, \tag{$*$}$$

($p_i(x)$ linear), we find solutions in the polyhedron

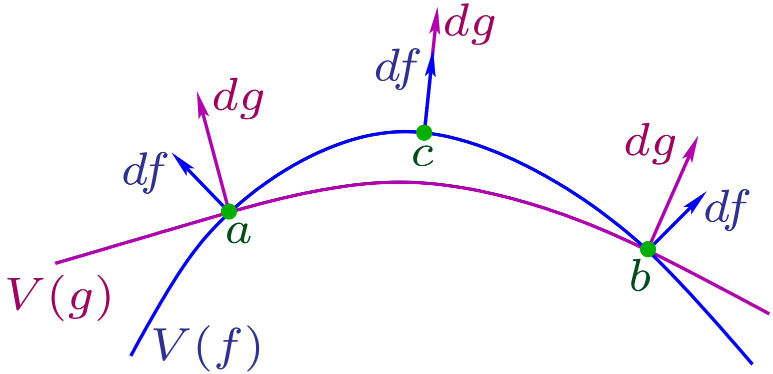
$$\Delta \;:=\; \{x \in \mathbb{R}^\ell \mid p_i(x) > 0\}\;.$$

The Khovanskii-Rolle Theorem (next slide) reduces solving $(*)$ to solving low degree polynomial systems, together with path continuation.

This is our new algorithm, which we now explain.

Khovanskii-Rolle Theorem

Theorem. *Between any two zeroes of g along the curve $V(f)\colon\ f=0$, lies at least one zero of the Jacobian $df \wedge dg$.*



Starting where $V(f)$ meets the boundary of the polyhedron Δ and where the Jacobian vanishes on $V(f)$, tracing the curve $V(f)$ in both directions finds all solutions $f = g = 0$.

Degree reduction ($\ell = 2$)

A system of master functions

$$\prod_{i=1}^{2+n} p_i(x)^{a_{i,j}} \;=\; 1 \qquad j = 1, 2$$

in logarithmic form

$$\varphi_j \;:=\; \sum_{i=1}^{2+n} a_{i,j} \log p_i(x) \;=\; 0 \qquad j = 1, 2\,,$$

has Jacobians of low degree

$$J_2 \;:=\; \mathrm{Jac}(\varphi_1, \varphi_2) \qquad J_1 \;:=\; \mathrm{Jac}(\varphi_1, J_2)\ .$$

Here, $n = \deg(J_2)$ and $2n = \deg(J_1)$.

An example with $n = 4$ and $\ell = 2$.

Consider the system of equations

$$\begin{aligned} t^{-1}u^{691}v^{5}w^{5} &= 3500(3-t) \\ t^{-1}u^{463}v^{5}w^{5} &= 3500(7-2t-v) \\ t^{1}u^{492} &= v+2t-4 \\ w &= 9-2t-2v\,. \end{aligned}$$

This has 7663 non-zero complex solutions.

Under $x = t$ and $y = t^1u^{492}$, this is Gale-dual to the system:

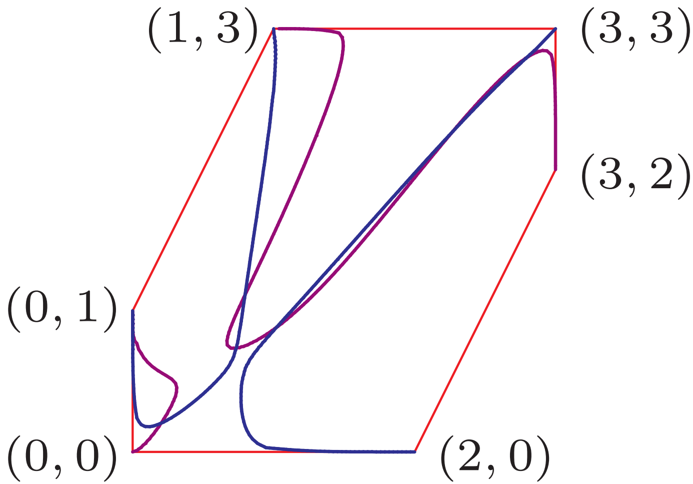
$$\begin{aligned} \frac{(3500)^{12}x^{27}(3-x)^{8}(3-y)^{4}}{y^{15}(4-2x+y)^{60}(2x-y+1)^{60}} &= 1\,, \\ \frac{(3500)^{12}x^{8}y^{4}(3-y)^{45}}{(3-x)^{33}(4-2x+y)^{60}(2x-y+1)^{60}} &= 1\,. \end{aligned}$$

The Gale Dual system

We display this Gale Dual system

$$f_1 \; := \; \frac{(3500)^{12}x^{27}(3-x)^8(3-y)^4}{y^{15}(4-2x+y)^{60}(2x-y+1)^{60}} \; = \; 1\,,$$

$$f_2 \; := \; \frac{(3500)^{12}x^{8}y^4(3-y)^{45}}{(3-x)^{33}(4-2x+y)^{60}(2x-y+1)^{60}} \; = \; 1\,.$$



Low-Degree Jacobians

If $\varphi_i := \log(f_i)$, then $J_2 := \mathrm{Jac}(\varphi_1, \varphi_2) \cdot \prod p_i(x,y) \;=$

$$2736 - 15476x + 2564y + 32874x^2 - 21075xy + 6969y^2 - 10060x^3$$
$$-7576x^2y + 8041xy^2 - 869y^3 \;+\; 7680x^3y - 7680x^2y^2 + 1920xy^3\,.$$

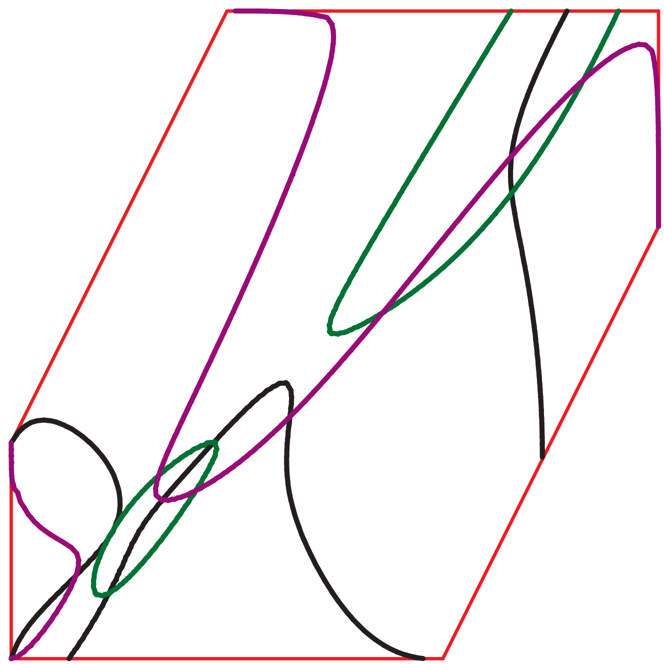
(polynomial of degree $n = 4$.) $J_1 := \mathrm{Jac}(\varphi_1, \Gamma_2) \cdot \prod p_i(x,y)^2 \;=$

$$8357040x - 2492208y - 25754040x^2 + 4129596xy - 10847844y^2$$
$$-37659600x^3 + 164344612x^2y - 65490898xy^2 + 17210718y^3 \;+ 75054960x^4$$
$$-249192492x^3y + 55060800x^2y^2 + 16767555xy^3 - 2952855y^4 \;- 36280440x^5$$
$$+143877620x^4y + 35420786x^3y^2 - 80032121x^2y^3 + 19035805xy^4 - 1128978y^5$$
$$+5432400x^6 - 33799848x^5y - 62600532x^4y^2 + 71422518x^3y^3 - 13347072x^2y^4$$
$$-1836633xy^5 + 211167y^6 + 2358480x^6y + 21170832x^5y^2 - 13447848x^4y^3$$
$$-8858976x^3y^4 + 7622421x^2y^5 - 1312365xy^6 - 1597440x^6y^2 - 1228800x^5y^3$$
$$+4239360x^4y^4 - 2519040x^3y^5 + 453120x^2y^6\,.$$

(A polynomial of degree $8 = 2n$.)

First step in Khovanskii-Rolle algorithm

We solve $J_2 = J_1 = 0$ and $V(J_2) \cap \partial\Delta$, then follow these 5+2 points along $V(J_2)$ to find the four solutions $J_2 = \varphi_1 = 0$.



Second Step in Khovanskii-Rolle

$V(\varphi_1) \cap \partial\Delta$ is (4) vertices of Δ. Tracing $V(\varphi_1)$ from these and the 4 points $\varphi_1 = J_2 = 0$ that we just found, we get the six solutions $\varphi_1 = \varphi_2 = 0$.