Generalized Witness Sets

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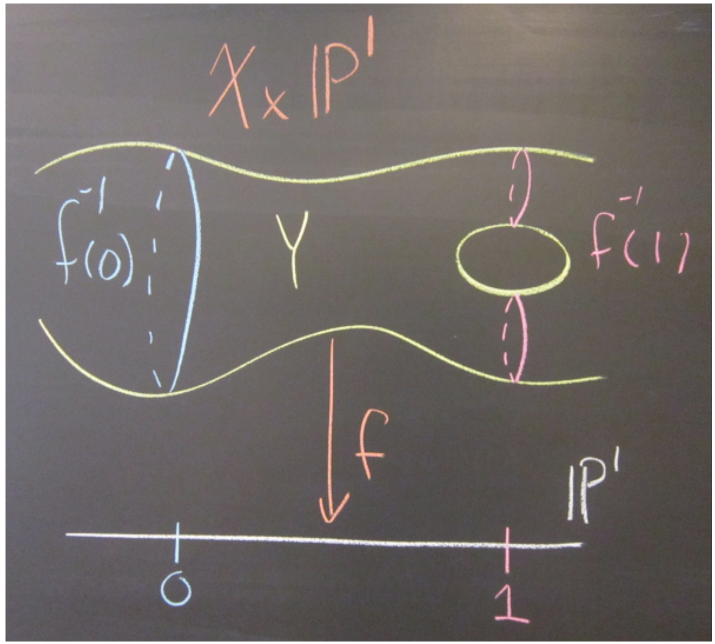


Numerical Algebraic Geometry and Intersection Theory

Numerical algebraic geometry operates on the geometric side of the dictionary between algebra and geometry underlying algebraic geometry

Numerical algebraic geometry computes points on varieties, and many algorithms are based on geometric constructions such as directly computing monodromy

Numerical algebraic geometry also has connections to intersection theory: Computing Segre classes or exploiting Newton-Okounkov bodies

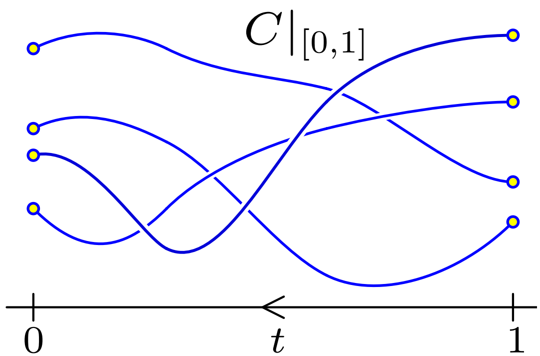
I will talk about further connections to intersection theory:

- Relating rational equivalence to homotopy
- Generalizing witness sets

Homotopy Continuation

Let $C := \mathcal{V}(H(x;t)) \subset X \times \mathbb{C}_t$ be a curve whose projection $C \to \mathbb{C}_t$ is dominant and regular at $t = 0, 1$

Restricting C to $[0, 1]$ gives arcs



connecting points of *start system* $H(x;1) = 0$ to points of the *target system* $H(x;0) = 0$

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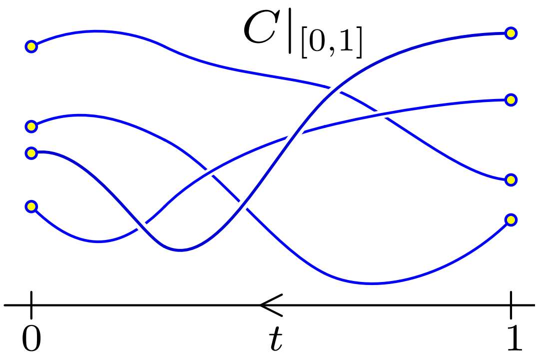

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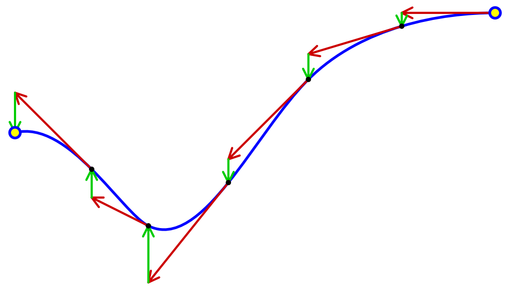
Numerical path-tracking along these arcs computes solutions to the target system from solutions to the start system



Rational Equivalence

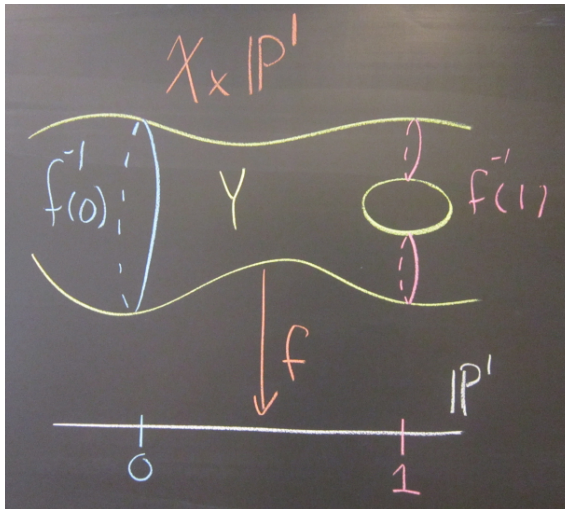
The group Z_kX of k-cycles on an algebraic variety X is the free abelian group with basis the irreducible subvarieties of X of dimension k

The fundamental cycle $[V] \in Z_kX$ of a k-dimensional subscheme $V \subset X$ is the sum of its components, with coefficients their multiplicities in V

For a subvariety $Y \subset X \times \mathbb{P}^1$ of dimension $k+1$ with $f\colon Y \to \mathbb{P}^1_t$ dominant and regular at $t = 0, 1$,

$[f^{-1}(0)] - [f^{-1}(1)] \ \in \ Z_kX$,

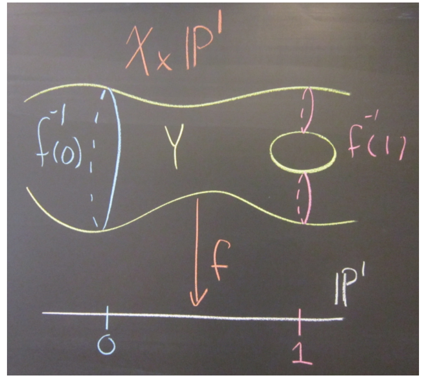
is an *elementary rational equivalence*.



These generate $\mathrm{Rat}_kX \subset Z_kX$, the group of rational equivalences

$A_kX := Z_kX/\mathrm{Rat}_kX$ is the *Chow group* of X

Rational Equivalence to Homotopy



Let $[f^{-1}(0)] - [f^{-1}(1)] \in \mathrm{Rat}_k X$ be an elementary rational equivalence induced by $Y \subset X \times \mathbb{P}^1$

Suppose that $\Lambda \subset X$ has codimension k and Λ meets $f^{-1}(0)$ and $f^{-1}(1)$ transversally.

$\Rightarrow$ $Y \cap \Lambda$ contains a curve $C \subset X \times \mathbb{P}^1$ with $C \to \mathbb{P}^1$ regular at $t = 0, 1$

Consequently, this is a homotopy with start system $f^{-1}(1) \cap \Lambda$ and target system $f^{-1}(0) \cap \Lambda$

(Need $\Lambda, Y, f : Y \to \mathbb{P}^1$ to be given explicitly by polynomials to get a traditional homotopy ...)

Witness sets

Witness sets are the fundamental data structure in numerical algebraic geometry.

For $V \subset \mathbb{P}^n$ of dimension k (a component of $\mathcal{V}(F)$) and $\Lambda \subset \mathbb{P}^n$ a general linear subspace of codimension k, the intersection $W := V \cap \Lambda$ is a collection of points. Then (W, Λ, F) is a *witness set* for V

Note that $\deg(W) := \#W = \text{degree}(V)$

Point de départ: W represents $[V]$ in $A_k\mathbb{P}^n$, the kth Chow group of $\mathbb{P}^n$

Indeed, for $L \subset \mathbb{P}^n$ a linear space of dimension k,

$$[V] \;=\; \deg(W) \cdot [L]\,,$$

in $A_k\mathbb{P}^n = \mathbb{Z}[L]$

Witness Sets, More Generally?

Goal: Develop a notion of witness set for subvarieties of a variety X that represents a subvariety V in $A_k X$ from intersection data

General witness sets are needy:

1 Finitely generated Chow groups $A_\bullet X$

2 Ability to recover $[V]$ from its intersection data
$\longrightarrow$ Requires a form of (Poincaré) duality

3 Transversality and a moving lemma

$1'$ Effectivity. Basis of $A_k X$ fundamental classes of effective cycles

1 and 2 hold if X is smooth and projective, and if rational equivalence is replaced by numerical equivalence

Detailed discussion: "General witness sets for numerical algebraic geometry" `arXiv.org/2002.00180`

Multiprojective Witness Sets

Let $X = \mathbb{P}^m \times \mathbb{P}^n$

$A_k X = \mathbb{Z}\{[L_1^p \times L_2^q] \mid p+q=k\}$,

$L_1^p \subset \mathbb{P}^m$, $L_2^q \subset \mathbb{P}^n$ are linear subspaces of dimensions p, q

A *multiprojective witness set* for $V \subset \mathbb{P}^m \times \mathbb{P}^n$ is $(W_\bullet, \Lambda_\bullet, F)$

F : bihomogeneous forms with V a component of $\mathcal{V}(F)$

$\Lambda_\bullet := \{\Lambda_1^{m-p} \times \Lambda_2^{n-q} \mid p+q=k\}$,

$\Lambda_1^a \subset \mathbb{P}^m$, $\Lambda_2^b \subset \mathbb{P}^n$ are general linear subspaces of dimensions a, b

$W_\bullet := \{V \cap (\Lambda_1^{m-p} \times \Lambda_2^{n-q}) \mid p+q=k\}$

Much smaller than witness sets for $X \subset \mathbb{P}^{mn+m+n}$

As flexible as a traditional witness set

Developed by Hauenstein, Rodriguez, Leykin, and S. in
`arXiv:1507.07069`, `arXiv:1608.00540`, `arXiv:1908.00899`

General Witness Sets

$X:$ smooth, projective, $\dim X = n$
$A_kX:$ finitely generated with an effective basis and nondegenerate intersection pairing
$\Rightarrow\ b_k := \text{rank} A_kX$ ($=$ rankA_{n-k} by duality)

Let $L_1^{(k)}, \ldots, L_{b_k}^{(k)}$ be subvarieties such that
$A_kX = \mathbb{Z}\{[L_1^{(k)}], \ldots, [L_{b_k}^{(k)}]\}$ as a free abelian group

A *witness set* for a k-dimensional $V \subset X$ is $(W_\bullet, \Lambda_\bullet)$,

$$\Lambda_\bullet = \{\Lambda_1^{(n-k)}, \ldots, \Lambda_{b_k}^{(n-k)} \mid [\Lambda_i^{(n-k)}] = [L_i^{(n-k)}]\},$$

such that $W_i := V \cap \Lambda_i^{(k)}$ is transverse

$$W_\bullet = \{W_1, \ldots, W_{b_k}\}$$

Then $[V] = \sum c_i(W)[L_i^{(k)}]$, where, if
$M = (m_{i,j})$ with $m_{i,j} := \deg(L_i^{(k)} \cap \Lambda_j^{(n-k)})$, then
$c_\bullet(W) = (\deg(W_1), \ldots, \deg(W_{b_k}))M^{-1}$

Schubert Witness Sets

Flag varieties have all the best properties for general witness sets

Even specialized algorithms, such as regeneration, hold for their witness sets

Example : Let $\mathbb{G}(1,\mathbb{P}^4)$ be the Grassmannian of lines in $\mathbb{P}^4$

Fix a flag $E_\bullet : E_0 \in E_1 \subset E_2 \subset E_3 \subset E_4 = \mathbb{P}^4$ of linear subspaces $\dim E_i = i$

For any $0 \leq i < j \leq 4$, we have a *Schubert variety*,

$$L_{ij}E_\bullet := \{\ell \in \mathbb{G}(1,\mathbb{P}^4) \mid \emptyset \neq \ell \cap E_i \,,\ \ell \subset E_j\}.$$

Its dimension is $i+j-1$

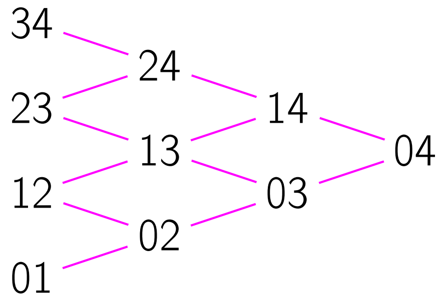
Schubert's Basis Theorem.

Classes $[L_{ij}E_\bullet]$ of Schubert varieties form a basis for $A_\bullet\mathbb{G}(1,\mathbb{P}^4)$

Schubert Witness Sets for $\mathbb{G}(1,\mathbb{P}^4)$

$$L_{ij}E_\bullet := \{\ell \in \mathbb{G}(1,\mathbb{P}^4) \mid \emptyset \neq \ell \cap E_i\,,\ \ell \subset E_j\}.$$

$L_{ij}E_\bullet \subset L_{ab}E_\bullet$ if $i \leq a$ and $j \leq b$.
Set $\widehat{ij} = 4{-}j, 4{-}i$.
Then $ij \leftrightarrow \widehat{ij}$ is vertical reflection in the line of symmetry, and $\widehat{13} = 13$, $\widehat{04} = 04$



This expresses duality. Suppose $E_\bullet$ and $E'_\bullet$ are general. Then $L_{ij}E_\bullet \cap L_{\widehat{ij}}E'_\bullet$ is a line, $\langle E_i \cap E'_{4-i}, E_j \cap E'_{4-j}\rangle$

For a smooth quadric Q in $\mathbb{P}^4$, let V_Q be the set of lines on Q. This is 3-dimensional. A Schubert witness set for V_Q has the form

$$(W_{13}, W_{04}, L_{13}E_\bullet, L_{04}E_\bullet) \qquad E_\bullet \text{ general}$$

$W_{ij} := V_Q \cap L_{ij}E_\bullet$. (In fact, $W_{04} = \emptyset$, W_{13} is 4 lines)