Equivariant Cohomology and the Pattern Map

Combinatorics of Symmetric Functions

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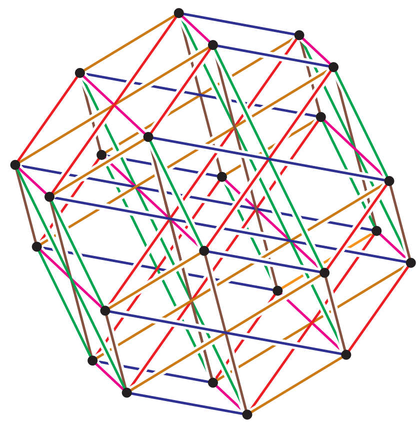
Equivariant Cohomology of Flag Manifolds

The (torus) equivariant cohomology of flag manifold $\mathcal{F} = G/B$ has three algebraic/combinatorial presentations.

- Borel: $H_T^*(\mathcal{F}) = S \otimes_{S^W} S$, where W is the Weyl group $S =$ symmetric algebra of character group of T (polynomials).

- GKM: $H_T^*(\mathcal{F}) \subset \text{Functions}(W, S)$, subspace of ϕ satisfying GKM-relations:

For every edge $u \to v$ in the moment graph, $\phi(v) - \phi(u)$ is divisible by $v - u$, the linear form giving the edge direction.



$W = S_4$

Schubert Basis

• Schubert: $H_T^*(\mathcal{F}) = \bigoplus_{w\in W} S\cdot\mathfrak{S}_w$, where $\mathfrak{S}_w$ is the equivariant class of a Schubert variety, X_w.

Schubert classes have (known) expressions in the other presentations, which generalize Schur polynomials.

Expanding $\mathfrak{S}_\alpha\cdot\mathfrak{S}_\beta$ in the Schubert basis for $H_T^*(\mathcal{F})$,

$$\mathfrak{S}_\alpha\cdot\mathfrak{S}_\beta \;=\; \sum_{\gamma\in W} c^\gamma_{\alpha,\beta}\,\mathfrak{S}_\gamma\,,$$

defines equivariant *Schubert structure constants* $c^\gamma_{\alpha,\beta}\in S$.

These generalizations of Littlewood-Richardson coefficients are positive in the sense of Graham.

Geometry of Permutation Patterns

Billey-Braden ('03): G: Semisimple linear algebraic group.
Let $\mathcal{F}$ be the flag variety of G, parametrizing Borel subgroups.
Let $\eta \in G$ be semisimple. Set $G_\eta := Z_G(\eta)$.

$B \mapsto B_\eta := B \cap G_\eta$ defines the *geometric pattern map*, π_η,

$$\mathcal{F}^\eta := \{B \in \mathcal{F} \mid \eta \in B\} \xrightarrow{\pi_\eta} \mathcal{F}_\eta := G_\eta / B_\eta\,.$$

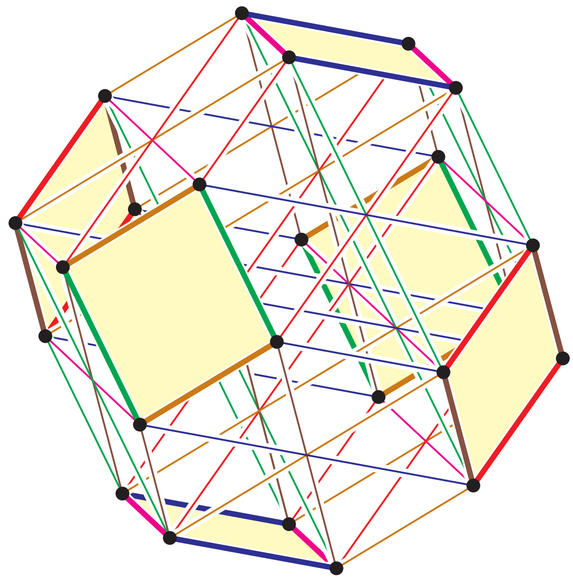
Let W, W_η be the Weyl groups of G, G_η. If $\pi_\eta \colon W \to W_\eta$ is the Billey-Postnikov generalised pattern map, then we have

Theorem [BB]. $\pi_\eta(X_w \cap \mathcal{F}^\eta) \;=\; X_{\pi_\eta(w)}$.

$\mathbb{F}\ell(2) \times \mathbb{F}\ell(2) \hookrightarrow \mathbb{F}\ell(4)$

Set $\eta = \begin{pmatrix} \alpha I_2 & 0 \\ 0 & \beta I_2 \end{pmatrix}$, so that $GL(4)_\eta = GL(2) \times GL(2)$.

$\mathbb{F}\ell(4)_\eta = GL(4)_\eta / B_\eta = \mathbb{P}^1 \times \mathbb{P}^1$.
Moment graph of $\mathbb{P}^1 \times \mathbb{P}^1$ is a square.



$\mathbb{F}\ell(4)^\eta$ is six $= \binom{4}{2}$ copies of $\mathbb{P}^1 \times \mathbb{P}^1$.

Each section $\iota_\varsigma \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{F}\ell(4)^\eta$ of pattern map is given by a shuffle ς, which is a minimal right coset representative of W_η in W.

We compute ι_ς^* in each of the three presentations.

Pattern Map: Borel & GKM

In the Borel presentation, $H^*_T(\mathcal{F}) = S \otimes_{S^W} S$, the left copy of S is the coefficient ring $H^*_T(pt)$, and the right copy is generated by equivariant Chern classes.

Given a section of the pattern map ι_ς, we have

$$\iota^*_\varsigma(f \otimes g) \;=\; f \otimes \varsigma(g) \in S \otimes_{S^{W_\eta}} S = H^*_T(\mathcal{F}_\eta)\,.$$

In the GKM presentation, the map $\iota^*_\varsigma\colon \text{Functions}(W, S) \to \text{Functions}(W_\eta, S)$ is simply restriction of functions:

$$\iota^*_\varsigma(\phi)(v) \;=\; \phi(\iota_\varsigma(v)) = \phi(v\varsigma)\,,$$

for $\phi\colon W \to S$ and $v \in W_\eta$.

Pattern Map: Schubert Basis

Expanding $\iota_\varsigma^*\mathfrak{S}_w$ in the Schubert basis for $H_T^*(\mathcal{F}_\eta)$,

$$\iota_\varsigma^*\mathfrak{S}_w \;=\; \sum_{v\in W_\eta} d_{w,\varsigma}^v\,\mathfrak{S}_v\,,$$

defines *decomposition coefficients* $d_{w,\varsigma}^v \in S$.

Using the formula $\pi_\eta(X_w \cap \mathcal{F}^\eta) \;=\; X_{\pi_\eta(w)}$, we obtain

Theorem. $d_{w,\varsigma}^v \;=\; c_{w,\varsigma}^{v\varsigma}$.

Algorithm:

Expand the product $\mathfrak{S}_w\cdot\mathfrak{S}_\varsigma$ in Schubert basis for $H_T^*(\mathcal{F})$.

Restrict to terms of the form $\mathfrak{S}_{v\varsigma}$ for $v\in W_\eta$.

Replace $\mathfrak{S}_{v\varsigma}$ by $\mathfrak{S}_v$ to obtain formula for $\iota_\varsigma^*(\mathfrak{S}_w)$.

Example

$G = C_4$, $S = \mathbb{Q}[t_1, \ldots, t_4]$, $G_\eta = A_3$, and $\varsigma = \overline{2}\,\overline{1}\,3\,4$

$$\mathfrak{C}_{3\,\overline{1}\,4\,2} \cdot \mathfrak{C}_{\overline{2}\,\overline{1}\,3\,4} \;=\; 2(t_1^2{+}t_1t_3)\mathfrak{C}_{\overline{3}\,\overline{1}\,4\,2} \;+\; 2(t_1{+}t_3)\mathfrak{C}_{\overline{1}\,\overline{3}\,4\,2}$$
$$+\; 2t_1\mathfrak{C}_{\overline{4}\,\overline{1}\,3\,2} \;+\; 2(t_1{+}t_2{+}t_3)\mathfrak{C}_{\overline{3}\,\overline{2}\,4\,1}$$
$$+\; 2(t_1{+}t_2)\mathfrak{C}_{3\,\overline{2}\,4\,\overline{1}} \;+\; \mathfrak{C}_{\overline{3}\,\overline{2}\,4\,\overline{1}} \;+\; 2\mathfrak{C}_{2\,\overline{3}\,4\,\overline{1}}$$
$$+\; 2\mathfrak{C}_{\overline{4}\,\overline{3}\,1\,2} \;+\; 2\mathfrak{C}_{\overline{2}\,\overline{3}\,4\,1} \;+\; 2\mathfrak{C}_{\overline{1}\,\overline{4}\,3\,2} \;+\; 2\mathfrak{C}_{\overline{4}\,\overline{2}\,3\,1}\,.$$

As only the first and last four indices have the form $v\varsigma$,

$$\iota_\varsigma^*\left(\mathfrak{C}_{3\,\overline{1}\,4\,2}\right) \;=\; 2(t_1^2 + t_1t_3)\mathfrak{S}_{1342} \;+\; 2(t_1 + t_3)\mathfrak{S}_{3142}$$
$$+\; 2t_1\mathfrak{S}_{1432} \;+\; 2(t_1 + t_2 + t_3)\mathfrak{S}_{2341}$$
$$+\; 2\mathfrak{S}_{3412} \;+\; 2\mathfrak{S}_{3241} \;+\; 2\mathfrak{S}_{4132} \;+\; 2\mathfrak{S}_{2431}\,.$$