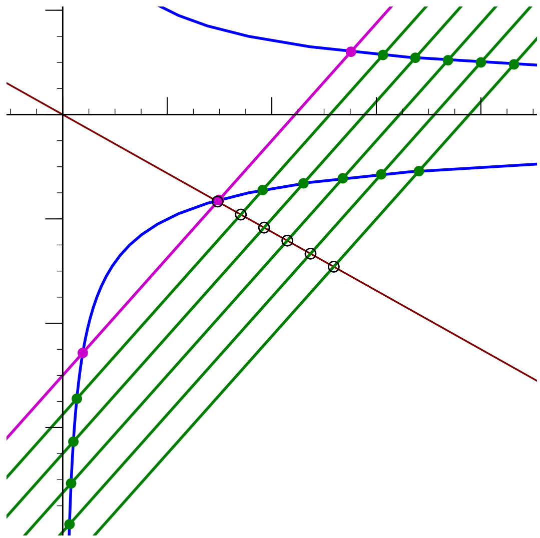
Trace Test in Numerical Algebraic Geometry

CARGO Lab 15-year Event

31 March 2017



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Numerical Algebraic Geometry

nu·mer·i·cal al·ge·bra·ic ge·om·e·try
[nōōˈmerəkəl ˌaljəˈbrāik jēˈämətrē]

1 : The use of techniques from numerical analysis to study algebraic varieties.

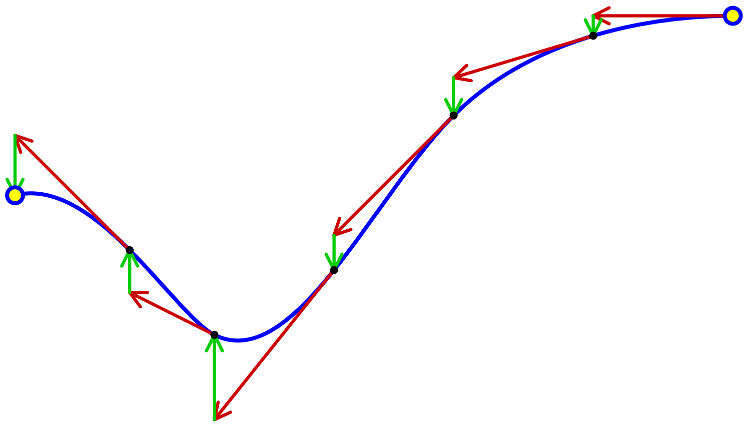
2 : A symbolic-numerical approach to computing in algebraic geometry that exploits modern parallelism and refinable approximations to treat questions that are out of reach of purely symbolic methods.

3 : The future of computation in algebraic geometry.

Core Numerical Methods

Numerical algebraic geometry rests upon two core numerical methods, *Newton's method* to refine approximate solutions to a system of equations, and *discretization* (e.g. Euler or Runge-Kutta) from numerical PDE.

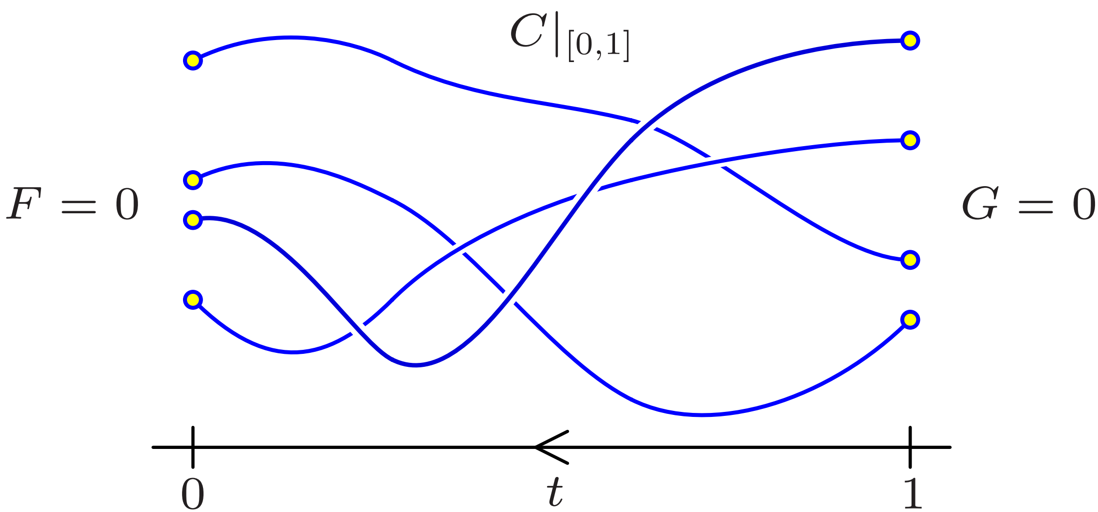
These are combined to give robust *predictor-corrector* methods for following implicitly defined curves, such as the Euler predictor shown below.



Homotopy

A *homotopy* is a one-parameter family of polynomial systems connecting a system F you want to solve with one G whose solutions are known.

Formally, $H\colon \mathbb{C}^n \times \mathbb{C}_t \to \mathbb{C}^m$ with $H(\bullet, 1) = G$ and $H(\bullet, 0) = F$, and $H^{-1}(0)$ is a curve C over $\mathbb{C}_t$. Then $C|_{t\in[0,1]}$ consists of arcs connecting the unknown points $F = 0$ to known points in $G = 0$.



Bézout Homotopy

The universal *Bézout homotopy* is easy to describe.

Suppose that we have a square polynomial system

$$F = (F_1, \ldots, F_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

with $\deg(F_i) = d_i$.

Set $G = (G_1, \ldots, G_n)$ where $G_i := x_i^{d_i} - 1$, and then define

$$H := t \cdot G + (1-t) \cdot F\,.$$

The solutions to G are known, and every solution to F is connected to a solution to G along some arc of $H^{-1}(0)|_{[0,1]}$.

More sophisticated homotopy algorithms are exceptionally powerful and efficient.

Witness Set

Numerical algebraic geometry uses the ability to solve systems of polynomial equations to study algebraic varieties on a computer.

Its key data structure for representing a variety $V \subset \mathbb{C}^n$ is a *witness set*. This is a triple (F, Λ, W), where

1. $F\colon \mathbb{C}^n \to \mathbb{C}^m$ is a polynomial system with V a component of $F^{-1}(0)$.
2. $\Lambda\colon \mathbb{C}^n \to \mathbb{C}^k$ is a general affine linear map with $k = \dim V$.
3. $W := V \cap L$ is transverse and a finite set of points, where we have $L := \Lambda^{-1}(0)$.

Observe that W is among the solutions to the augmented system $[F, \Lambda]$.

The set W is considered to be a generic point of V in the sense of Weil.

Changing Witness Sets and Sampling

Witness sets are used in many algorithms to study a variety $V \subset \mathbb{C}^n$.

Let (F, Λ, W) be a witness set for V, and set $k := \dim V$.

Let $\Lambda' \colon \mathbb{C}^n \to \mathbb{C}^k$ be another map and $\Lambda(t)$ for $t \in \mathbb{C}$ be a family of maps interpolating between Λ and Λ' (so that $\Lambda(1) = \Lambda$ and $\Lambda(0) = \Lambda'$).

The augmented system $[F, \Lambda(t)]$ is a homotopy between W and $W' := V \cap L'$, where $L' = (\Lambda')^{-1}(0)$.

When Λ' is sufficiently general so that $W' := V \cap L'$ is transverse, then W' is another witness set for V.

Even if Λ' is not general, then W' consists of points of V.

Moving Λ in this manner enables us to sample points of V.

Membership and Monodromy

Let (F, Λ, W) be a witness set for $V \subset \mathbb{C}^n$.

We may test if $x \in \mathbb{C}^n$ lies in V:

Let $\Lambda'\colon \mathbb{C}^n \to \mathbb{C}^k$ be a general linear map with $x \in L' := (\Lambda')^{-1}$.

Choose a family $\Lambda(t)$ interpolating between Λ and Λ',
and compute $W' := V \cap L'$ as before.

Then $x \in V \iff x \in W'$.

Suppose that $\Lambda = \Lambda'$ and $\Lambda(t)$ is not constant.
Then $W = W'$, and the arcs in the homotopy given by $\Lambda(t)$
define a permutation of W.

Computing such *monodromy permutations* is a standard operation
in numerical algebraic geometry.

Numerical Irreducible Decomposition

Let $V = V_1 \cup \cdots \cup V_s$ be a union of components of $F^{-1}(0)$, all of the same dimension.

Given a general slice $W = V \cap \Lambda^{-1}(0)$ of V, a *numerical irreducible decomposition* is the partition $W = W_1 \cup \cdots \cup W_s$ of W where $W_i := V_i \cap \Lambda^{-1}(0)$.

Monodromy maps points of V_i to V_i, preserving the component W_i.

The partition of W given by cycles in a monodromy permutation is finer than this numerical irreducible decomposition.

Computing more monodromy permutations coarsens this orbit partition.

Needed for this is a stopping criterion.

The Trace Test

Given a partition $W = U_1 \cup \cdots \cup U_r$ of the slice $W = V \cap L$, a stopping criterion for numerical irreducible decomposition would tell us if each component U_i forms the witness set for a component of V.

This reduces to the basic problem: Given a subset $W' \subset W$ of a witness set, how to certify that $W' = W$?

Trace Test: *Suppose that* $L(t)$ *for* $t \in \mathbb{C}$ *is a general pencil of affine-linear spaces with* $L(0) = L$. *Use continuation to follow points of* W' *along* t, *obtaining sets* $W'(t)$. *Then the trace of points in* $W'(t)$,

$$\mathit{Tr}(W'(t)) \ := \ \sum \{w \mid w \in W'(t)\} \,,$$

is an affine function of t *if and only if* $W' = W$.

Proof of Trace Test

A general irreducible curve in $\mathbb{C}^2$ is defined by a dense irreducible polynomial $f \in \mathbb{C}[x, t]$ of degree d. Normalize f so that $1 =$ coefficient of x^d.

$f \in \mathbb{C}(t)[x]$ is irreducible and monic. The negative sum of its roots is its coefficient of x^{d-1}. Thus

$$\text{trace}(K/\mathbb{C}(t))(x) \;=\; c_0 t + c_1 \qquad c_0, c_1 \in \mathbb{C}\,, \tag{1}$$

where K contains the roots of f.

A general pencil $L(t)$ spans a codimension $m-1$ plane M with $M \cap V$ a curve, and M has coordinates $(\underline{x}, t)$. By (1), $\text{Tr}(W(t))$ is an affine function when W is a witness set.

This does not hold for $\text{Tr}(W'(t))$ if $W' \subsetneq W$, as the monodromy in t is the full symmetric group.

Explain

Multihomogeneous Witness Sets

A subvariety $V \subset \mathbb{P}^A \times \mathbb{P}^B$ of dimension m has *multidegrees* $d_{a,b}$ for $a+b = m$: For a general codimension a plane $L \subset \mathbb{P}^A$ and a general codimension b plane $M \subset \mathbb{P}^B$,

$$d_{a,b}(V) \;=\; \#V \cap (L \times M)\,.$$

Definition (Hauenstein-Rodriguez) An intersection $W_{a,b} = V \cap (L \times M)$ is a *multihomogeneous witness set* of bidimension (a, b) for V.

Advantages:

(1) Reflects the structure of V in $\mathbb{P}^A \times \mathbb{P}^B$.

(2) Smaller than alternatives. Embedding V into $\mathbb{P}^{AB+A+B}$ via Segre σ,

$$\deg(\sigma(V)) \;=\; \sum_{a+b=m} \binom{m}{a} d_{a,b}\,.$$

This is huge.

Using Multihomogeneous Witness Sets

Hauenstein and Rodriguez showed that many algorithms in numerical algebraic geometry work well with multihomogeneous witness sets. These include regeneration, membership, and using a multihomogeneous witness set in one bidimension to populate another.

What does not work well is the trace test.

Fact. If $L(t) \subset \mathbb{P}^A$ and $M(s) \subset \mathbb{P}^B$ are pencils of affine spaces of codimensions a and b, respectively, then $\mathrm{Tr}(V \cap (L(t) \times M(s)))$ is not a bilinear function in s and t.

We cannot even fix t and let s vary for irreducible decomposition, for $V \cap L$ could be reducible even if V is irreducible.

Dimension Reduction

Let $V \subset \mathbb{P}^A \times \mathbb{P}^B$ be irreducible of dimension $m \geq 2$, $a+b = m$ with $d_{a,b}(V) \neq 0$, $L' \subset \mathbb{P}^A$ a general linear space of codimension $a-1$, and $M' \subset \mathbb{P}^B$ a general linear space of codimension $b-1$.

$U := V \cap (L' \times M')$ is irreducible of dimension 2 with multidegrees

$$d_{0,2} = d_{a-1,b+1}(V)\,, \quad d_{1,1} = d_{a,b}(V)\,, \quad d_{2,0} = d_{a+1,b-1}(V)\,.$$

Either (1) $d_{0,2} = d_{2,0} = 0 \;\Rightarrow\; U$ is a product of curves. Then V is also a product and we may treat each factor separately.

Or (2) a further linear slice is possible, reducing V to a curve in a product of projective spaces.

The cases are detected from the tangent spaces at general points of V or of U.

A Multihomogeneous Trace Test

Assume that V is not a product. Given nonzero adjacent multidegrees $d_{\alpha+1,\beta}$ and $d_{\alpha,\beta+1}$, $L' \subset \mathbb{P}^A$ and $M' \subset \mathbb{P}^B$ of codimensions α and β containing hyperplanes $L \subset L'$ and $M \subset M'$, then

$$W_{10} := V \cap (L \times M') \text{ and } W_{01} := V \cap (L' \times M)$$

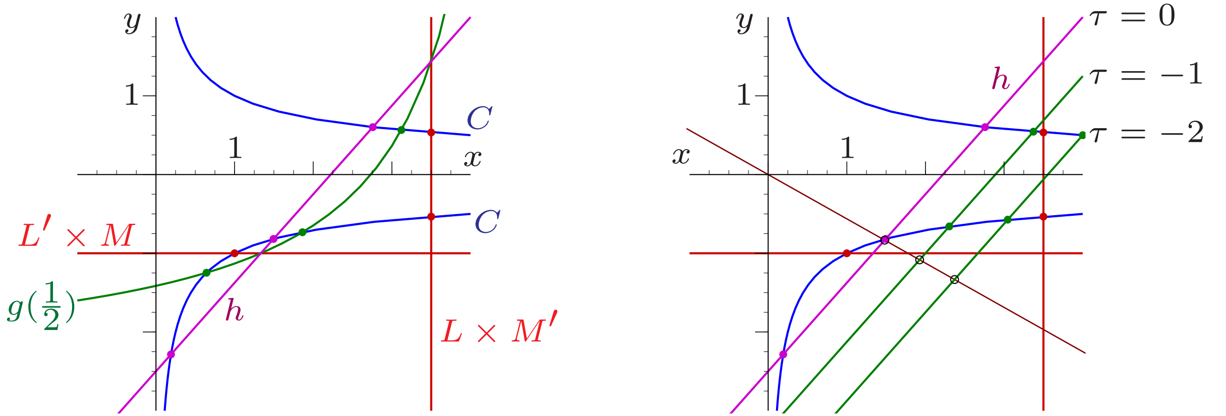
are the corresponding multihomogeneous witness sets.

Then $C := V \cap (L' \times M')$ is an irreducible curve with multidegrees $d_{10} = d_{\alpha+1,\beta}$ and $d_{01} = d_{\alpha,\beta+1}$ having witness sets W_{10} and W_{01}.

Working in an affine patch $\mathbb{C}^n \oplus \mathbb{C}^m$ on $L' \times M'$, C has degree $d_{10} + d_{01}$ and $W_{01} \cup W_{10}$ can be used to get a witness set $W = C \cap H$, which we may use for a trace test in the affine space $\mathbb{C}^n \oplus \mathbb{C}^m$.

Example

Suppose that $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ is defined locally by $y^2x = 1$.



Left: Linear spaces $x = x_0$ and $y = y_0$, line $H : h = 0$, and the curve $g(\frac{1}{2})$, where $g(t) := (x-x_0)(y-y_0)(1-t) + th$. These are $g(t)$ at $t = 0,\ \frac{1}{2},\ 1$.

Right: the parallel slices $h = \tau$ are in green, and the averages of witness points ($\frac{1}{3}$ of the trace) lies on the brown line.

Congradulations to CARGO Lab on Your 15th Anniversary