Toric Compactifications and Discrete Periodic Operators

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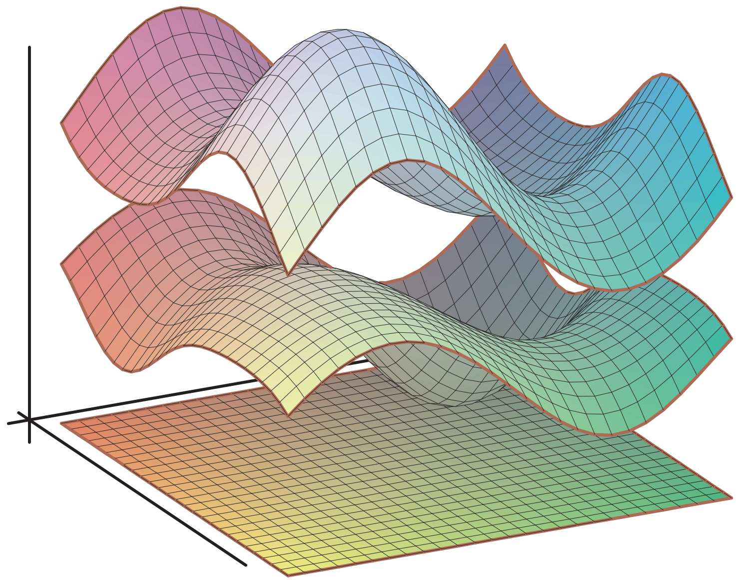


Work with Matthew Faust

Overview

The spectrum of a self-adjoint difference operator L on a $\mathbb{Z}^d$-periodic graph Γ is represented by the *dispersion relation* of its Floquet transform, which is a hypersurface in $\mathbb{T}^d \times \mathbb{R}$. ($\mathbb{T} \subset \mathbb{C}^\times$ is the unit circle.)

Complexifying gives an algebraic hypersurface in $(\mathbb{C}^\times)^d \times \mathbb{C}$.

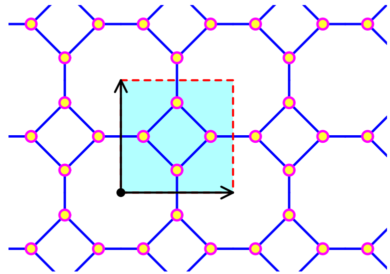
This has a natural compactification as a hypersurface in a projective *toric variety* X associated to the graph Γ.

This construction clarifies the structure of the complex dispersion relation at different infinities, as well as the real algebraic geometry nature of the original dispersion relation.

It also raises the question of extending the transformed operator to the 'boundary' of X.

Operator on a Discrete Periodic Graph

Let Γ be a $\mathbb{Z}^d$-periodic locally finite graph,
with adjacency relation $\sim$.
Write $v+\alpha$ for the translation of vertex
v by $\alpha \in \mathbb{Z}^d$
Choose a fundamental domain
$W \subset \text{vertices}(\Gamma)$.



Fix $\mathbb{Z}^d$-periodic labels.
$V : \text{vertices}(\Gamma) \to \mathbb{R}$ and $c : \text{edges}(\Gamma) \to \mathbb{R}$.

These parameters give the *Laplace-Beltrami* difference operator
$L = L(V, c)$ on functions $f : \text{vertices}(\Gamma) \to \mathbb{R}$,

$$Lf(v) \ := \ \sum_{v\sim u} c_{u,v}\big(f(v) - f(u)\big) \ + V(v)f(v)\,.$$

This is self-adjoint, and we want to study its spectrum.

Floquet Transformation

Floquet (Fourier) transform leads to an operator L on functions $f\colon \text{vertices}(\Gamma)\times\mathbb{T}^d\to\mathbb{R}$ such that

$$f(v+\alpha, z) \;=\; z^{\alpha} f(v,z) \qquad (z^{\alpha}=z_1^{\alpha_1}\cdots z_d^{\alpha_d}=e^{\alpha\cdot k2\pi i})\,.$$

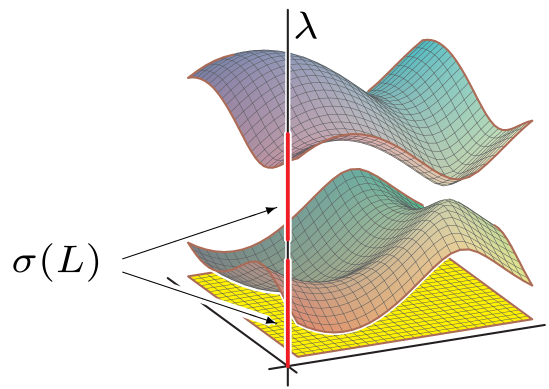
f is determined by $f(v,z)$ for $v\in W$, the fundamental domain. This is a vector of functions, $f(v)\colon\mathbb{T}^d\to\mathbb{R}$, for $v\in W$. Then

$$Lf(v) \;=\; \sum_{v\sim(u+\alpha)} c_{v,u+\alpha}\big(f(v)-z^{\alpha}f(u)\big) \;+V(v)f(v)\,.$$

$L(z)$ is a $W\times W$ matrix of Laurent polynomials in Floquet parameters z. The dispersion relation is defined by

$$0=\det(L(z)-\lambda I_W)\,.$$

It is a subset of $\mathbb{T}^d\times\mathbb{R}_\lambda$.
Its projection to $\mathbb{R}_\lambda$ is the spectrum of L.



Floquet and Fermi Varieties

It makes good algebraic sense to allow complex parameters
V : vertices$(\Gamma) \to \mathbb{C}$ and c: edges$(\Gamma) \to \mathbb{C}$,
complex-valued functions f, and complex Floquet parameters, $z \in (\mathbb{C}^\times)^d$.

We still have

$$Lf(v) \;=\; \sum_{v\sim(u+\alpha)} c_{v,u+\alpha}\big(f(v) - z^\alpha f(u)\big) \;\; + V(v)f(v)\,.$$

Define $\Phi = \Phi_{V,c}(z,\lambda) \;:=\; \det(L(z) - \lambda I_W)$, a Laurent polynomial whose coefficients depend upon the parameters V, c.

We have the (complex) Floquet variety

$$\mathcal{V}(\Phi) \;:=\; \{(z,\lambda) \mid \Phi(z,\lambda) = 0\} \;\subset\; (\mathbb{C}^\times)^d \times \mathbb{C}_\lambda\,,$$

and the (complex) Fermi hypersurface $\mathcal{V}(\Phi) \subset (\mathbb{C}^\times)^d$ (λ is fixed).

Several talks discuss properties of these varieties (Faust, Matos, Villalobos).

Compactifications

Compactification is an old theme in algebraic geometry (19^{th}c.), as well as in spectral theory (1990's).

(∗) Oftentimes compactifying gives the 'correct perspective'.

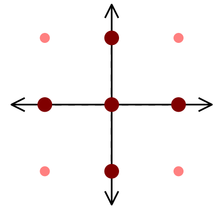
(∗) Compactifying allows global methods/constructions.

(∗) Information at infinity is sometimes easier to see.

(∗) A well-chosen compactification organizes the information at infinity.

Interchanging Nonlinearities

A common method in algebraic geometry is to replace a nonlinear equation by an equivalent linear equation.

Example. Let $\Phi = 3z_1^{-1} - 5z_2^{-1} + 7 - 5z_2 + 3z_1$.
Exponent vectors of monomials in Φ are its support, $\mathcal{A} =$  .

These monomials define a map from $(\mathbb{C}^\times)^2$,

$$\varphi\colon (z_1, z_2) \longmapsto [z_1^{-1}, z_2^{-1}, 1, z_2, z_1] \,.$$

If $y_1, \ldots, y_5$ are the coordinates of the codomain and
$\Lambda := 3y_1 - 5y_2 + 7y_3 - 5y_4 + 3y_5$, then $\Phi = \Lambda \circ \varphi$.

When the codomain of φ is a projective space, the closure of the image is a compactification, $X_{\mathcal{A}}$, of $(\mathbb{C}^\times)^2$.

This turns the nonlinear polynomial Φ on $(\mathbb{C}^\times)^2$ into a linear polynomial on $X_{\mathcal{A}}$.

Toric Compactification I

In general, a hypersurface $\mathcal{V}(\Phi)$ in $(\mathbb{C}^\times)^d$ often has an optimal compactification in a projective toric variety associated to Φ.

Exponents of monomials in Φ are its *support* $\mathcal{A} = \mathcal{A}(\Phi) \subset \mathbb{Z}^d$, and

$$\varphi_{\mathcal{A}}\colon (\mathbb{C}^\times)^d \ni z \longmapsto [z^\alpha \mid \alpha \in \mathcal{A}] \in \mathbb{P}^{\mathcal{A}},$$

where $\mathbb{P}^{\mathcal{A}}$ is the projective space with coordinates y_α indexed by $\mathcal{A}$.

If we write $\Phi = \sum_{\alpha\in\mathcal{A}} b_\alpha z^\alpha$, then the linear form

$$\Lambda := \sum_{\alpha\in\mathcal{A}} b_\alpha y_\alpha$$

pulls back to Φ in that $\Phi = \Lambda \circ \varphi_{\mathcal{A}}$.

Thus the nonlinear function Φ on $(\mathbb{C}^\times)^d$ contains the same information as the linear function Λ on the nonlinear set $\varphi_{\mathcal{A}}((\mathbb{C}^\times)^d)$.

Toric Compactification II

Recall that $\Phi = \sum_{\alpha\in\mathcal{A}} b_\alpha z^\alpha$ and

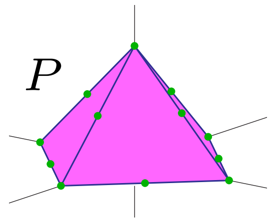
$$\varphi_{\mathcal{A}}\colon (\mathbb{C}^\times)^d \ni z \longmapsto [z^\alpha \mid \alpha\in\mathcal{A}] \in \mathbb{P}^{\mathcal{A}}\,.$$

Definition. The toric variety $X_{\mathcal{A}}$ is the closure in $\mathbb{P}^{\mathcal{A}}$ of $\varphi_{\mathcal{A}}(\mathbb{C}^\times)^d$.

This is compact and has a finite stratification by orbits of $(\mathbb{C}^\times)^d$:

$P :=$ convex hull of $\mathcal{A}(\Phi)$, the *Newton polytope* of Φ.

Orbits correspond to faces F of P, with $\varphi_{\mathcal{A}\cap F}(\mathbb{C}^\times)^d$ the orbit for the face F of P.



The closure of this orbit is the toric variety $X_{\mathcal{A}\cap F}$.

Thus the boundary $X_{\mathcal{A}} \setminus \varphi_{\mathcal{A}}(\mathbb{C}^\times)^d$ of $X_{\mathcal{A}}$ is the union of toric varieties $X_{\mathcal{A}\cap F}$ for the proper faces F of P.

Consequences for $\mathcal{V}(\Phi)$

Recall $\Phi = \sum_{\alpha\in\mathcal{A}} b_\alpha z^\alpha$ and $\Lambda = \sum_{\alpha\in\mathcal{A}} b_\alpha y_\alpha$ with $\Phi = \Lambda \circ \varphi_{\mathcal{A}}$.

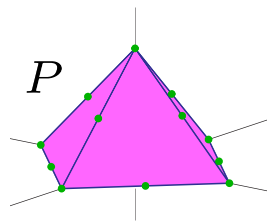
Thus $\varphi_{\mathcal{A}}(\mathcal{V}(\Phi)) \subset X_{\mathcal{A}} \cap \mathcal{V}(\Lambda)$, and

$\overline{\mathcal{V}(\Phi)} := X_{\mathcal{A}} \cap \mathcal{V}(\Lambda)$ is a (tropical) compactification of $\mathcal{V}(\Phi)$.

The points added at infinity have an elegant description:

For a face F of $P = \text{conv}(\mathcal{A}(\Phi))$, we have the *facial form*

$$\Phi|_F \ := \sum_{\alpha\in\mathcal{A}\cap F} b_\alpha z^\alpha \,.$$



Then $\overline{\mathcal{V}(\Phi)} \cap X_{\mathcal{A}\cap F} = \overline{\varphi_{\mathcal{A}\cap F}(\mathcal{V}(\Phi|_F))}$.

$\rightsquigarrow$ Links facial forms of Φ to points of $\mathcal{V}(\Phi)$ at infinity.

Floquet and Fermi, Really

This applies to Floquet and Fermi varieties, and it underlies Faust's results. It also clarifies the structure of these varieties at infinity.

Reality. $\sigma\colon (\mathbb{C}^\times)^d \ni z \mapsto \overline{z^{-1}}$ is a complex structure on $(\mathbb{C}^\times)^d$.

The complex structure σ extends to $X_{\mathcal{A}(\Phi)}$,
showing it to be an *arithmetic toric variety*.

As Γ is undirected, $L(z)^T = L(z^{-1})$, so it is Hermitian.
When V, c are real, ${}^\sigma\Phi(z) = \overline{\Phi(z^{-1})} = \Phi(z)$.
Thus the Fermi and Floquet varieties are stable under σ,
so that they are real algebraic varieties.

In particular, the real dispersion relation and Fermi surfaces are the real points of real algebraic varieties.

Extensions/Future Directions ?

- Realize L as an operator on $X_{\mathcal{A}(\Phi)}$ (and not just $(\mathbb{C}^\times)^d$).
  (This is necessary to complete this project.)

- The toric variety $X_{\mathcal{A}(\Phi)}$ may be singular.
  Seek a desingularization adapted to the Floquet and Fermi varieties.

- Generalize the 1991 "directional compactification" of Bättig, Knörrer, and Trubowitz to arbitrary complex Fermi hypersurfaces.