Algebraic Combinatorics

Forward Looking Session
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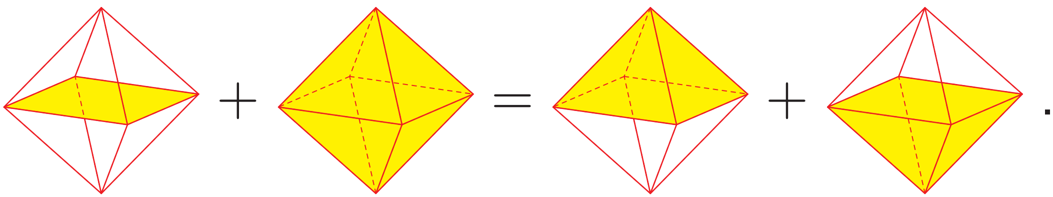


What is Algebraic Combinatorics?

$\rightarrow$ One might call algebraic combinatorics that part of discrete mathematics that communicates with algebra (and related areas such as algebraic geometry, representation theory, and ...), or that which arises in these areas.

- That is, combinatorics (yes, often counting) that uses algebra and related fields.
  It has an interest in exact results.

Sessions at DM10 related to algebraic combinatorics

- Combinatorial Hopf Algebras
- Combinatorial Representation Theory
- Discrete Mathematical Biology
- Enumerative Combinatorics
- Geometric Combinatorics
- Schubert Calculus
- Topological Combinatorics
- Linear Series on Tropical Curves
  (Chip-Firing on Graphs)

Future Trends

Increasing use of sophisticated tools from other areas.

Evidence for this is seen in the Schubert Calculus.

When I got my PhD in 1994, most people successfully working in this area were trained in combinatorics, and it was felt that combinatorics would solve the main questions.

In the ensuing decade, the balance shifted to those trained in algebraic geometry, whose techniques proved most successful. E.g. group actions, geometric degenerations, intersection theory, subtle cohomological properties (Cohen-Macaulay).

An Example: Matroid Invariants

The Tutte polynomial $T_M(x,y)$ of a matroid encodes all matroid invariants f that are additive w.r.t. deletion-contraction, that is,

$$f(M) \;=\; f(M|_v) \;+\; f(M/_v)\,.$$

This is fundamentally important, comes with combinatorial formulas, and has a sophisticated reason to exist:

$\mathbb{Z}[x,y]$ is the Grothendieck ring of the exact category of matroids.

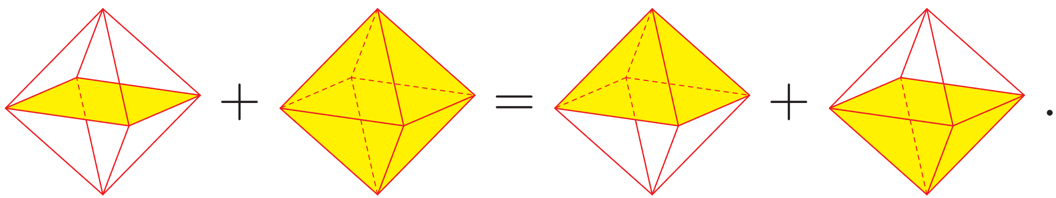
Valuative Invariants

Recently, new matroid invarints were found:

Billera-Jia-Reiner: An quasi-symmetric function valued invariant that arises from combinatorial Hopf algebras.

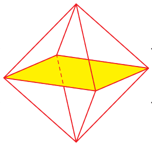
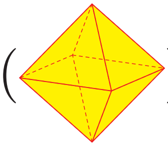
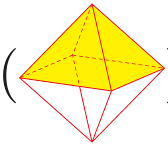
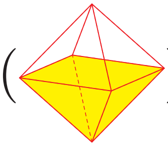
Speyer: An invariant which comes from the topological K-theory of a Grassmannian.

These are *valuative*, in that they are additive on matroid decompositions of matroid polytopes:



Derksen-Fink polynomial

A valuative invariant F satisfies

$F($  $) + F($  $) = F($  $) + F($  $)$.

Derksen defined a quasi-symmetric function valued matroid invariant that specializes to the Tutte polynomial, as well as to the Billera-Jia-Reiner and Speyer invariants, and together with Alex Fink, showed that this is the universal valuative invariant and the space of valuative invariants is $\mathbb{Z}\langle u, v\rangle$.

This has a sophisticated reason to exist, comes with combinatorial formulas, and I believe will be fundamentally important.