Galois Groups of Schubert Problems

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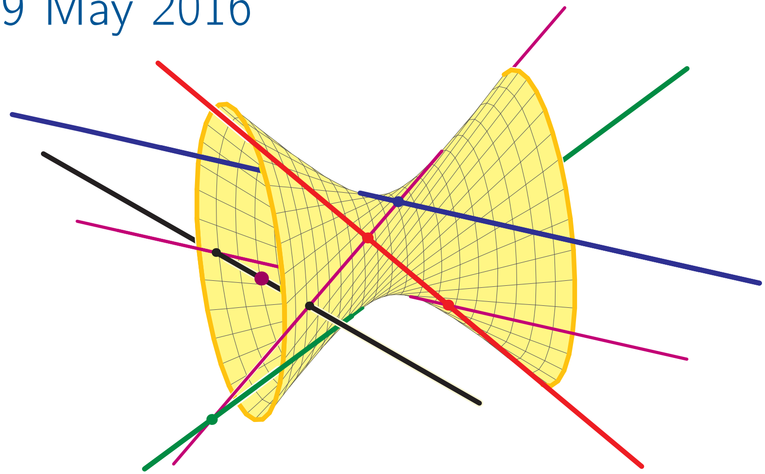

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Work with: Ravi Vakil, Jan Verschelde, Anton Leykin, Jacob White, Abraham Martín del Campo, Timo de Wolff, Robert Williams, Ata Pir, Christopher Brooks, Zach Maril, Aaron Moore, Runshi Geng, Taylor Brysiewicz, . . .

Galois Theory and the Schubert Calculus

Galois theory originated by studying the symmetries of roots of polynomials. Later, Galois groups came to be understood as encoding all symmetries of field extensions. It is now a pillar of number theory and arithmetic geometry.

Galois groups also appear in enumerative geometry, encoding subtle intrinsic structure of geometric problems. This is not well-developed, for such geometric Galois groups are very hard to determine. Until recently, they were almost always expected to be the full symmetric group.

I will describe a project to shed more light on Galois groups in enumerative geometry. It is focussed on Galois groups in the *Schubert calculus*, a well-studied class of geometric problems involving linear subspaces.

It is best to begin with examples.

The Problem of Four Lines

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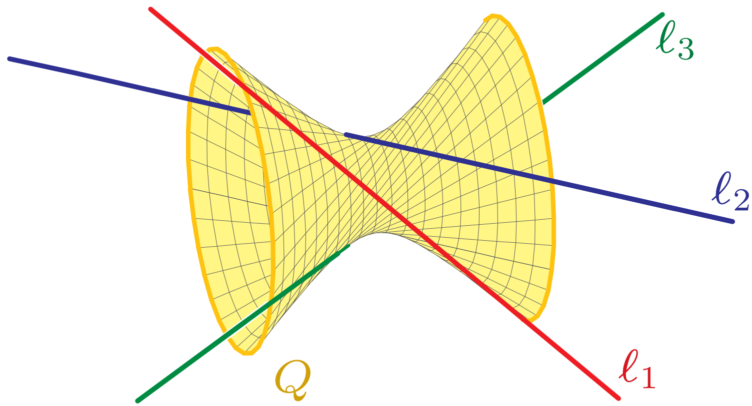

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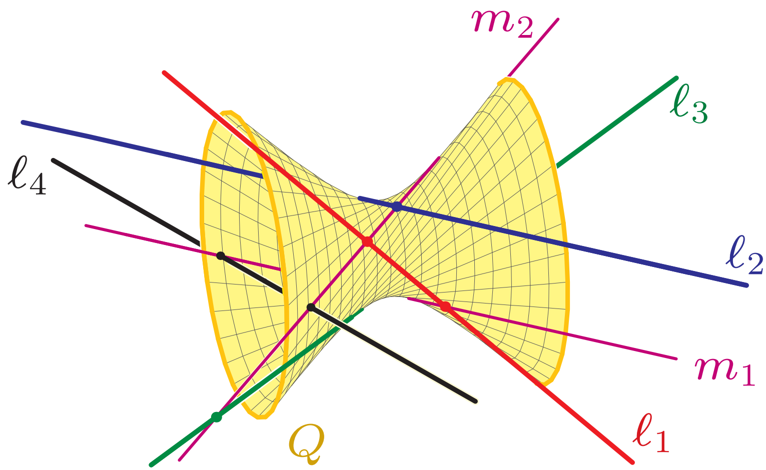
ℓ_1, ℓ_2, and ℓ_3 lie on a unique hyperboloid Q of one sheet, and the lines meeting them form one ruling of Q. Thus the solutions m_i are the lines in that ruling passing through the points of intersection $\ell_4 \cap Q$.



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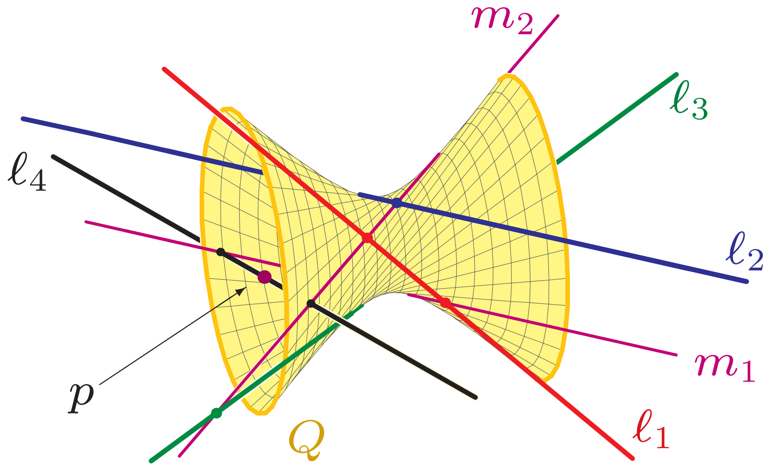
Rotating the line ℓ_4 180° around the point p interchanges the two solution lines m_1, m_2.



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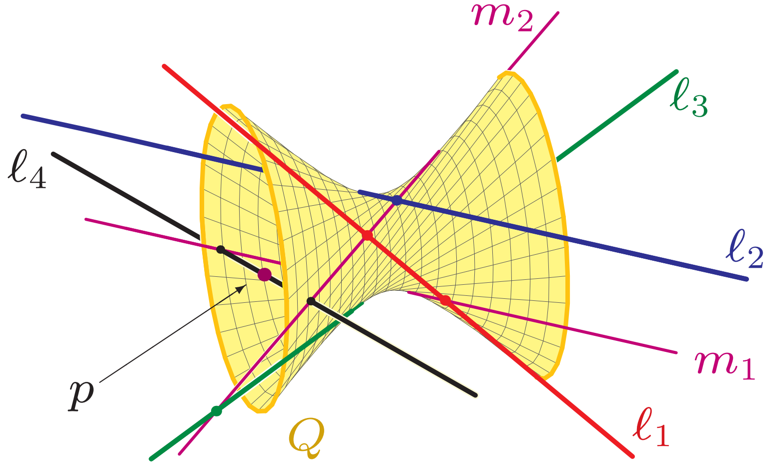


This shows that the Galois group of the problem of four lines is the symmetric group $\mathcal{S}_2$.

A Problem with Exceptional Geometry

Q: What 4-planes H in $\mathbb{C}^8$ meet four general 4-planes K_1, K_2, K_3, K_4 in a 2-dimensional subspace of each?

Auxiliary problem: There are four (h_1, h_2, h_3, h_4) 2-planes in $\mathbb{C}^8$ meeting each of K_1, K_2, K_3, K_4.

Fact: All solutions H to our problem have the form $H_{i,j} = \langle h_i, h_j \rangle$ for $1 \leq i < j \leq 4$.

It follows that the two problems have the same Galois group, which is the symmetric group $\mathcal{S}_4$. This permutes the 2-planes in the auxillary problem and is the induced action on the six solutions $H_{i,j}$ of the original problem.

This action is *not* two-transitive.

Galois Groups of Enumerative Problems

In 1870, Jordan explained how *algebraic* Galois groups arise naturally from problems in enumerative geometry; earlier (1851), Hermite showed that such an algebraic Galois group coincides with a geometric monodromy group.

This Galois group of a geometric problem is a subtle invariant. When it is *deficient* (not the full symmetric group), the geometric problem has some exceptional, intrinsic structure.

Hermite's observation, work of Vakil, and some number theory together with modern computational tools give several methods to study Galois groups.

I will describe a project to study Galois groups for problems coming from the Schubert calculus using numerical algebraic geometry, symbolic computation, combinatorics, and more traditional methods (Theorems).

Some Theory

A degree e surjective map $E \xrightarrow{\pi} B$ of equidimensional irreducible varieties (up to codimension one, $E \to B$ is a covering space of degree e)
$\rightsquigarrow$ degree e extension of fields of rational functions $\pi^* K(B) \subset K(E)$. Define the Galois group $\text{Gal}(E/B) \subset \mathcal{S}_e$ to be the Galois group of the Galois closure of this extension.

Hermite's Theorem. (Work over $\mathbb{C}$.) Restricting $E \to B$ to open subsets over which π is a covering space, $E' \to B'$, the Galois group is equal to the monodromy group of deck transformations.

This is the group of permutations of a fixed fiber induced by analytically continuing the fiber over loops in the base.

Point de départ: Such monodromy permutations are readily and reliably computed using methods from numerical algebraic geometry.

Enumerative Geometry

"Enumerative Geometry is the art of determining the number e of geometric figures x having specified positions with respect to other, fixed figures b." — Hermann Cäser Hannibal Schubert, 1879.

B := configuration space of the fixed figures, and X := the space of the figures x we count. Then $E \subset X \times B$ consists of pairs (x, b) where $x \in X$ has given position with respect to $b \in B$.

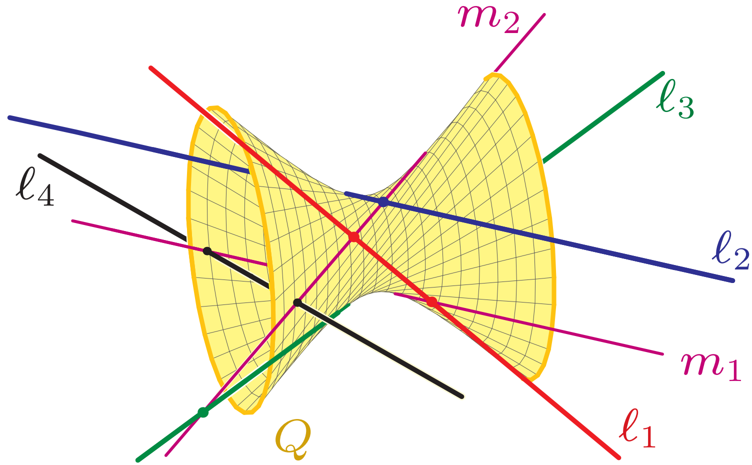
The projection $E \to B$ is a degree e cover outside of some discriminant locus, and the *Galois group of the enumerative problem* is $\mathrm{Gal}(E/B)$.

In the problem of four lines, B = four-tuples of lines, X = lines, and E consists of 5-tuples $(m, \ell_1, \ell_2, \ell_3, \ell_4)$ with m meeting each ℓ_i. We showed that this has Galois group the symmetric group $\mathcal{S}_2$.

Schubert Problems

The Schubert calculus is an algorithmic method promulgated by Schubert to solve a wide class of problems in enumerative geometry.

Schubert problems involve linear subspaces of a vector space incident upon other linear spaces, such as the problem of four lines, and the problems of 2-planes and 4-planes in $\mathbb{C}^8$.



As there are many millions of computable Schubert problems, many with their own unique geometry, they provide a rich and convenient laboratory for studying Galois groups of geometric problems.

Proof-of-Concept Computation

Leykin and I used off-the-shelf numerical continuation software to compute Galois groups of *simple Schubert problems*, which are formulated as the intersection of a skew Schubert variety with Schubert hypersurfaces.

In every case, we found monodromy permutations generating the full symmetric group (determined by `Gap`). This included one Schubert problem with $e = 17,589$ solutions.

We conjectured that all simple Schubert problems have the full symmetric group as Galois group.

White and I have just shown that these Galois groups all contain the alternating group.

The bottleneck to studying more general problems numerically is that we need numerical methods to solve *one* instance of the problem.

Numerical Project

Recent work, including certified continuation (Beltrán-Leykin, Hauenstein-Liddell), Littlewood-Richardson homotopies (Vakil, Verschelde, and S.), regeneration (Hauenstein), implementation of Pieri and of Littlewood-Richardson homotopies (Martín del Campo and Leykin) and new algorithms in the works will enable the reliable numerical computation of Galois groups of more general problems.

We plan to use a supercomputer to investigate many of the millions of computable Schubert problems. We intend to build a library of Schubert problems (expected to be very few) whose Galois groups are deficient.

These data will help us to classify Schubert problems with deficient Galois groups and to showcase the possibilities of numerical computation.

Problem. Software/algorithm development takes a lot of time.

Vakil's Criteria

A Schubert problem is *at least alternating* if its Galois group contains the alternating group. Vakil introduced two combinatorial criteria for showing that a Schubert problem is at least alternating. The first is simple combinatorics, which was used to prove:

Theorem. (Brooks, Martín del Campo, S.) *The Galois group of any Schubert problem involving $\mathbf{2}$-planes in $\mathbb{C}^n$ is at least alternating.*

Vakil's second criterion requires 2-transitivity. By it, to show high-transitivity (S_e or A_e), we often only need 2-transitivity. Interestingly, all known Galois groups of Schubert problems are either at least alternating or fail to be 2-transitive.

Vakil's Criteria II

White and I are studying 2-transitivity using geometry and combinatorics.

Theorem. [S.-White]

Every Schubert problem involving $\mathbf{3}$*-planes in* $\mathbb{C}^n$ *is 2-transitive.*
Every special Schubert problem is 2-transitive.

$\rightsquigarrow$ The proof suggests that not 2-transitive implies imprimitive.

Vakil's geometric Littlewood-Richardson rule, his criteria, and some 2-transitivity give an algorithm that can show a Schubert problem has at least alternating monodromy. Using the simpler geometric Pieri rule, we show:

Theorem. [S.-White] *Every simple Schubert problem is at least alternating.*

We are also able to prove that several infinite families of simple Schubert problems have full symmetric Galois group.

Specialization Lemma

Given $\pi: E \to B$ with B rational, and $b \in B(\mathbb{Q})$ the fiber $\pi^{-1}(b)$ has a minimal polynomial $p_b(t) \in \mathbb{Q}[t]$. In this situation, the algebraic Galois group of $p_b(t)$ is a subgroup of Gal(E/B).

Working modulo a prime, the minimal polynomial of such a fiber is easy to compute when $e \lesssim 600$. The degrees of its irreducible factors give the cycle type of a Frobenius element in the Galois group.

This quickly determines the Galois group when it is the full symmetric group, and allows the estimation of the Galois group when it is not.

Using Vakil's criteria and this method, we have nearly determined the Galois groups of all Schubert problems involving 4-planes in $\mathbb{C}^8$ and $\mathbb{C}^9$. (The first interesting case.) The deficient Schubert problems fall into a few easily-identified families, which suggests the possibility of classifying all deficient Schubert problems and identifying their Galois groups.

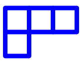
Combinatorial Shadows of Deficient Probems

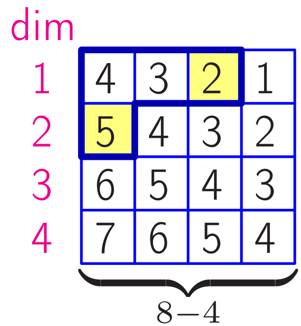
Many deficient problems have

- Restrictions on numbers of real solutions (unsurprising).
- Combinatorics reflecting structure of Schubert problem/Galois group.

Partitions encode Schubert conditions:

Eg. in $G(4,8)$, (4-planes in $\mathbb{C}^8$)

 $\longleftrightarrow$ the set of 4-planes meeting a 2-plane L_2 in a 1-plane and sharing a 2-plane with a 5-plane L_5, where $L_2 \subset L_5$.



Reprise: A Problem with Exceptional Geometry

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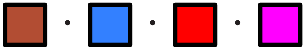
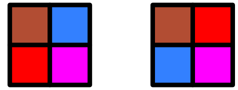
Schematically, $\square\!\square\!\square^{\,4} = 4$.

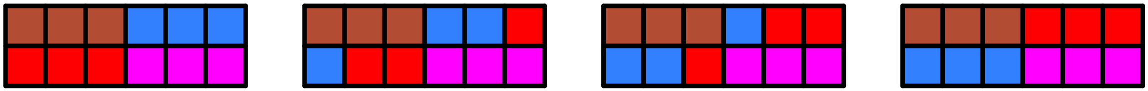
Fact: All solutions H to our problem have the form $H_{i,j} = \langle h_i, h_j \rangle$ for $1 \leq i < j \leq 4$.
Schematically, $\boxplus^{\,4} = 6$.

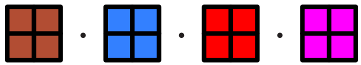
Fillings Give Numbers of Solutions

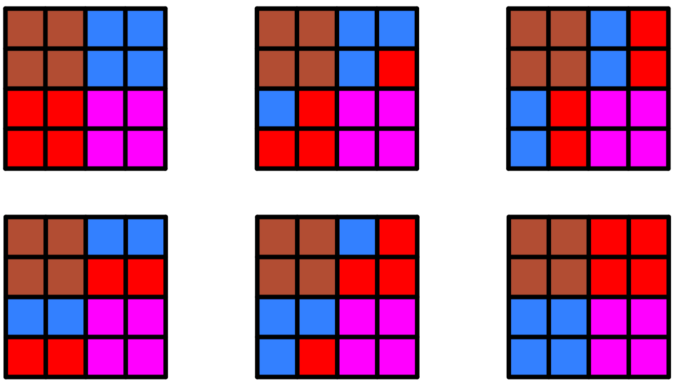
 is the problem of four lines.

Its two solutions correspond to two fillings: 

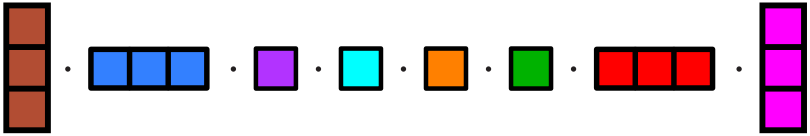
The auxillary problem, , has four

fillings: 

For the deficient problem, , there are six

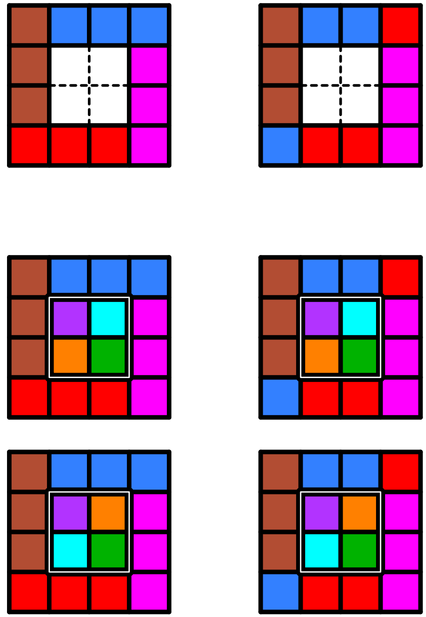
fillings: 

A Deficient Schubert Problem



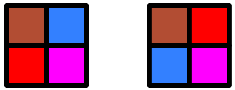
$= 4$ in $G(4,8)$

Fillings



The first two and last two conditions give an auxillary problem of four lines.

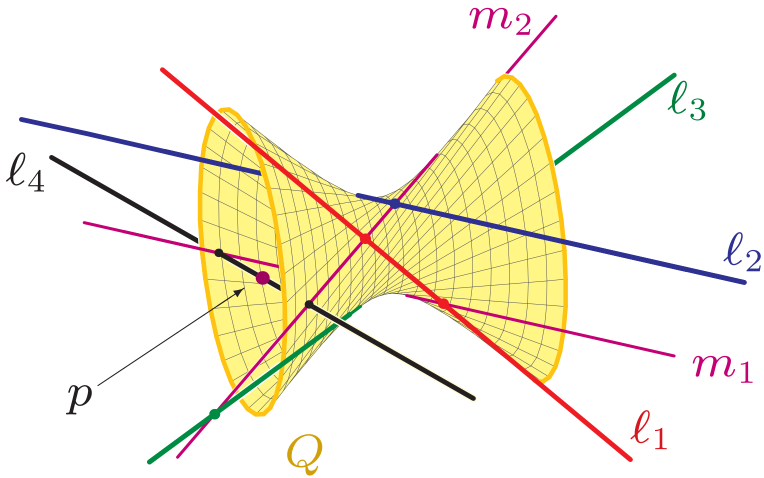
Look at the four corners:



For each solution of the auxillary problem, the middle four conditions give another problem of four lines, and this is reflected by the possible fillings.

The Galois group is $S_2 \wr S_2$, which is the dihedral group of symetries of a square.

Thank You!



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