Experimentation in the Schubert Calculus

Lecture 1: The Shapiro Conjecture and its Proof.

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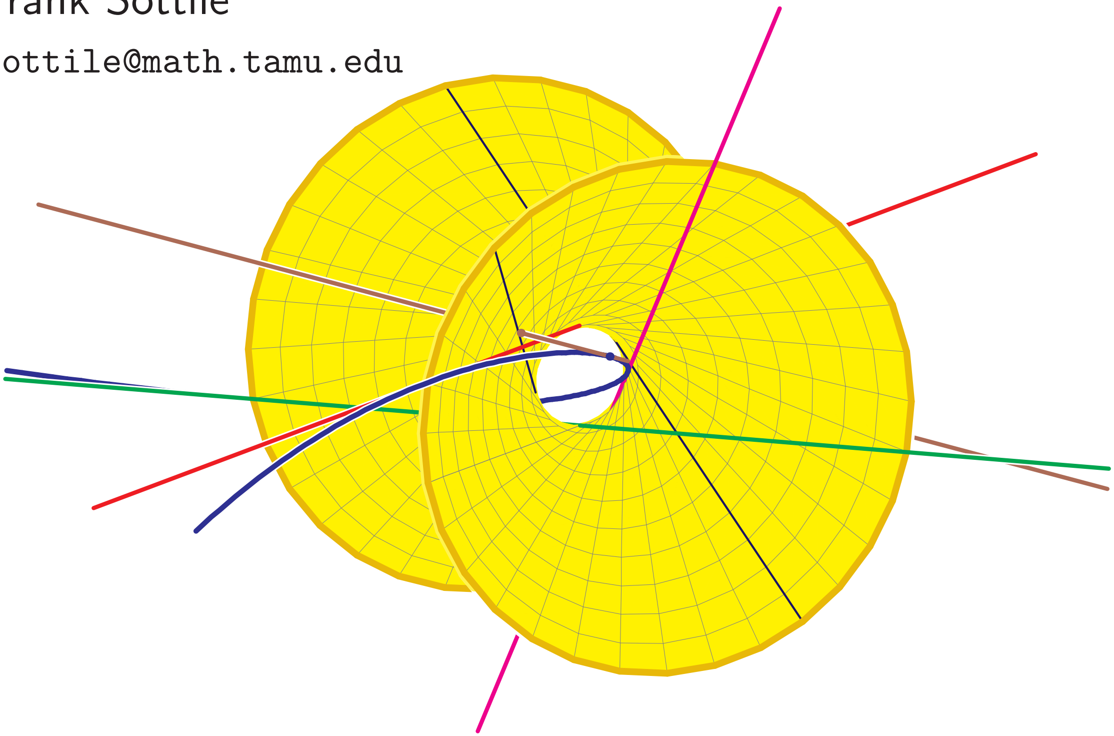


Experimentation in the Schubert Calculus

The Schubert calculus provides a rich and well-structured family of problems in enumerative geometry which may be used as a laboratory for exploring ill-understood phenomena, as it is very easy to model moderate-sized Schubert problems on a computer.

These lectures will discuss two such phenomena: reality of solutions to systems of equations from geometry and Galois groups of enumerative problems.

We will discuss a number of proofs, many conjectures, and hints of additional structure that can be seen in data which has been amassed through massive (several tera Hertz years) computations exploring these phenomena.

Wronskian and the MTV Theorem

The Wronskian of polynomials $f_1, \ldots, f_m \in \mathbb{C}[t]$ is

$$Wr := \det \begin{pmatrix} f_1(t) & f_2(t) & \cdots & f_m(t) \\ f_1'(t) & f_2'(t) & \cdots & f_m'(t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(m-1)}(t) & f_2^{(m-1)}(t) & \cdots & f_m^{(m-1)}(t) \end{pmatrix} .$$

When $\deg(f_i) = m+p-1$, $\deg(Wr) = mp$. Moreover, up to scalar, Wr depends only on the linear span P of the f_i, and only finitely many spans P have a given Wronskian.

Theorem. (Mukhin, Tarasov, Varchenko) *If $Wr(P)$ has only real roots, then P has a basis of real polynomials.*

Those spans P with a given Wronskian are the solutions to a system of polynomial equations, so this is an example of a system of equations with *only* real solutions.

Polynomial systems with only real solutions

Among the roots of a real univariate polynomial f, some are real and the rest occur in complex-conjugate pairs.

Rarely are all roots of f real.

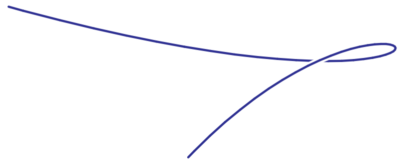
A primary example that comes to mind is when f is the characteristic polynomial of a real symmetric matrix, which only has real eigenvalues.

Similarly, a first example of a system of multivariate polynomials with only real solutions is the system for the eigenvalues/eigenvectors of a symmetric matrix.

It will turn out that this elementary fact from linear algebra is behind the unexpected reality of the MTV Theorem, whose proof I will sketch.

MTV Theorem in 3–space $(m = p = 2)$

Let $\gamma(t) = (t, t^2, t^3)$ be the rational normal curve



A cubic polynomial $f(t)$ $\iff$ affine function applied to γ

$\iff$ affine hyperplane, $f^\perp$

Two polynomials f, g $\iff$ two affine hyperplanes

$\iff$ a line $f^\perp \cap g^\perp$

$Wr(f, g)(s) = 0$ $\iff$ the line $f^\perp \cap g^\perp$ meets tangent line to γ at $\gamma(s)$.

$Wr(f, g)$ is a given quartic F $\iff$ $f^\perp \cap g^\perp$ meets tangents to γ at the four roots of F

View Animation

Numerical accident ?

The proof begins with a numerical accident.

In 1884, Schubert (essentially) determined that

$$\# Wr^{-1}(F(t)) \;=\; \frac{(mp)!\,0!1!2!\cdots m-1!}{p!(p{+}1)!\cdots(p{+}m{-}1)!}\,.$$

Call this number $\deg(m,p)$.

This number is also the dimension of the space of invariants

$$\deg(m,p) \;\stackrel{!}{=}\; \dim\left((\mathbb{C}^m)^{\otimes mp}\right)^{\mathfrak{sl}_m\mathbb{C}}\,.$$

Strengthening this coincidence is at the heart of our story.

A remarkable function

For $i = 1, \ldots, m$ let $y_i(t)$ be a polynomial of degree ip. Define the master function

$$\Phi \quad := \quad \prod_{i=1}^{m} \mathrm{Discr}(y_i) \Bigg/ \prod_{i=1}^{m-1} \mathrm{Res}(y_i, y_{i+1}) \ ,$$

where Discr and Res are the classical discriminant and resultant.

Writing Φ in terms of the roots $s_{i,j}$ of y_i gives,

$$\Phi(s) \ = \ \prod_{i=1}^{m} \prod_{j \neq k} (s_{i,j} - s_{i,k})^{2} \cdot \prod_{i=1}^{m-1} \prod_{j,k} (s_{i,j} - s_{i+1,k})^{-1} \ .$$

(The exponents come from the Cartan matrix of type A.)

Remarkably, $\deg(m, p)$ counts the (orbits of) critical points of $\Phi(s)$.

Schematic of proof

Critical points of Φ

Schematic of proof

$$\begin{tikzcd}
 & \text{Critical points of } \Phi \arrow[dl, "\text{Fuchsian diff. eq.}"'] \arrow[dr, "\text{Bethe Ansatz}"] & \\
Wr^{-1}(y_m(t)) & & \text{Rep. Theory}
\end{tikzcd}$$

Schematic of proof

$$
\begin{tikzcd}[column sep=small]
 & \text{Critical points of } \Phi \arrow[dl, "\substack{\text{Fuchsian}\\ \text{diff. eq.}}"'] \arrow[dr, "\substack{\text{Bethe}\\ \text{Ansatz}}"] & \\
Wr^{-1}(y_m(t)) \arrow[rr, leftrightarrow, dashed, "\text{! new connection}"] & & \text{Rep. Theory}
\end{tikzcd}
$$

Schematic of proof

Critical points of Φ

Fuchsian diff. eq.

Bethe Ansatz

! new connection

$$Wr^{-1}(y_m(t)) \longleftarrow - - - - \longrightarrow \text{Rep. Theory}$$

Eigenvalues of a symmetric matrix

(forces reality)

Schematic of proof

Critical points of Φ

Fuchsian diff. eq.

Bethe Ansatz

! new connection

$Wr^{-1}(y_m(t)) \longleftarrow - - - - - \longrightarrow$ Rep. Theory

Real spaces of polynomials $\left(Wr^{-1}(y_m(t)) \right)$

Eigenvalues of a symmetric matrix (forces reality)

Schematic of proof

$$
\begin{tikzcd}
 & \text{Critical points of } \Phi \arrow[dl, "\text{Fuchsian diff. eq.}"'] \arrow[dr, "\text{Bethe Ansatz}"] & \\
Wr^{-1}(y_m(t)) \arrow[rr, leftrightarrow, dashed, "\text{! new connection}"] & & \text{Rep. Theory} \\
\begin{array}{l}\text{Real spaces}\\ \text{of polynomials}\\ \left(Wr^{-1}(y_m(t))\right) \blacksquare\end{array} & & \begin{array}{c}\text{Eigenvalues of a}\\ \text{symmetric matrix}\\ \text{(forces reality)}\end{array} \arrow[u] \arrow[ll]
\end{tikzcd}
$$

Critical points of master function

For $i = 1, \ldots, m$, let $y_i(t)$ be a polynomial of degree ip. Recall the master function

$$\Phi \ := \ \prod_{i=1}^{m} \mathrm{Discr}(y_i) \Big/ \prod_{i=1}^{m-1} \mathrm{Res}(y_i, y_{i+1}) \ .$$

Fix $y_m(t)$ to be a polynomial of degree mp with roots $\mathbf{s} := (s_1, \ldots, s_{mp})$. This will be our Wronski polynomial.

The master function $\Phi_{\mathbf{s}}(\mathbf{x})$ depends on the roots $\mathbf{x}$ of the other y_i.

Let $\mathbf{x}$ be a critical point of $\Phi_{\mathbf{s}}(\mathbf{x})$, and $\mathbf{y} := (y_1, \ldots, y_{m-1})$ the corresponding polynomials whose roots are $\mathbf{x}$.

Theorem. (MV) *There are* $\deg(m, p)$ *such critical points* $\mathbf{y}$*.*

Spaces of polynomials from $\mathbf{y}$

For polynomials $\mathbf{y} = (y_1, \ldots, y_{m-1})$, with $\deg y_i = ip$, the *fundamental differential operator* $D_{\mathbf{y}}$ is

$$\left(\frac{d}{dt} - \ln'\left(\frac{y_m}{y_{m-1}}\right)\right) \cdots \left(\frac{d}{dt} - \ln'\left(\frac{y_1}{y_1}\right)\right)\left(\frac{d}{dt} - \ln'(y_1)\right) \ .$$

Let $V_{\mathbf{y}}$ be the kernel of $D_{\mathbf{y}}$.

Theorem. (MV)

1. *$V_{\mathbf{y}}$ is a space of polynomials iff $\mathbf{y}$ is a critical point of $\Phi_{\mathbf{s}}$.*
2. *If $f_1, \ldots, f_m$ span $V_{\mathbf{y}}$ with $\deg f_i = p+i$, then*

$$\begin{aligned} y_1 &= f_1 \ , \\ y_2 &= Wr(f_1, f_2) \ , \\ &\vdots \\ y_m &= Wr(f_1, f_2, \ldots, f_m) \ . \end{aligned}$$

Finite-dimensional $\mathfrak{sl}_m\mathbb{C}$-modules

Finite-dimensional $\mathfrak{sl}_m\mathbb{C}$-modules V are sum of *weight spaces* $V[\mu]$.

The *singular vectors*, $\mathrm{sing}V[\mu]$, in $V[\mu]$ are those annihilated by $\mathfrak{n}^+$.

We have the direct sum decomposition

$$V \;=\; \bigoplus_{\mu} U\mathfrak{sl}_m\mathbb{C}.\mathrm{sing}V[\mu]\,,$$

where $U\mathfrak{sl}_m\mathbb{C}$ is the universal enveloping algebra.

Each singular vector generates an irreducible summand of V.

Consequently, finding a basis of the singular vectors decomposes V into irreducible submodules.

Periodic Gaudin model

V := dual of vector representation of $\mathfrak{sl}_m\mathbb{C}$ (also $\mathfrak{gl}_m\mathbb{C}$).

Let $e_{i,j} \in \mathfrak{gl}_m\mathbb{C}$ be the elementary matrix with 1 in i, j position.

Define an operator $X_{i,j}(t) \; := \delta_{i,j}\frac{d}{dt} - \sum_{k=1}^{mp} \frac{e_{i,j}^{(k)}}{t - s_k}$,

where $e_{i,j}^{(k)}$ acts on the kth factor in $V^{\otimes mp}$.

Formal conjugate of the expansion of the row determinant of $(X_{i,j})$ is

$$\frac{d^m}{dt^m} + K_1(t)\frac{d^{m-1}}{dt^{m-1}} + \cdots + K_{m-1}(t)\frac{d}{dt} + K_m(t)\,.$$

$K_1(t), \ldots, K_m(t)$ are the Gaudin Hamiltonians. They form a family of commuting operators on $V^{\otimes mp}$, centralizing $\mathfrak{gl}_m\mathbb{C}$.

Bethe Ansatz for Gaudin model

In the theory of integrable systems, Bethe Ansätze are conjectural methods to find the joint eigenvectors and spectra of families of commuting operators.

As the Gaudin Hamiltonians centralize the action of $\mathfrak{sl}_m\mathbb{C}$, the Bethe Ansatz also gives a precise way to understand $\left(V^{\otimes mp}\right)^{\mathfrak{sl}_m\mathbb{C}}$.

The idea is to define a (rational) universal weight function

$$\beta \;:\; \begin{array}{c}\text{spaces of roots of}\\ \underbrace{y_1,\ldots,y_{m-1}}_{\mathbf{x}},\ \underbrace{y_m}_{\mathbf{s}}\end{array} \longrightarrow \text{0-weight space of } V^{\otimes mp},$$

and then address for which values $(\mathbf{x},\mathbf{s})$ is $\beta(\mathbf{x},\mathbf{s})$ $\mathfrak{sl}_m\mathbb{C}$-invariant.

Completeness of the Bethe Ansatz

Theorem. (MTV) *Let $\mathbf{x}$ be a critical point of the master function $\Phi_{\mathbf{s}}$.*

1. *$\beta(\mathbf{x},\mathbf{s})$ is well-defined, non-zero, and a joint eigenvector of the Gaudin Hamiltonians.*
2. *$\beta(\mathbf{x},\mathbf{s})$ is a $\mathfrak{sl}_m\mathbb{C}$-invariant.*
3. *When $\mathbf{s}$ is general, the vectors $\beta(\mathbf{x},\mathbf{s})$ for $\mathbf{x}$ a critical point form a basis of $\left(V^{\otimes mp}\right)^{\mathfrak{sl}_m\mathbb{C}}$.*
4. *When $\mathbf{s}$ is general, the Gaudin Hamiltonians have simple spectrum.*
5. *The eigenvalues $\lambda_i(t)$ of $K_i(t)$ on $\beta(\mathbf{x},\mathbf{s})$ satisfy*

$$\frac{d^m}{dt^m}+\lambda_1(t)\frac{d^{m-1}}{dt^{m-1}}+\cdots+\lambda_{m-1}(t)\frac{d}{dt}+\lambda_m(t) \;=\; \left(\tfrac{d}{dt}-\ln'\left(\tfrac{y_m}{y_{m-1}}\right)\right)\cdots\left(\tfrac{d}{dt}-\ln'\left(\tfrac{y_2}{y_1}\right)\right)\left(\tfrac{d}{dt}-\ln'(y_1)\right),$$

the fundamental differential operator of the critical point $\mathbf{x}$.

Proof of the Shapiro Conjecture

Usual Euclidean inner product on V induces the Shapovalov form on $V^{\otimes mp}$, which is $\mathfrak{sl}_m\mathbb{C}$-invariant.

Gaudin Hamiltonians are symmetric w.r.t the Shapovalov form.

Therefore, when $\mathbf{s}$ and t are real, their eigenvalues $\lambda_i(t)$ on a vector $\beta(\mathbf{x},\mathbf{s})$ for a critical point $\mathbf{x}$ are real.

Then the fundamental differential operator is real, and thus its kernel $V_{\mathbf{x}}$ is also real.

This implies the MTV Reality Theorem, as spaces $V_{\mathbf{x}}$ for $\mathbf{x}$ a critical point give all spaces of polynomials whose Wronskian has roots $\mathbf{s}$.

Thank you for your attention!