8.3 Central Projection

Chapter 9

Schubert Calculus

9.1 Grassmann Varieties

Grassmann varieties (or Grassmannians) are a class of algebraic varieties which are extremely important in mathematics and its applications. One reason for this is that many problems involving matrices have a geometric reformulation in terms of the Grassmannian. This leads to the use of the Grassmannian in some applications, which we will cover in in Part III. This geometric reformulation often turns a question about matrices into a question about certain Schubert subvarieties of the corresponding Grassmannian.

Fortunately for us, these Grassmann varieties have been intensively studied. They have many interesting structures which have useful properties. In this chapter, we will describe some of these structures which will be important in Part III.

Definition 9.1 Let $0 < p, m$ be integers, set $n := m + p$ and suppose V is an n-dimensional $\mathbb{F}$-vector space. The set of all p-dimensional linear subspaces of V is called the *Grassmannian of p-planes in V*. We will write $\text{Grass}(p, V)$ or $\text{Grass}(p, n)$ for this set. By Definition 8.1 of projective space, $\text{Grass}(1, V) = \mathbb{P}(V)$.

The row space of a $p \times n$ matrix $X = (x_{i,j})$ with full rank is a p-dimensional linear subspace M of $\mathbb{F}^n$. Let $\text{Mat}^\circ_{p\times n}\mathbb{F}$ be the collection of matrices with full rank, also called the affine Steifel manifold. The association

$$X \longmapsto \text{row space } X$$

defines a surjective map

$$\text{Mat}^\circ_{p\times n}\mathbb{F} \longrightarrow\!\!\!\!\rightarrow \text{Grass}(p, n)\,. \tag{9.1}$$

This map is not injective. If $T \in GL_p\mathbb{F}$, then TX and X have the same row space. Conversely, if matrices X and X' have the same row space M, then $X = TX'$, where $T \in GL_p\mathbb{F}$ is the matrix transforming the basis of M given by the row vectors of X' into the basis of row vectors of X. In this way, we identify the Grassmannian $\text{Grass}(p, n)$ as the set of orbits of $GL_p\mathbb{F}$ acting on the affine Steifel manifold $\text{Mat}^\circ_{p\times n}\mathbb{F}$.

We realize $\text{Grass}(p, n)$ as a projective variety via its Plücker embedding. Suppose a p-plane M is the row space of a $p \times n$ matrix X. The $\binom{n}{p}$ maximal $(p \times p)$ minors of X are the *Plücker coordinates* of M. (A maximal minor of X is the determinant of any $p \times p$-submatrix of X:

$$z_\alpha(X) \ := \ \det \begin{pmatrix} x_{1,\alpha_1} & \cdots & x_{1,\alpha_p} \\ \vdots & \ddots & \vdots \\ x_{p,\alpha_1} & \cdots & x_{p,\alpha_p} \end{pmatrix}$$

where $1 \leq \alpha_1 < \cdots < \alpha_p \leq n$ are the columns of X involved in this determinant.) Any other matrix with row space M has the form TX, for some $T \in GL_p\mathbb{F}$, and the $p \times p$ minors of TX differ from those of X by the scalar multiple $\det T$. Thus the Plücker coordinates of $M \in \mathrm{Grass}(p, n)$ determine a point in the projective space $\mathbb{P}^{\binom{n}{p}-1}$.

Write $\binom{[n]}{p}$ for the set of increasing sequences $\alpha : 1 \leq \alpha_1 < \cdots < \alpha_p \leq n$ which index these maximal minors. For each $\alpha \in \binom{[n]}{p}$, let z_α be the corresponding Plücker coordinate of $\mathbb{P}^{\binom{n}{p}-1}$.

We give another description of the Plücker embedding of $\mathrm{Grass}(p, V)$ using the pth exterior power $\bigwedge^p V$ of V. If M is a linear subspace of V, then $\bigwedge^p M$ is a linear subspace of $\bigwedge^p V$. When M is p-dimensional, $\bigwedge^p M$ is 1-dimensional, and hence is a point in $\mathbb{P}(\bigwedge^p V)$. In this way, we get the Plücker embedding

$$\begin{array}{rcl} \mathrm{Grass}(p, V) & \longrightarrow & \mathbb{P}(\bigwedge^p V) \\ M & \longmapsto & \bigwedge^p M \end{array}$$

To see that this coordinate-free description gives the same map as before, suppose V has a basis $e_1, \ldots, e_n$. Recall that $\bigwedge^p V$ has a basis consisting of the decomposable tensors

$$e_\alpha \ := \ e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_p} \quad \text{for } \alpha \in \binom{[n]}{p}.$$

Let $X = (x_{ij})$ be a $p \times n$ matrix with row space $M \in \mathrm{Grass}(p, V)$. Then M has a basis given by the row vectors $v_i \ := \ \sum_{j=1}^n x_{ij} e_j$ of X. Using the multilinearity and alternating relations, we see that

$$v_1 \wedge \cdots \wedge v_p \ = \ \sum_{\alpha \in \binom{[n]}{p}} z_\alpha(X) e_\alpha\,.$$

Since the decomposable tensor $v_1 \wedge \cdots \wedge v_p$ spans the 1-dimensional subspace $\bigwedge^p M$, the $z_\alpha(X)$ provide homogeneous coordinates for $\bigwedge^p M$ with respect to the basis $\{e_\alpha\}$ of $\bigwedge^p V$. We call them the Plücker coordinates of M. This shows the equality of the two descriptions of the Plücker embedding, and also that the image of the Grassmannian is the collection of all decomposable tensors.

We now prove that this Plücker map is actually an embedding.

Theorem 9.2 *The association of a p-plane M to its Plücker coordinates defines an injective map* $\mathrm{Grass}(p, n) \to \mathbb{P}^{\binom{n}{p}-1}$ *whose image is a smooth irreducible subvariety of dimension $mp = p(n-p)$.*

We will identify $\mathrm{Grass}(p, n)$ with its image under the Plücker embedding.

Proof: We show that the intersection of the image with each affine piece $\mathcal{U}_\alpha := \{[z_\beta] \in \mathbb{P}^{\binom{n}{p}-1} \mid z_\alpha \neq 0\}$ for $\alpha \in \binom{[n]}{p}$ of Plücker space is a smooth irreducible subvariety isomorphic to $\mathrm{Mat}_{p\times m}$, and this intersection is in bijection with those $M \in \mathrm{Grass}(p, n)$ whose αth Plücker coordinate is non-zero. It suffices to do this for the index $\alpha = (1, 2, \ldots, p)$, as the general case follows by permuting the basis $e_1, \ldots, e_n$ of $\mathbb{F}^n = V$.

Let $\mathcal{G}_\alpha \subset \mathrm{Grass}(p, n)$ be the subset consisting of those p-planes whose αth Plücker coordinate is non-zero. Then $\mathcal{G}_\alpha$ is the inverse image of the affine piece $\mathcal{U}_\alpha$ under the Plücker map. If we represent a p-plane $M \in \mathcal{G}_\alpha$ as the row space of a $p \times n$ matrix X, then the first p columns of X form an invertible matrix $T \in GL_p\mathbb{F}$, as its (non-vanishing) determinant is the αth Plücker coordinate of M. Replacing X by $T^{-1}X$, we see that we may assume that a p-plane $M \in \mathcal{G}_\alpha$ is the row space of a matrix of the form

$$[I_p : Y]\,, \tag{9.2}$$

where I_p is the $p \times p$ identity matrix and $Y \in \mathrm{Mat}_{p\times m}$. Conversely, given a matrix $Y \in \mathrm{Mat}_{p\times m}$, the row space M of the matrix (9.2) is a p-plane in $\mathcal{G}_\alpha$. Thus the association

$$\begin{array}{rcl} \mathrm{Mat}_{p\times m} & \longrightarrow & \mathcal{G}_\alpha \\ Y & \longmapsto & \text{row space } [I_p : Y] \end{array}$$

defines a bijection of $\mathrm{Mat}_{p\times m}$ with $\mathcal{G}_\alpha$.

For each $i = 1, \ldots, p$ and $j = 1, \ldots, m$, let $\beta(i, j) \in \binom{[n]}{p}$ be the index

$$\beta(i, j) \;=\; 1, 2, \ldots, i{-}1, i{+}1, \ldots, p, p + j\,.$$

Then the $\beta(i, j)$th maximal minor of the matrix (9.2) is $(-1)^{p-i}y_{ij}$, where y_{ij} is the i, jth entry in the matrix Y. In this way, we see that the composition

$$\mathrm{Mat}_{p\times m} \;\longrightarrow\; \mathcal{G}_\alpha \;\longrightarrow\; U_\alpha \tag{9.3}$$

is one to one, and hence $\mathcal{G}_\alpha$ is in bijection with its image. Since the maximal minors of the matrix (9.2) are polynomials in the entries of $Y \in \mathrm{Mat}_{p\times m}$, the composition (9.3) is a regular map. We claim that its image is an affine subvariety of U_α, which proves that the image of $\mathrm{Grass}(p, n)$ in $\mathbb{P}^{\binom{n}{p}-1}$ is a closed subvariety.

For this, we identify $\mathrm{Mat}_{p\times m}$ with the coordinate subspace of U_α spanned by the $z_{\beta(i,j)}$ for $i = 1, \ldots, p$ and $j = 1, \ldots, m$, where y_{ij} corresponds to $(-1)^{p-i}z_{\beta(i,j)}$ and set $\mathbb{A}^N$ to be the complementary coordinate subspace. Then the image of $\mathcal{G}_\alpha$ in $U_\alpha \simeq \mathrm{Mat}_{p\times m} \times \mathbb{A}^N$ is the graph of the map obtained by following the composition (9.3) with the projection $U_\alpha \twoheadrightarrow \mathbb{A}^N$. This shows that the image of $\mathcal{G}_\alpha$ is a closed subvariety of U_α, as the graph of a regular map is Zariski closed, by Lemma 8.15.

Since the image of $\mathcal{G}_\alpha$ is the graph of a regular map $\mathrm{Mat}_{p\times m} \to \mathbb{A}^N$, it is isomorphic to $\mathrm{Mat}_{p\times m}$, and hence smooth and irreducible. Since each $\mathcal{G}_\alpha$ is dense in the Grassmannian and the Grassmannian is the union of the $\mathcal{G}_\alpha$, this completes the proof of the theorem. $\square$

Remark 9.3 Another way to see that $\mathrm{Grass}(p,n)$ is smooth and irreducible uses group actions. The general linear group acts on $\bigwedge^p V$ by $g(v_1 \wedge \cdots \wedge v_p) = (gv_1) \wedge (gv_2) \wedge \cdots \wedge (gv_p)$. This functoriality of exterior powers gives and action of $GL(V)$ in $\mathbb{P}(\bigwedge^p V)$, and $\mathrm{Grass}(p,V)$ is a single orbit of this action. Thus it is smooth, by Theorem 6.58. Similarly, the irreducibility of $GL(V)$ implies that $\mathrm{Grass}(p,V)$ is irreducible. Since $\mathrm{Grass}(p,V)$ is a closed subvariety of Plücker space, it is a minimal orbit of this action, by Theorem 6.58.

Remark 9.4 In the proof of Theorem 9.2, we identify $\mathrm{Mat}_{p\times m}$ with an affine open subset $\mathcal{G}_{(1,2,\ldots,p)}$ of $\mathrm{Grass}(p,n)$ via

$$\mathrm{Mat}_{p\times m} \ni Y \ \longmapsto\ \text{row space } [I_p : Y] \in \mathrm{Grass}(p,n)\,. \tag{9.4}$$

This shows that the set of $p \times m$ matrices give a system of local coordinate charts for the Grassmannian.

The identification (9.4) of $\mathrm{Mat}_{p\times m}$ with an open subset of $\mathrm{Grass}(p,n)$ shows that $\mathrm{Grass}(p,n)$ is a compactification of the set of $p\times m$ matrices. Under this identification, the Plücker coordinates are all minors of all possible sizes of the $p \times m$ matrix Y. This suggests that the Grassmannian is a good choice of compactification for $\mathrm{Mat}_{p\times m}$ when we are studying equations involving various minors of matrices in $\mathrm{Mat}_{p\times m}$.

This identification of an open subset of $\mathrm{Grass}(p,V)$ as a set of matrices has a coordinate-free description. Choose a p-plane $H \in \mathrm{Grass}(p,V)$ and a complementary m-plane $K \subset V$ with $K \cap H = \{0\}$ so that $H \oplus K = V$. Given a linear map $\varphi : H \to K$, its graph

$$\Gamma_\varphi \ :=\ \{(x, \varphi(x)) \mid x \in H\}$$

is a p-plane in V which is complementary to K, and every p-plane complementary to K arises as the graph of a linear map $\varphi : H \to K$. In this way, we identify $\{M \in \mathrm{Grass}(p,V) \mid M \cap K = \{0\}\}$ with the space $\mathrm{Hom}(H,K)$ of linear maps from H to K, which is isomorphic to $\mathrm{Mat}_{p\times m}$.

Thus we see that the (Zariski) tangent space $T_h\mathrm{Grass}(p,V)$ to the Grassmannian $\mathrm{Grass}(p,V)$ at a point H is isomorphic to $\mathrm{Hom}(H,K)$ or to $\mathrm{Mat}_{p\times m}$. We remove the dependence of this identification on the choice of K by observing that if K is complementary to H, then the composition

$$K \ \hookrightarrow\ V \ \twoheadrightarrow\ V/H$$

is an isomorphism, and so we may canonically identify $\mathrm{Hom}(H,K)$ with $\mathrm{Hom}(H,V/H)$, and hence we obtain the following proposition.

Proposition 9.5 *The Zariski tangent space to the Grassmannian* $\mathrm{Grass}(p,V)$ *at a point H is naturally identified with* $\mathrm{Hom}(H,V/H)$.

Remark 9.6 In the proof of Theorem 9.2, we identified $\mathrm{Mat}_{p\times m}$ with a coordinate subspace of $\mathcal{U}_{(1,2,\ldots,p)}$ and we also showed that $\mathrm{Grass}(p,n) \cap \mathcal{U}_{(1,2,\ldots,p)}$ is the graph of the map $\mathrm{Mat}_{p\times m} \to \mathbb{A}^N$ given by the $p \times p$ maximal minors of the matrix (9.4) involving at least 2 columns in the range $p+1,\ldots,p+m$. This gives the following inhomogeneous system of equations for $\mathrm{Grass}(p,n)\cap\mathcal{U}_{(1,2,\ldots,p)}(=\mathcal{G}_{(1,2,\ldots,p)}$:

$$z_\gamma \;=\; \gamma\text{th maximal minor of } [I_p : Y]\,, \tag{9.5}$$

under the substitution $y_{ij} = (-1)^{p-i} z_{\beta(i,j)}$. These maximal minors include minors of all sizes of the matrix X. Homogenizing these equations with the coordinate function $z_{(1,2,\ldots,p)}$, we obtain a set of valid equations for $\mathrm{Grass}(p,n)$, which defines $\mathrm{Grass}(p,n)\cap\mathcal{U}_{(1,2,\ldots,p)}$. For each index $\alpha\in\binom{[n]}{p}$, we obtain a similar set of valid equations for $\mathrm{Grass}(p,n)$ by considering equations for $\mathrm{Grass}(p,n)\cap\mathcal{U}_\alpha$.

We close with a discussion of the homogeneous ideal $\mathcal{I}_{m,p}$ of $\mathrm{Grass}(p,n)$ in $\mathbb{P}^{\binom{n}{p}-1}$, called the *Plücker ideal*. By Remark 9.6, the Plücker ideal has generators of degree at most $\min\{m,p\}$.

When $p=2$ we have $\beta(1,j)=(2,2{+}j)$ and $\beta(2,j)=(1,2+j)$, and the equations (9.5) are relevant when γ is not equal to $(1,2)$ or to some $\beta(i,j)$. That is, when $\gamma = k<l$ with $2<k$. Then we obtain the equation.

$$z_{kl} \;=\; \det\begin{pmatrix} y_{1,k-2} & y_{1,l-2}\\ y_{2,k-2} & y_{2,l-2}\end{pmatrix} \;=\; \det\begin{pmatrix} -z_{2k} & -z_{2l}\\ z_{1k} & z_{1l}\end{pmatrix} \;=\; -z_{1l}z_{2k}+z_{1k}z_{2l}\,.$$

Homogenizing with respect to the coordinate z_{12}, we obtain $z_{1l}z_{2k}-z_{1k}z_{2l}+z_{12}z_{kl}=0$. If we permute the indices (changing α), then we obtain the system of equations

$$0 \;=\; \underline{z_{il}z_{jk}} - z_{ik}z_{jl} + z_{ij}z_{kl} \qquad \text{for } 1\leq i<j<k<l\leq n\,. \tag{9.6}$$

For example, when $n=4$, we obtain the equation (4.3) of Section 4.3. When $n=5$, the Grassmannian $\mathrm{Grass}(2,3)$ is defined by the 5 equations

$$\begin{array}{c}
\underline{z_{14}z_{23}} - z_{13}z_{24} + z_{12}z_{34}\\
\underline{z_{15}z_{23}} - z_{13}z_{25} + z_{12}z_{35}\\
\underline{z_{15}z_{24}} - z_{14}z_{25} + z_{12}z_{45}\\
\underline{z_{15}z_{34}} - z_{14}z_{35} + z_{13}z_{45}\\
\underline{z_{25}z_{34}} - z_{24}z_{35} + z_{23}z_{45}
\end{array} \tag{9.7}$$

This collection of quadratic trinomials turns out to be the reduced Gröbner basis for the Plücker ideal, which we analyze in detail in the next section.

9.2 Equations for the Grassmannian

We describe a quadratic Gröbner basis for the homogeneous Plücker ideal of the Grassmannian which generalizes the equations given in the last section when $p=2$. The combinatorics of this Gröbner basis enable the computation of the Hilbert series of the Grassmannian.

A first step is to identify the Plücker ideal of the Grassmannian. For each $\alpha\in\binom{[n]}{p}$, let z_α be an indeterminate—these are the Plücker coordinates. Let $x_{i,j}$ for $i=1,\ldots,p$ and $j=1,\ldots,n$ be indeterminates, which we view as entries of a generic $p\times n$ matrix, $X=(x_{i,j})$. Let $\phi_{m,p}$ be the map

$$\phi_{m,p} \;:\; \mathbb{F}[z_\alpha] \;\longmapsto\; \mathbb{F}[x_{i,j}]\,,$$

which sends a coordinate z_α to the αth maximal minor of the matrix $X=(x_{i,j})$.

Theorem 9.7 *The Plücker ideal $\mathcal{I}_{m,p}$ is the kernel of the map $\phi_{m,p}$.*

Proof: Under the algebraic-geometric dictionary of Theorem 6.39, the map $\phi_{m,p}$ corresponds to the map $\mathrm{Mat}_{p\times n} \to \mathbb{A}^{\binom{n}{p}}$ which sends a matrix to its $\binom{n}{p}$-tuple of maximal minors. The image of this map is the affine cone over the Grassmannian, and thus it is the subvariety of $\mathbb{A}^{\binom{n}{p}}$ defined by the Plücker ideal $\mathcal{I}_{m,p}$. □

Thus the Plücker ideal is the ideal of algebraic relations among the maximal minors of a generic matrix. It has an important corollary for Linear Algebra, which is valid over any field.

Corollary 9.8 *An $\binom{n}{p}$-tuple of numbers $(z_\alpha \mid \alpha \in \binom{[n]}{p})$ are the $\binom{n}{p}$ maximal minors of a matrix $Y \in \mathrm{Mat}_{p\times n}$ if and only if every polynomial in the Plücker ideal vanishes at that $\binom{n}{p}$-tuple.*

Thus a finite list of generators for $\mathcal{I}_{m,p}$, gives an effective algorithm to decide when a given list of numbers are the maximal minors of a matrix. We deduce another, algebraic corollary to Theorem 9.7.

Corollary 9.9 *The coordinate ring of the Grassmannian is isomorphic to the subring of $\mathbb{F}[x_{i,j}]$ generated by the maximal minors of the matrix $X = (x_{i,j})$.*

We will study this subring and use it to show that a certain collection of monomials in $\mathbb{F}[z_\alpha]$ are linearly independent in the coordinate ring of the Grassmannian. After we construct a Gröbner basis for the Plücker ideal, we will see that these monomials are the standard monomials from the theory of Gröbner bases, and thus are a basis for the coordinate ring of the Grassmannian.

We introduce some algebraic combinatorics to help describe this Gröbner basis. The set $\binom{[n]}{p}$ of sequences $\alpha : 1 \leq \alpha_1 < \cdots < \alpha_p \leq n$ has a natural partial order, called the *Bruhat order*

$$\alpha \leq \beta \iff \alpha_i \leq \beta_i \qquad \text{for } i = 1, \ldots, p\,.$$

We write $\mathbb{Y}_{m,p}$ for the resulting partially ordered set (or poset) and call it *Young's lattice*. Figure 9.1 shows $\mathbb{Y}_{3,2}$.

A monomial $z_\alpha \cdot z_\beta \cdots z_\gamma$ in the Plücker coordinates is *standard* if it is sorted, that is, if $\alpha \leq \beta \leq \cdots \leq \gamma$ in the Bruhat order on $\mathbb{Y}_{m,p}$. We derive some facts about these standard monomials to show that we have a Göbner basis for the Plücker ideal.

Theorem 9.10 *The set of standard monomials are linearly independent in the coordinate ring $\mathbb{F}[z_\alpha]/\mathcal{I}_{m,p}$ of the Grassmannian.*

We work in the ring $\mathbb{F}[x_{i,j}]$ to prove this. Linearly order the variables $x_{i,j}$ in this ring by the lexicographic order on $(i, -j)$. Thus

$$x_{1,n} < \cdots < x_{1,2} < x_{1,1} < x_{2,n} \cdots x_{p,n} < \cdots < x_{p,2} < x_{p,1}\,.$$

Let $\prec_{drl}$ be the resulting degree reverse lexicographic monomial order on the ring $\mathbb{F}[x_{i,j}]$.

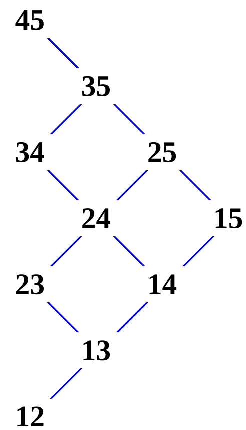


Figure 9.1: Young's Lattice, $\mathbb{Y}_{3,2}$, for $p = 2$ and $m = 3$.

Lemma 9.11 *The initial term of a maximal minor* $\phi_{m,p}(z_\alpha) \in \mathbb{F}[x_{i,j}]$ *is*

$$x_{1,\alpha_1}\, x_{2,\alpha_2} \cdots x_{p,\alpha_p}\,.$$

Proof: Consider the minor $\phi_{m,p}(z_\alpha)$

$$\phi_{m,p}(z_\alpha) \;=\; \det\begin{pmatrix} x_{1,\alpha_1} & \cdots & x_{1,\alpha_p} \\ \vdots & \ddots & \vdots \\ x_{p,\alpha_1} & \cdots & x_{p,\alpha_p} \end{pmatrix}.$$

In any term of this minor, the largest variable is the one from the last row. Thus the largest term in the reverse lexicographic order must include the smallest such variable x_{p,α_p}. Among all such terms, the largest term must include the smallest next variable $x_{p-1,\alpha_{p-1}}$. Continuing in this fashion, we see that the largest term of $\phi_{m,p}(z_\alpha)$ is $x_{1,\alpha_1}x_{2,\alpha_2}\cdots x_{p,\alpha_p}$. $\square$

Let $f = c \cdot z_\alpha z_\beta \cdots z_\gamma$ with $z_\alpha z_\beta \cdots z_\gamma$ a standard monomial (so that $\alpha \leq \beta \leq \cdots \leq \gamma$ in the Bruhat order). By Lemma 9.11, the initial term of $\phi_{m,p}(f)$ is

$$c \cdot x_{1,\gamma_1} \cdots x_{1,\beta_1} x_{1,\alpha_1}\, x_{2,\gamma_2} \cdots x_{p-1,\alpha_{p-1}}\, x_{p,\gamma_p} \cdots x_{p,\beta_p} x_{p,\alpha_p}\,,$$

and this term is written in order from smallest to largest variable.

Observe that we can recover the standard monomial f from this initial term. Thus the polynomials $\{\phi_{m,p}(f) \mid f \text{ is a standard monomial}\}$ have distinct initial terms, which implies they are linearly independent. By Corollary 9.9, this proves Theorem 9.10.

We derive a collection of valid quadratic relations in the Plücker ideal, called the Van der Waerden syzygies. To simplify our notation, we extend our indexing of Plücker coordinates to arbitrary sequences $i_1, \ldots, i_p$ of integers between 1 and n by setting

$$z_{i_1,\ldots,i_p} \;=\; -z_{i_1,\ldots,i_{j-1},i_{j+1},i_j,i_{j+2},\ldots,i_p}\,.$$

That is, if we permute the indices $i_1, \ldots, i_p$, the coordinate $z_{i_1,\ldots,i_p}$ changes by the sign of the corresponding permutation, and if there are repeated indices, then the coordinate is zero.

Let A, B, and C be sequences of numbers from $[n]$ of respective lengths $t-1$, $p+1$, and $p-t$. For a subset $I \subset [p+1]$ let B_I be the subsequence of B consisting of the elements in the positions indexed by I, and B_{I^c} the complementary subsequence of B. Define the *Van der Waerden syzygy* $[A\dot{B}C]$ to be

$$[A\dot{B}C] \ := \sum_{I\in\binom{[p+1]}{t}} (-1)^{\sum_j i_j - j}\, z_{A,\,B_{I^c}} \cdot z_{B_I,\,C}\,.$$

Example 9.12 When $A = 1$, $B = 2, 3, 4, 5$, and $C = 6$, we have $p = 3$ and $t = 2$ and so $[A\dot{B}C]$ is

$$z_{145}\, z_{236} - z_{135}\, z_{246} + z_{134}\, z_{256} + z_{125}\, z_{346} - z_{124}\, z_{356} + z_{123}\, z_{456}\,. \tag{9.8}$$

Theorem 9.13 *Each Van der Waerden syzygy is a valid relation in the Plücker ideal.*

Proof: Let X, Y, and Z be sequences of vectors in $\mathbb{F}^p$ of respective lengths $t-1, p+1$, and $p-t$. For each $I \in \binom{[p+1]}{t}$, define x_I and y_I to be the determinants

$$x_I \ := \ \det[X : Y_{I^c}] \quad \text{and} \quad y_I \ := \ \det[Y_I : Z]\,,$$

where the matrices are the concatenation of the given lists of column vectors, Y_I is the subsequence of Y consisting of the elements in the positions indexed by I, and Y_{I^c} the complementary subsequence.

Consider the following expression

$$\Phi(X, Y, Z) \ := \sum_{I\in\binom{[p+1]}{t}} (-1)^{\sum_j i_j - j}\, x_I \cdot y_I\,.$$

If the vectors in X and Z are fixed, then $\Phi(X, Y, Z)$ is a function of the sequence of vectors $Y \subset \mathbb{F}^p$. This is in fact a multilinear $p+1$ form, as each summand is multilinear, being a product of determinants. We claim that it is alternating, which implies it is identically zero. (Recall from multilinear algebra that an alternating $p+1$-form on p-dimensional space is identically zero.)

To show that Φ is alternating, suppose $Y_a = Y_{a+1}$. There are three kinds of summands in $\Phi(X, Y, Z)$:

1. Those in which $a \in I$ and $a+1 \notin I$,
2. those in which $a \notin I$ and $a+1 \in I$, and
3. those in which either $a, a+1 \in I$ or else $a, a+1 \notin I$.

Switching the positions of a and $a+1$ pairs each term of the first kind with a term of the second. These terms are equal, but have opposite signs, and so they cancel. Terms of the third kind are zero, since they involve a determinant with two equal columns. Thus $\Phi(X, Y, Z) = 0$, for every choice of X, Y, Z.

To see this implies that $\phi_{m,p}([A\dot{B}C]) = 0$ for any choice of A, B, C, let M be a $p \times n$ matrix and let X, Y, and Z be column vectors of M from columns A, B, and C, respectively. Under this specialization, $\phi_{m,p}([A\dot{B}C])(M) = \Phi(X, Y, Z) = 0$. Thus $\phi_{m,p}([A\dot{B}C])$ vanishes on any $p \times n$ matrix, and so $[A\dot{B}C] \in \mathcal{I}_{m,p}$, as claimed. $\square$

These Van der Waerden syzygies in fact form a Gröbner basis for the Plücker ideal, and a subset gives a minimal Gröbner basis. We describe that subset. Suppose we are given a pair $\alpha, \beta \in \mathbb{Y}_{m,p}$ with $\alpha \not\leq \beta$. There there is some index t with $\alpha_i \leq \beta_i$ for $i < t$ but $\alpha_t > \beta_t$. We say that the pair (α, β) has *violation* t. The violation of a pair (γ, δ) with $\gamma \leq \delta$ is $p + 1$.

Given $\alpha \not\leq \beta$, set $A := \alpha_1 < \cdots < \alpha_{t-1}$, $B := \beta_1 < \cdots < \beta_t < \alpha_t \cdots < \alpha_p$, and $C := \beta_{t+1} < \cdots < \beta_p$. Then the *straightening syzygy* $S(\alpha, \beta)$ of the pair α, β is the Van der Waerden syzygy $[A\dot{B}C]$. For example, the syzygy of Example 9.12 is the straightening syzygy when $\alpha = 145$ and $\beta = 236$. Also the polynomial in (9.6) is the straightening syzygy $S((i, l), (j, k))$. These are called straightening syzygies because of the following lemma.

Lemma 9.14 *Suppose $\alpha, \beta \in \mathbb{Y}_{m,p}$ with $\alpha \not\leq \beta$. Then every term of the straightening syzygy $S(\alpha, \beta)$ has the form $\pm z_\gamma z_\delta$ with $\gamma, \delta \in \mathbb{Y}_{m,p}$ where $\gamma \leq \alpha$ and $\beta \leq \delta$. Furthermore, the violation of the pair (γ, δ) is at least that of (α, β), with equality only when $(\gamma, \delta) = (\alpha, \beta)$.*

Thus if $\alpha \not\leq \beta$, $S(\alpha, \beta)$ rewrites the term $z_\alpha z_\beta$ as a linear combination of products $z_\gamma z_\delta$ with later violation, modulo the Plücker ideal. Rewriting any resulting incomparable pairs using their straightening syzygies, and so on, we see that $z_\alpha z_\beta$ is equal to a linear combination of terms $z_\gamma z_\delta$ with $\gamma \leq \delta$, that is, a linear combination of standard monomials.

Proof: Suppose $\alpha, \beta \in \mathbb{Y}_{m,p}$ with $\alpha \not\leq \beta$. Consider the term of $S(\alpha, \beta)$ indexed by $I \in \binom{[p+1]}{t}$

$$(-1)^{\sum_j i_j - j} \; z_{A,\, B_{I^c}} \cdot z_{B_I, C}\,,$$

where A, B, C are as given in the definition of straightening syzygy.

Let $z_\gamma := \pm z_{A,\, B_{I^c}}$ with $\gamma \in \binom{[n]}{p}$, so that γ is obtained by sorting the list (A, B_{I^c}). Let $i \in [p]$. If $i < t$, then γ_i is the ith smallest element of (A, B_{I^c}), which is at most α_i, the ith smallest element of A. If $i \geq t$, then γ_i is the $(p+1-i)$th largest element of (A, B_{I^c}), which is at most α_i, the $(p+1-i)$th largest element of (A, B). Thus $\gamma \leq \alpha$ in the Bruhat order. Let δ be obtained by sorting the list (B_I, C) so that $z_\delta = \pm z_{B_I, C}$. Similar arguments show that $\beta \leq \delta$ in the Bruhat order.

Let s be the violation of (γ, δ) and t the violation of (α, β). We must have $s \geq t$ as $\gamma_i \leq \alpha_i \leq \beta_i \leq \delta_i$ for $i < t$. Assume that $\alpha \neq \gamma$ and so $\beta \neq \delta$. Let c be the smallest (first) number in B_{I^c} and d the largest (last) number in B_I. Since $\alpha \neq \gamma$, we must have $c \leq \beta_t < \alpha_t \leq d$. Either $\gamma_t = c$ or else $\gamma_t \leq \alpha_{t-1} (< \beta_t)$, as bubble-sorting the sequence (A, B_{I^c}) shows. Similarly, either $\delta_t = d$ or else $\beta_{t+1} \leq \delta_t$. In each of the four possibilities, we have $\gamma_t < \delta_t$ and so $t < s$. $\square$

We use Lemma 9.14 and Theorem 9.10 to show that the straightening syzygies constitute a Gröbner basis for the Plücker ideal. We must first endow $\mathbb{F}[z_\alpha]$ with a monomial order. Linearly order the Plücker coordinates using lexicographic order $<_l$ on their indices. For example, with $p = 2$ and $m = 3$, the Plücker coordinates are, in increasing order,

$$z_{12},\ z_{13},\ z_{14},\ z_{15},\ z_{23},\ z_{24},\ z_{25},\ z_{34},\ z_{35},\ z_{45}\,.$$

We assume that any monomial in the Plücker coordinates is written with the variables in order from least to greatest in this lexicographic order. Since $<_l$ is compatible with the Bruhat order (it is a linear extension of the Bruhat order) this conforms to our convention for writing standard monomials.

Let $\succ_{drl}$ be the resulting degree reverse lexicographic monomial order on $\mathbb{F}[z_\alpha]$, that is

$$z_{\alpha^{(1)}} \cdots z_{\alpha^{(m)}} \ \succ_{drl}\ z_{\beta^{(1)}} \cdots z_{\beta^{(s)}}\ ,$$

if either $m > s$ or else $m = s$ and we have some $j \leq m$ with

$$\alpha^{(m)} = \beta^{(m)}, \ldots, \alpha^{(j+1)} = \beta^{(j+1)}, \quad \text{but} \quad \alpha^{(j)} <_l \beta^{(j)}\,.$$

The underlined terms in the generators of $\mathcal{I}_{2,3}$ in (9.7) are the initial terms in this monomial order.

Theorem 9.15 *The straightening syzygies $S(\alpha, \beta)$ with $\alpha <_l \beta$ and α, β incomparable in the Bruhat order ($\alpha \not\leq \beta$) form a minimal Gröbner basis for the Plücker ideal $\mathcal{I}_{m,p}$ with respect to the monomial order $\succ_{drl}$.*

Let $\alpha <_l \beta$ with $\alpha \not\leq \beta$ in the Bruhat order. By Lemma 9.14, any term $\pm z_\gamma z_\delta$ in the straightening syzygy $S(\alpha, \beta)$ satisfies $\gamma \leq \alpha$ and $\beta \leq \delta$ so in particular $\gamma <_l \alpha <_l \beta <_l \delta$. This implies that the initial term of $S(\alpha, \beta)$ is $z_\alpha z_\beta$.

Let IC be the set of pairs (α, β) with $\alpha <_l \beta$ and α, β incomparable ($\alpha \not\leq \beta$) in $\mathbb{Y}_{m,p}$. Let NS be the monomial ideal of the non-standard monomials. Then we have

$$NS \ = \ \langle z_\alpha z_\beta \mid (\alpha, \beta) \in \mathrm{IC} \rangle \ = \ \langle \mathrm{in}_{\succ_{drl}} S(\alpha, \beta) \mid (\alpha, \beta) \in \mathrm{IC} \rangle\,.$$

Since the straightening syzygies lie in the ideal $\mathcal{I}_{m,p}$ of the Grassmannian, NS is a subset of the initial ideal $\mathrm{in}_{\succ_{drl}} \mathcal{I}_{m,p}$ of the Plücker ideal. Thus we obtain maps of $\mathbb{F}$-vector spaces

$$\mathbb{F}[z_\alpha]/NS \ \longrightarrow\ \mathbb{F}[z_\alpha]/\mathrm{in}_{\succ_{drl}} \mathcal{I}_{m,p} \ \longrightarrow\ \mathbb{F}[z_\alpha]/\mathcal{I}_{m,p}\,.$$

(The second is the isomorphism of Theorem 7.12.) The first map is a surjection, and the second is an isomorphism. Since the standard monomials give a $\mathbb{F}$-basis for the quotient ring $\mathbb{F}[z_\alpha]/NS$, Theorem 9.10 implies that the composition is injective, and hence both maps are isomorphisms. Thus the straightening syzygies constitute a Gröbner basis for the Plücker ideal. This completes the proof of Theorem 9.15.

We now describe the form of the reduced Gröbner basis of the Plücker ideal $\mathcal{I}_{m,p}$. Because the straightening syzygies $S(\alpha, \beta)$ for $\alpha <_l \beta$ with α, β incomparable have distinct initial terms $z_\alpha z_\beta$, they are linearly independent. Thus we obtain an element

$R(\alpha, \beta)$ of the reduced Gröbner basis with initial term the non-standard monomial $z_\alpha z_\beta$ by reducing the straightening syzygy $S(\alpha, \beta)$ modulo the other straightening syzygies.

The reduction is accomplished more generally for any straightening syzygy $S(\alpha, \beta)$ using the *straightening algorithm* that rewrites any $z_\alpha z_\beta$ with $\alpha \not\leq \beta$ in terms of standard monomials. Given $\alpha \not\leq \beta$, let t be the violation of the pair and construct $S(\alpha, \beta)$. Then all terms $\pm z_\delta z_\gamma$ of $S(\alpha, \beta)$ have violation $s > t$ (except the term $z_\alpha z_\beta$). Initialize $f := S(\alpha, \beta)$. Given any non-standard term $c z_\gamma z_\delta$ of f with violation $s > t$, replace f by $f - cS(\delta, \gamma))$, canceling this term. Continuing in this fashion, we straighten $z_\alpha z_\beta$ and obtain a quadratic polynomial $R(\alpha, \beta) \in \mathcal{I}_{m,p}$ of the form

$$z_\alpha z_\beta - \text{linear combination of standard monomials}\,.$$

For example, suppose $\alpha = 245$ and $\beta = 136$. This pair has violation at position 1. Then

$$S(245, 136) \;=\; z_{245}z_{136} - z_{145}z_{236} - z_{125}z_{346} + z_{124}z_{356}\,.$$

We now straighten the non-standard term $z_{145}z_{236}$ (with violation 2) using the straightening syzygy $S(145, 236)$ of (9.8) to obtain

$$z_{245}z_{136} - z_{135}z_{246} + z_{134}z_{256} + z_{123}z_{456}\,. \tag{9.9}$$

Since all terms (besides the initial term) are standard, this is the element $R(245, 136)$ of the reduced Gröbner basis.

By Lemma 9.14, any standard term $c z_\gamma z_\delta$ of $R(\alpha, \beta)$ with $\gamma \leq \delta$ satisfies $\gamma \leq \alpha$ and $\beta \leq \delta$. Suppose that we straighten the pair α, β in the reverse order to obtain $R(\beta, \alpha)$. Then $R(\alpha, \beta) = R(\beta, \alpha)$, as the standard monomials are linearly independent. For example, $R(136, 245)$ is equal to $S(136, 245)$, and this equals the polynomial $R(245, 136)$ computed above. Thus if α, β are incomparable, the non-initial terms $c z_\gamma z_\delta$ of $R(\alpha, \beta)$ satisfy $\gamma \leq \alpha, \beta$ and $\alpha, \beta \leq \delta$.

To complete our description of the elements $R(\alpha, \beta)$ of the reduced Gröbner basis, we describe the lattice structure of Young's lattice, which is a distributive lattice. Given $\alpha, \beta \in \mathbb{Y}_{m,p}$ let $\alpha \wedge \beta$ be the meet, or greatest lower bound of α and β, and $\alpha \vee \beta$ be the join, or least upper bound of α and β in the Bruhat order. These lattice operations are defined as follows.

$$\begin{aligned}(\alpha \wedge \beta)_i &:= \min\{\alpha_i, \beta_i\}\,,\\ (\alpha \vee \beta)_i &:= \max\{\alpha_i, \beta_i\}\,.\end{aligned}$$

For example, if $m = p = 4$ and $\alpha = 1458$ and $\beta = 2367$, then

$$\alpha \wedge \beta \;=\; 1357 \quad \text{and} \quad \alpha \vee \beta \;=\; 2468\,.$$

For $\alpha, \beta, \gamma \in \mathbb{Y}_{m,p}$, they are distributive

$$\begin{aligned}\alpha \vee (\beta \wedge \gamma) &= (\alpha \vee \beta) \wedge (\alpha \vee \gamma)\,,\\ \alpha \wedge (\beta \vee \gamma) &= (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)\,.\end{aligned}$$

Theorem 9.16 *With respect to the monomial order $\succ_{drl}$, the Plücker ideal has a reduced Gröbner basis with generators $R(\alpha,\beta)$ indexed by incomparable pairs α,β in Young's lattice $\mathbb{Y}_{m,p}$ whose initial two terms are*

$$z_\alpha \cdot z_\beta - z_{\alpha\wedge\beta} \cdot z_{\alpha\vee\beta}\,.$$

Moreover, if $\lambda z_\gamma z_\delta$ is any non-initial term in $R(\alpha,\beta)$, then we have $\gamma \le \alpha\wedge\beta$ and $\alpha\vee\beta \le \delta$.

Proof: The only part left to prove is the statement that $R(\alpha,\beta)$ contains the term $-z_{\alpha\wedge\beta}\cdot z_{\alpha\vee\beta}$. Observe that if $z_\gamma z_\delta \succ_{drl} z_\rho z_\kappa$, then $\mathrm{in}\phi_{m,p}(z_\gamma z_\delta) \succ_{drl} \mathrm{in}\phi_{m,p}(z_\rho z_\kappa)$.

Thus the largest possible initial term $\mathrm{in}(z_\delta z_\gamma)$ for $\delta \le \alpha,\beta$ and $\alpha,\beta \le \gamma$ occurs with $\mathrm{in}\phi_{m,p}(z_{\alpha\wedge\beta}\cdot z_{\alpha\vee\beta})$. By Lemma 9.11 this is

$$x_{1,\alpha_1}x_{1,\beta_1}x_{2,\alpha_2}x_{2,\beta_2}\cdots x_{p,\alpha_p}x_{p,\beta_p}\,,$$

which equals $\mathrm{in}\phi_{m,p}(z_\alpha z_\beta)$. Thus the term $-z_{\alpha\wedge\beta}\cdot z_{\alpha\vee\beta}$ must occur in $R(\alpha,\beta)$, as it is the only possible term that can cancel this monomial in $\phi_{m,p}(z_\alpha z_\beta)$. $\square$

The form of this Gröbner basis implies a stronger result concerning different monomial orders.

Theorem 9.17 *Let $<$ be any linear extension of the Bruhat order and let $\succ_{drl}$ be the resulting degree reverse lexicographic monomial order on $\mathbb{F}[z_\alpha]$. Then the polynomials $R(\alpha,\beta)$ for α,β incomparable in the Bruhat order constitute the reduced Gröbner basis for the Plücker ideal with initial term $z_\alpha z_\beta$.*

Proof: Suppose α and β are incomparable and let $R(\alpha,\beta)$ be the polynomial of Theorem 9.16. Then every term $c\cdot z_\gamma z_\delta$ of $R(\alpha,\beta)$ satisfies $\delta \le \alpha$ and $\beta \le \delta$, as these are comparable in the Bruhat order. Thus $z_\alpha z_\beta \succeq_{drl} z_\gamma z_\delta$, and so $z_\alpha z_\beta$ is the initial term of $R(\alpha,\beta)$. The same arguments as in the proof of Theorem 9.15 suffice to establish this theorem. $\square$

By Theorem 9.16, the initial ideal $\mathrm{in}(\mathcal{I}_{m,p})$ of the Plücker ideal is generated by all monomials $z_\alpha z_\beta$ with $\alpha,\beta \in \binom{[n]}{p}$ incomparable in Young's Lattice. This combinatorial description enables us to write the initial ideal as an intersection of prime ideals, each the ideal of a coordinate linear subspace of dimension mp. For this, let $Ч_{m,p}$ be the set of (saturated) chains in Young's lattice. Examining Figure 9.1, we see that

$$\begin{aligned} Ч_{3,2} \;=\; \{&\{12,13,14,15,25,35,45\}\,,\ \{12,13,14,24,25,35,45\}\,,\\ &\{12,13,14,24,34,35,45\}\,,\ \{12,13,23,24,34,35,45\}\,,\\ &\{12,13,23,24,25,35,45\}\,\}\end{aligned}$$

Lemma 9.18 *Let $\mathcal{I}_{m,p}$ be the Plücker ideal. Then*

$$\mathrm{in}_{\succ_{drl}}(\mathcal{I}_{m,p}) \;=\; \bigcap_{ч\in Ч_{m,p}} \langle z_\gamma \mid \gamma \notin ч\rangle\,.$$

Proof: If $z_\alpha z_\beta$ is a generator of $\mathrm{in}(\mathcal{I}_{m,p})$, then α and β are incomparable in Young's lattice. Thus if ч $\in$ Ч$_{m,p}$ is a saturated chain, at most one of α or β lies in ч, and so $z_\alpha z_\beta$ lies in the ideal $\langle z_\gamma \mid \gamma \notin ч\rangle$.

Suppose now that z is a monomial not in $\mathrm{in}_{\succ_{drl}}(\mathcal{I}_{m,p})$. Then z is standard, and so we have $z = z_{\alpha^{(1)}} \cdot z_{\alpha^{(2)}} \cdots z_{\alpha^{(m)}}$ with $\alpha^{(1)} \leq \alpha^{(2)} \leq \cdots \leq \alpha^{(m)}$. Then there is some chain ч $\in$ Ч$_{m,p}$ containing the indices $\alpha^{(1)}, \ldots, \alpha^{(m)}$ and so the monomial z does not lie in the ideal $\langle z_\gamma \mid \gamma \notin ч\rangle$. This proves the equality of the two monomial ideals of the lemma, by Proposition 7.5. $\square$

Each ideal $\langle z_\gamma \mid \gamma \notin ч\rangle$ defines the coordinate subspace of Plücker space spanned by the coordinates z_α with $\alpha \in ч$, which is isomorphic to $\mathbb{P}^{mp}$. Thus $\mathcal{V}(\mathrm{in}_{\succ_{drl}}(\mathcal{I}_{m,p}))$ is the union of these coordinate subspaces, and so it has degree equal to their number. We deduce

Corollary 9.19 *The degree of the Grassmannian in its Plücker embedding is the number of maximal chains in Young's Lattice.*

Proof: The degree of the Grassmannian is the degree of its ideal $\mathcal{I}_{m,p}$. By Macaulay's Theorem??????, $\deg(\mathcal{I}_{m,p}) = \deg(\mathrm{in}(\mathcal{I}_{m,p}))$, and this is equal to the number of chains in Young's lattice, by Lemma 9.18. $\square$

The number $d_{m,p}$ of chains in Young's Lattice has a closed formula

$$d_{m,p} \;=\; \frac{1!\,2!\,3!\cdots(p-2)!\,(p-1)!\cdot(mp)!}{m!\,(m+1)!\,(m+\,2)!\cdots(m+p-1)!}\,. \tag{9.10}$$

When $p = 2$, we have $d_{m,2} = \frac{1}{m+1}\binom{2m}{m}$, which is a Catalan number. (See Table 7.1.)

Hilbert Series

Notes.

The results of this section are due to W.V.D. Hodge [20], although he did not use the language of Gröbner bases. Hodge invented the term standard monomial in this context. B. Sturmfels and N. White were inspired by Hodge's use of the term and introduced it into the theory of Gröbner bases [43].

Exercise 9.1 Prove that the operations $\wedge$ and $\vee$ on endow $\mathbb{Y}_{m,p}$ with the structure of a distributive lattice, that is, they give the greatest lower bound and least upper bound, and they are distributive.

Exercise 9.2 A *linear extension* of a poset P is a linear order on the elements of the poset that is compatible with the given order. Show that the lexicographic order $<_l$ on $\mathbb{Y}_{m,p}$ is a linear extension of $\mathbb{Y}_{m,p}$.

Exercise 9.3 Do the following in any computer algebra system.

(a) Write a program which generates a given Van der Waerden syzygy $[A\dot{B}C]$.

(b) Write a program that, given m and p, generates the collection of straightening syzygies $S(\alpha,\beta)$ for $(\alpha,\beta)\in IC$.

(c) Write a program that, given m and p, generates the full reduced Gröbner basis for the Plücker ideal.

Exercise 9.4 The straightening algorithm may be used more generally to rewrite any polynomial $f\in\mathbb{F}[z_\alpha]$ as a linear combination of standard monomials modulo $\mathcal{I}_{m,p}$. Each step proceeds as follows.

Given f, let cz^A be the largest (in $\succ_{drl}$) non-standard monomial in f and write $cz^A = cz^B\cdot z_\alpha z_\beta$ where $(\alpha,\beta)\in IC$ are the minimal incomparable variables appearing in z^A. Replace f by $f-cz^B S(\alpha,\beta)$, and repeat until all terms of f are standard.

(a) Prove that this algorithm terminates, and hence rewrites f as a linear combination of standard monomials.

(b) Implement this algorithm in any computer algebra system.

This more general algorithm is the classical straightening algorithm of invariant theory due to A. Young [49].

Exercise 9.5 The examples in the text (9.6) and (9.9) of elements of the reduced Gröbner bases for Plücker ideals have coefficients ±1. Prove or find a counterexample to the following statement:

The coefficients that occur in every $R(\alpha,\beta)$ all have absolute value 1.

Exercise 9.6 When $m=p=3$, compare the equations given in Remark 9.6 for $\mathrm{Grass}(p,n)$ with any of the Gröbner bases given in this section.

9.3 Schubert Decomposition

Let $\mathcal{U}_\alpha$ be the principal affine piece of Plücker space where the αth Plücker coordinate z_α is non-zero. In the proof of Theorem 9.2, we identified $\mathcal{G}_\alpha := \mathrm{Grass}(p,n)\cap\mathcal{U}_\alpha$ with $\mathrm{Mat}_{p\times m}$, showing that a p-plane in $\mathcal{G}_\alpha$ is represented uniquely as the row space of a matrix whose columns $\alpha_1,\ldots,\alpha_p$ form the identity matrix I_p. Such a matrix has the form

$$\begin{bmatrix} * & \cdots & * & 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * \\ * & \cdots & * & 0 & * & \cdots & * & 1 & * & \cdots & * & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 1 & * & \cdots & * \end{bmatrix}, \tag{9.11}$$

where the entries marked with $*$ are arbitrary numbers from $\mathbb{F}$ and the 1 in row i occurs in column α_i. The mp arbitrary entries in this matrix gives coordinates for $\mathcal{G}_\alpha$. As α varies in $\binom{[n]}{p}$, we obtain coordinate charts for the Grassmannian, giving it the structure of a $\mathbb{F}$-manifold.

A different set of coordinates for the Grassmannian which gives each point a unique coordinate is provided by the Schubert cellular decomposition, itself a consequence of

Gaussian elimination. Let M be a p-plane which is the row space of a $p \times n$ matrix X. Let E be the echelon matrix obtained from X by Gaussian elimination as in Example 7.18[1]. Then E has the form:

$$\begin{bmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 \end{bmatrix}, \qquad (9.12)$$

where the entries marked with $*$ are arbitrary numbers from $\mathbb{F}$. This matrix E is the unique $p \times n$ echelon matrix with row space M. In this way, we obtain a bijection of the Grassmannian with $p \times n$ echelon matrices of rank p, and this bijection may be computed using Gaussian elimination.

Figure 9.2 shows the possible echelon matrices when $m = p = 2$. (We arrange them according to the Bruhat order on their indices.)

$$\begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \end{bmatrix} \qquad \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Figure 9.2: Echelon matrices when $m = p = 2$.

Definition 9.20 For each $\alpha \in \binom{[n]}{p}$, the *Schubert cell* Ω°_α is the collection of all p-planes whose associated echelon matrix has leading 1s in columns $\alpha_1, \ldots, \alpha_p$.

Comparing (9.11) with (9.12), we see that the Schubert cell Ω°_α is just the coordinate subspace of $\mathcal{G}_\alpha$ obtained by setting entries $x_{i,j} = 0$ when $\alpha_i < j$. This shows that the Schubert cell is a locally closed subvariety of the Grassmannian.

Since each p-plane in the Grassmannian lies in some Schubert cell, we obtain the Schubert decomposition of the Grassmannian

$$\mathrm{Grass}(p,n) \;=\; \coprod_{\alpha \in \binom{[n]}{p}} \Omega^\circ_\alpha \,.$$

[1] Change the convention in the Example to conform to those here

Echelon matrices (9.12) with leading 1s in columns $\alpha_1, \ldots, \alpha_p$ have $\alpha_i - i$ arbitrary entries ($*$s) in row i. If we let $|\alpha| := \sum \alpha_i - i$, this shows that $\Omega^\circ_\alpha \simeq \mathbb{A}^{|\alpha|}$.

We define the *Schubert variety* Ω_α to be the Zariski closure of the Schubert cell Ω°_α. Then we have

$$\Omega_\alpha \;=\; \coprod_{\beta\leq\alpha} \Omega^\circ_\beta\,. \tag{9.13}$$

To see this, let $\mathcal{X}$ be the union of Schubert cells on the right. For each $\beta \in \binom{[n]}{p}$, let $\mathcal{M}_\beta \subset \mathrm{Mat}^\circ_{p\times n}\mathbb{F}$ be the set of full rank $p \times n$ matrices $X = (x_{i,j})$ which satisfy

$$x_{i,j} \;=\; 0 \qquad \text{if } \alpha_i < j \text{ for } i = 1, 2, \ldots, p\,.$$

Note that $\mathcal{M}_\alpha$ contains all echelon matrices E parameterizing Schubert cells Ω°_β for $\beta \leq \alpha$, and these are the only echelon matrices in M_α.

Thus the image of the map $\phi\colon \mathcal{M}_\alpha \to \mathrm{Grass}(p, n)$ given by

$$X \;\longmapsto\; \text{row space } X$$

contains $\mathcal{X}$. To see that the image equals $\mathcal{X}$, note that if a matrix $X \in \mathcal{M}_\alpha$, then its associated echelon matrix E is also in $\mathcal{M}_\alpha$. Since $\mathcal{X}$ is the image of an irreducible variety under a regular map, it is irreducible. Of all the Schubert cells contained in $\mathcal{X}$, Ω°_α is the one with largest dimension, so Ω°_α is dense in $\mathcal{X}$. This implies that $\mathcal{X} \subset \Omega_\alpha$. For the other inclusion, observe that if $\beta \not\leq \alpha$, then z_β vanishes on Ω°_α. Since z_β is nowhere zero on Ω°_β, this implies that $\Omega_\alpha \cap \Omega^\circ_\beta = \emptyset$ for $\beta \not\leq \alpha$. This establishes (9.13).

These considerations show that the Bruhat order has a geometric interpretation and shows that Ω_α is parameterized by $\mathcal{M}_\alpha$.

Corollary 9.21 *$\Omega_\alpha \subseteq \Omega_\beta$ if and only if $\alpha \leq \beta$ in $\mathbb{Y}_{m,p}$.*

The Schubert variety Ω_α has another description in terms of Plücker coordinates. On the Schubert cell Ω°_α, observe that z_α is never zero, while the coordinate z_β is identically zero when $\beta \not\leq \alpha$. Together with (9.13), this shows that

$$\Omega_\alpha \;=\; \{z \in \mathrm{Grass}(p, n) \mid z_\beta = 0 \text{ for } \beta \not\leq \alpha\}\,.$$

Indeed, the condition on the coordinates exclude all Schubert cells Ω°_β for $\beta \not\leq \alpha$ from this set while including all Schubert cells Ω°_β for $\beta \leq \alpha$, whose union is the Schubert variety Ω_α.

This set-theoretic description shows that the ideal of the Schubert variety Ω_α equals the radical of the ideal

$$\mathcal{I}_{m,p} \;+\; \langle z_\beta \mid \beta \not\leq \alpha\rangle\,.$$

In the monomial order $\succ_{drl}$ of Section 9.2, this has initial ideal

$$\langle z_\gamma z_\delta \mid \gamma, \delta \leq \alpha \text{ and } \gamma, \delta \text{ incomparable}\rangle \;+\; \langle z_\beta \mid \beta \not\leq \alpha\rangle\,.$$

which is square-free. Thus the original ideal is radical, by the result of Example????????, and thus we have

$$\mathcal{I}(\Omega_\alpha) \;=\; \mathcal{I}_{m,p} \;+\; \langle z_\beta \mid \beta \not\leq \alpha\rangle\,.$$

Let $\hat{0}$ be the sequence $1, 2, \ldots, p$, the minimal element in Young's lattice. We write $[\hat{0}, \alpha]$ to denote the set $\{\beta \mid \beta \leq \alpha\}$, the interval in Young's lattice below α. Let $Ч(\alpha)$ be the set of chains in the interval $[\hat{0}, \alpha]$. We have the following analog of Lemma 9.18 and Corollary 9.19, whose proof we leave to the exercises.

Corollary 9.22 *For any $\alpha \in \binom{[n]}{p}$, we have*

(i) $\mathrm{in}_{\succ_{drl}} \mathcal{I}(\Omega_\alpha) \;=\; \bigcap_{ч \in Ч(\alpha)} \langle z_\gamma \mid \gamma \notin ч \text{ or } \gamma \not\leq \alpha \rangle.$

(ii) *The degree of the Schubert variety Ω_α in the Plücker embedding equals the number of chains in the interval $[\hat{0}, \alpha]$.*

We now give a coordinate-free description of the Schubert decomposition. Let $e_1, e_2, \ldots, e_n$ be an ordered basis of $\mathbb{F}^n = V$, where e_j is the row vector whose only non vanishing entry is in column j. For each $i = 1, \ldots, n$, define the i-dimensional subspace F_i of $\mathbb{F}^n$ by

$$F_i \;:=\; \langle e_1, e_2, \ldots, e_i \rangle \,.$$

We call this collection of subspaces $F_\bullet$ the standard flag. If we interpret $\alpha_0 = 0$ and $\alpha_{p+1} = n + 1$, then

$$\Omega^\circ_\alpha \;=\; \{H \in \mathrm{Grass}(p, V) \mid \dim H \cap F_j = i \;\text{ if } \alpha_i \leq j < \alpha_{i+1} \text{ for } j = 1, \ldots, n\}\,,$$

as an examination of the echelon matrix (9.12) shows. The Schubert variety has a simpler description

$$\Omega_\alpha \;=\; \{H \in \mathrm{Grass}(p, V) \mid \dim H \cap F_{\alpha_i} \geq i \text{ for } i = 1, \ldots, p\}\,.$$

To see this, note that the conditions exclude all Schubert cells Ω°_β for $\beta \not\leq \alpha$ from this set while including all Schubert cells Ω°_β for $\beta \leq \alpha$, whose union is the Schubert variety Ω_α.

We extend this notation. A (complete) flag $F_\bullet$ in a vector space V (of dimension n) is a sequence of linear subspaces

$$F_\bullet \;:\; 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = V\,,$$

where $\dim F_i = i$. An index $\alpha \in \binom{[n]}{p}$ and a flag $F_\bullet$ together determine a Schubert variety

$$\Omega_\alpha F_\bullet \;:=\; \{H \in \mathrm{Grass}(p, V) \mid \dim H \cap F_{\alpha_i} \geq i \text{ for } i = 1, \ldots, p\}\,.$$

All Schubert varieties with the same index are isomorphic. Given two flags $F_\bullet, E_\bullet$, there is an element g of $GL(V)$ with $gF_\bullet = E_\bullet$, and so $g(\Omega_\alpha F_\bullet) = \Omega_\alpha E_\bullet$.

We consider this for $\mathrm{Grass}(2, 4)$. A 2-dimensional subspace of $\mathbb{F}^4$ is equivalently a line in $\mathbb{P}^3$, which identifies $\mathrm{Grass}(2, 4)$ with the set of lines in $\mathbb{P}^3$. A complete flag in $\mathbb{P}^3$ is given by specifying a point p incident to a line ℓ lying on a plane H. In Figure 9.3, we display the possible relative positions of lines $\mu \in \mathrm{Grass}(2, 4)$ with respect to a given flag $p \in \ell \subset H$, that is, the Schubert cells. This describes the Schubert decomposition of $\mathrm{Grass}(2, 4)$. Each picture is labeled with the associated

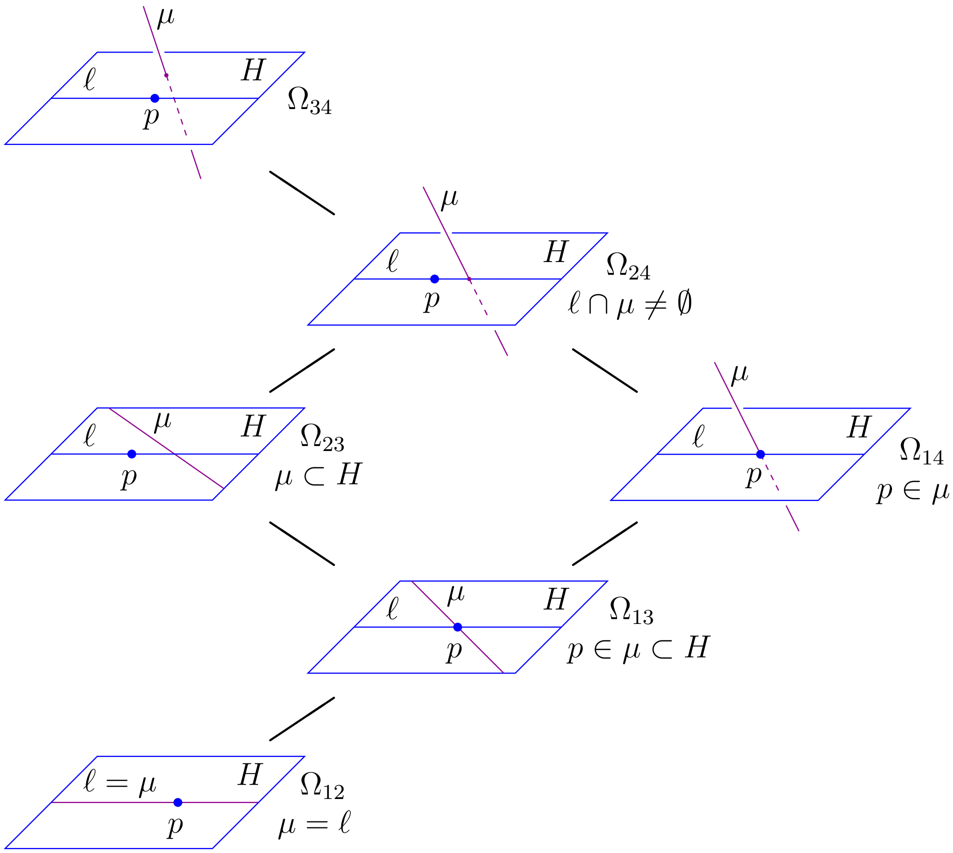


Figure 9.3: Schubert varieties of Grass(2, 4).

Schubert variety, as well as the essential conditions on lines μ defining that Schubert variety. It is very instructive to compare this with Figure 9.2.

We close this section with a description of the tangent spaces to Schubert varieties, at points in the dense Schubert cell. For this, we initially make a choice of coordinates, embodied by an ordered basis $e_1, e_2, \ldots, e_n$ for V, and a corresponding flag, $F_\bullet$. Let $H \in \Omega^\circ_\alpha$. Changing our basis, we may assume that $H = \langle e_{\alpha_1}, \ldots, e_{\alpha_p}\rangle$. Set $K := \langle e_j \mid j \notin \alpha\rangle$, the unique m-plane complementary to every p-plane in Ω°_α. Then the coordinate chart $\mathcal{G}_\alpha$ of (9.11) is isomorphic to $\operatorname{Hom}(H, K)$ via $\varphi \in \operatorname{Hom}(H, K) \mapsto \Gamma_\varphi$, the graph of φ. By our choices of H and K, we have $F_j = (H \cap F_j) \oplus (K \cap F_j)$. Thus if $x \in H$ and $\varphi \in \operatorname{Hom}(H, K)$, the point $x \oplus \varphi(x)$ lies in F_j only if both $x \in F_j$ and $\varphi(x) \in F_j$. This identifies

$$\Omega^\circ_\alpha \;=\; \{\Gamma_\varphi \mid \varphi \in \operatorname{Hom}(H, K) \text{ and } \varphi(H \cap F_{\alpha_i}) \subset K \cap F_{\alpha_i}, i = 1, \ldots, p\}\,.$$

Under the identification $T_H\mathrm{Grass}(p, V) = \operatorname{Hom}(H, K) \simeq \mathcal{G}_\alpha$, this is $T_H\Omega_\alpha$.

We remove the dependences on choices and give a coordinate-free description of the tangent space. Under the canonical isomorphism $K \simeq V/H$, which gives $T_H\mathrm{Grass}(p, V) = \operatorname{Hom}(H, V/H)$, we have $K \cap F_{\alpha_i} \simeq (F_{\alpha_i} + H)/H$, and so

$$T_H\Omega_\alpha F_\bullet \;=\; \{\varphi \in \operatorname{Hom}(H, V/H) \mid \varphi(H \cap F_{\alpha_i}) \subset (F_{\alpha_i} + H)/H, i = 1, \ldots, p\}\,.$$

Since $\dim H \cap F_{\alpha_i} = i$ and e_i spans $H \cap F_{\alpha_i}$ over $H \cap F_{\alpha_{i-1}}$, a tangent vector φ is given by the p vectors $\varphi(e_1), \ldots, \varphi(e_p)$ where $\varphi(e_i) \in (F_{\alpha_i} + H)/H$.

Exercise 9.7 Show directly, by computing the transition functions, that the coordinates for $\mathcal{G}_\alpha$ described at the beginning of Section 9.3 give the Grassmannian the structure of a $\mathbb{F}$-manifold.

Exercise 9.8 Let B_+ be the Borel subgroup of GL_n consisting of the invertible upper triangular matrices. Show that the orbits of B_+ acting on the Grassmannian Grass(p,n) are exactly the Schubert cells Ω_α°.

Exercise 9.9 Prove Corollary 9.22.

9.4 The Simple Schubert Calculus

We generalize the geometric problem of Chapter 4, studying quantitative aspects of intersections of Schubert varieties. This yields a large class of non trivial and important geometric problems which we will solve. In this section we formulate the general problem in the Schubert calculus of enumerative geometry and consider its basic aspects. Then we solve this problem for the important case of simple (codimension-1) Schubert varieties, which is used in Part III. We also discuss refinements, including the question of real solutions. In the next section (which is independent of the rest of the book), we solve the general problem using tools from Algebraic Topology and Intersection Theory, namely cohomology and Chow rings.

There is an alternative notation for Schubert varietites which is convenient when discussing intersections of Schubert varieties. A *partition* λ is a weakly decreasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$. Given a flag $F_\bullet$ and a partition λ with $m \geq \lambda_1$, set

$$\mathcal{X}_\lambda F_\bullet \ := \ \{H \in \mathrm{Grass}(p,V) \mid \dim H \cap F_{m+i-\lambda_i} \geq i \text{ for } i = 1,\ldots,p\}\,.$$

Since $\dim H \cap F_{m+i} \geq i$ for any p-plane H, λ_i measures how exceptionally H has a dimension-i intersection with a subspace of the flag. If we set

$$\alpha(\lambda) \ := \ m+1-\lambda_1 < m+2-\lambda_2 < \cdots < m+p-\lambda_p\,,$$

then $\mathcal{X}_\lambda F_\bullet = \Omega_{\alpha(\lambda)} F_\bullet$. Since $\dim \mathcal{X}_\lambda F_\bullet = |\alpha(\lambda)| = mp - \sum_i \lambda_i$, the Schubert variety $\mathcal{X}_\lambda F_\bullet$ has codimension $|\lambda| := \lambda_1 + \cdots + \lambda_p$ in Grass(p,V).

Given a collection of partitions $\lambda^1, \lambda^2, \ldots, \lambda^s$ with $\sum |\lambda^j| \leq mp$, the intersection

$$\mathcal{X}_{\lambda^1} F^1_\bullet \cap \mathcal{X}_{\lambda^2} F^2_\bullet \cap \cdots \cap \mathcal{X}_{\lambda^s} F^s_\bullet\,,$$

either is empty or it has dimension at least $mp - \sum |\lambda^j|$, for any choice of flags $F^1_\bullet, F^2_\bullet, \ldots, F^s_\bullet$. This minimum dimension occurs when the flags are in general position.

Theorem 9.23 *Let $\lambda^1, \lambda^2, \ldots, \lambda^s$ be partitions with $\sum |\lambda^i| \leq mp$. Then there is a non empty open subset $\mathcal{U} \subset [GL(V)]^s$ such that if $F_\bullet$ is any flag, then*

$$\mathcal{X}_{\lambda^1} g_1 F_\bullet \cap \mathcal{X}_{\lambda^2} g_2 F_\bullet \cap \cdots \cap \mathcal{X}_{\lambda^s} g_s F_\bullet\,, \tag{9.14}$$

either is empty or it has dimension $mp - \sum |\lambda^j|$, for all $(g_1,\ldots,g_s) \in \mathcal{U}$.

An intersection of subvarieties is called *proper* either if it is empty, or else if it has the expected dimension, that is if the codimension of the intersection is the sum of the codimensions of the varieties being intersected. Theorem 9.23 states that a general intersection of Schubert varieties is proper. We deduce this Theorem from the following more general lemma concerning transitive group actions on varieties.

Lemma 9.24 *Suppose G is an irreducible algebraic group acting transitively on a variety $\mathcal{X}$. If $\mathcal{Y}, \mathcal{Z} \subset \mathcal{X}$, then there is a non-empty open subset $\mathcal{U} \subset G$ such that the intersection $g\mathcal{Y} \cap \mathcal{Z}$ either is empty for each $g \in \mathcal{U}$, or else every component has codimension* $\mathrm{codim}_{\mathcal{X}}\mathcal{Y}$ *in $\mathcal{Z}$, for each $g \in \mathcal{U}$.*

Proof of Theorem 9.23: We use the following trick of realizing an intersection of several subvarieties as a single intersection with a diagonal. Given $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_s \subset \mathcal{X}$, let $\Delta = \{(x, \ldots, x) \mid x \in \mathcal{X}\} \subset \mathcal{X}^s$ be the (small) diagonal, which is identified with $\mathcal{X}$, and set $\mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_s \subset \mathcal{X}^s$. Then

$$\mathcal{Y} \cap \Delta \;=\; \{x \in \mathcal{X} \mid x \in \mathcal{Y}_i \; i = 1, \ldots, s\} \;=\; \mathcal{Y}_1 \cap \cdots \cap \mathcal{Y}_s\,.$$

(This equality is valid not just set-theoretically, but also at the level of ideals.) To apply Lemma 9.24, set $G := (GL(V))^s$, $\mathcal{X} := (\mathrm{Grass}(p, V))^s$, and $Z := \mathrm{Grass}(p, V)$, considered as the diagonal Δ in $\mathcal{X}$. Let $F_\bullet$ be any flag and set $\mathcal{Y} := \mathcal{X}_{\lambda^1}F_\bullet \times \cdots \times \mathcal{X}_{\lambda^1}F_\bullet \subset \mathcal{X}$. We see that if $g = (g_1, g_2, \ldots, g_s) \in G$, then

$$(g\mathcal{Y}) \cap \mathcal{Z} \;=\; \mathcal{X}_{\lambda^1} g_1 F_\bullet \cap \cdots \cap \mathcal{X}_{\lambda^1} g_s F_\bullet\;,$$

The intersection of (9.14). Since $\mathrm{codim}_{\mathcal{X}}\mathcal{Y} = \sum |\lambda^j|$, and G acts transitively on $\mathcal{X}$, Lemma 9.24 guarantees the existence of a non-empty open set $\mathcal{U} \subset [GL(V)]^s$ consisting of s-tuples $(g_1, \ldots, g_s)$ such that either the intersection (9.14) is empty, or else it has codimension $\sum |\lambda^j|$ in $\mathcal{Z} = \mathrm{Grass}(p, V)$. □

Proof of Lemma 9.24: Let $\mathcal{W} := \{(g, x) \mid x \in g\mathcal{Y}\}$ be the incidence variety and consider the diagrams

$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ G & & \mathcal{X} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & \mathcal{W}|_{\mathcal{Z}} := q^{-1}(\mathcal{Z}) \arrow[dl] \arrow[dr] & \\ G & & \mathcal{Z} \end{tikzcd}$$

The fibre $p^{-1}(g)$ is $\{g\} \times g\mathcal{Y}$, which is isomorphic to $\mathcal{Y}$. Thus $\mathcal{W}$ has dimension $\dim G + \dim \mathcal{Y}$. The fibre $\mathcal{W}_x := q^{-1}(x)$ is equal to $\{g \mid x \in g\mathcal{Y}\}$ and so we have $\mathcal{W}_{h\cdot x} = h\mathcal{W}_x$. Since G acts transitively on $\mathcal{X}$, $\mathcal{W} \to \mathcal{X}$ is a fibre bundle with isomorphic fibres, each of dimension $\dim \mathcal{W} - \dim \mathcal{X} = \dim G + \dim \mathcal{Y} - \dim \mathcal{X}$. This implies $\mathcal{W}|_{\mathcal{Z}} \to \mathcal{Z}$ is a fibre bundle with $\dim \mathcal{W}|_{\mathcal{Z}} = \dim G + \dim \mathcal{Y} + \dim \mathcal{Z} - \dim \mathcal{X}$.

Observe that the fibre of the map $\mathcal{W}|_{\mathcal{Z}} \to G$ over a point $g \in G$ is

$$\{z \in \mathcal{Z} \mid z \in g\mathcal{Y}\} \;=\; \mathcal{Z} \cap g\mathcal{Y}\,.$$

If the map $\mathcal{W}|_{\mathcal{Z}} \to G$ is not dominant, then let $\mathcal{U}$ be an open set in the complement of the image. For $g \in \mathcal{U}$, the fibre of $\mathcal{W}|_{\mathcal{Z}} \to G$ is empty, so $\mathcal{Z} \cap g\mathcal{Y} = \emptyset$. Otherwise

$\mathcal{W}|_{\mathcal{Z}} \to G$ is dominant. By Theorem????[2], there is an open subset $\mathcal{U}$ of G such that the fibres are equidimensional of dimension

$$\dim \mathcal{W}|_{\mathcal{Z}} - \dim G \;=\; \dim \mathcal{Z} + \dim \mathcal{Y} - \dim \mathcal{X}\,,$$

and so for $g \in \mathcal{U}$, $\mathcal{Z} \cap g\mathcal{Y}$ has codimension in $\mathcal{Z}$ equal to $\mathrm{codim}_{\mathcal{X}}\mathcal{Y}$. $\square$

The flag manifold $\mathbb{F}\ell_n$ (or $\mathbb{F}\ell(V)$) is the set of flags in $\mathbb{F}^n$ (respectively V). Its structure will be studied in Section 9.6. In particular, $GL(V)$ acts transitively on $\mathbb{F}\ell(V)$ and so we may equivalently talk of a general flag $F_\bullet \in \mathbb{F}\ell(V)$ or a general translate $gF_\bullet$ of a given flag.

Definition 9.25 Schubert data consists of a list of partitions $\lambda^1, \lambda^2, \ldots, \lambda^s$ satisfying $\sum_i |\lambda^i| = mp$. Given Schubert data and general flags in F^n, Theorem 9.23 guarantees that the intersection

$$\mathcal{X}_{\lambda^1}F^1_\bullet \cap \mathcal{X}_{\lambda^2}F^2_\bullet \cap \cdots \cap \mathcal{X}_{\lambda^s}F^s_\bullet \tag{9.15}$$

is zero-dimensional (or empty). The general problem in the Schubert calculus of enumerative geometry is to determine the number of points, $d(\lambda^1, \lambda^2, \ldots, \lambda^s)$, in the intersection. (This notation suppresses the possible dependence of this number on the field $\mathbb{F}$.)

This number is well-defined when $\mathbb{F}$ is algebraically closed; it is the degree of the map $\pi\colon \mathcal{W} \to (\mathbb{F}\ell_n)^s$, where $\mathcal{W}$ is the incidence variety of the intersection (9.15)

$$\mathcal{W} \;:=\; \{(H, F^1_\bullet, \ldots, F^s_\bullet) \mid H \in \mathcal{X}_{\lambda^i}F^i_\bullet \; i = 1, \ldots, s\}\,.$$

Suppose we consider the intersection (9.15) scheme-theoretically. That is, we take into account the equations defining the Schubert varieties, which define a zero-dimensional ideal, $\mathcal{I}$. Then the degree of $\mathcal{I}$ gives an algebraic count of the number of solutions, and we have

$$\deg \mathcal{I} \;\geq\; \deg \sqrt{\mathcal{I}} \;=\; d(\lambda^1, \lambda^2, \ldots, \lambda^s)\,,$$

as in Section 7.3. The first inequality is an equality when the intersection is transverse, so that the ideal $\mathcal{I}$ is radical. In Section 9.5 we describe the classical algorithms of the Schubert calculus which compute $\deg \mathcal{I}$. Interestingly, these are independent of the field, $\mathbb{F}$.

For algebraically closed fields of characteristic zero, Kleiman's Transversality Theorem [24] guarantees that the intersection (9.15) is transverse, when the flags are in general position, and so $\deg \mathcal{I} = d(\lambda^1, \lambda^2, \ldots, \lambda^s)$. For fields with positive characteristic, is it not known whether general Schubert subvarieties meet transversally, and the following Transversality Conjecture remains open.

Conjecture 9.26 *Let $\mathbb{F}$ be any infinite field. Given any collection of Schubert data $\lambda^1, \ldots, \lambda^s$, there is a non-empty open subset $\mathcal{U}$ of $(\mathbb{F}\ell_n)^s$ consisting of flags $(F^1_\bullet, \ldots, F^s_\bullet)$ such that the intersection (9.15) is transverse.*

[2] This will be given earlier in a result on dominant maps

Even if Conjecture 9.26 were true, we are not guaranteed $\deg \mathcal{I}$ solutions unless $\mathbb{F}$ is algebraically closed. In particular, when $\mathbb{F} = \mathbb{R}$ (so (9.15) is transverse for general flags), we may ask: which numbers $0 \leq d \leq \deg \mathcal{I}$ of real solutions can occur (for some choice of general flags)?

In the remainder of this section we show that Conjecture 9.26 holds when the partitions λ^i are all equal to $(1, 0, \ldots, 0)$. The corresponding Schubert varieties are hypersurfaces, called simple Schubert varieties. The same proof also shows that it is possible for all of the solutions to be real. Lastly, we give a closed formula for this number[3].

We abbreviate $\mathcal{X}_{(1,0,\ldots,0)}F_{\bullet}$ to $\mathcal{X}_1 F_{\bullet}$. We have $\alpha(1, 0, \ldots, 0) = m < m + 2 < \cdots < m + p$. Thus, when $F_{\bullet}$ is the standard flag,

$$X_1 F_{\bullet} \ = \ \mathcal{V}(z_{\hat{1}}) \,,$$

where $\hat{1} = m + 1 < \cdots < m + p$ is the maximal element in the Bruhat order. Thus the Schubert variety $\mathcal{X}_1 F_{\bullet}$ is a hyperplane section of the Grassmannian in its Plücker embedding. Because this Schubert variety has codimension 1, it is called a simple Schubert variety. Of the p conditions defining $\mathcal{X}_1 F_{\bullet}$, all except one are forced by linear algebraic reasons, leaving only the single condition

$$\mathcal{X}_1 F_{\bullet} \ = \ \{H \in \mathrm{Grass}(p, V) \mid \dim H \cap F_m \geq 1\} \,.$$

Note that $\dim H \cap F_m \geq 1 \Leftrightarrow H \cap F_m \neq \{0\}$.

The simple Schubert calculus concerns the enumerative problems in the Schubert calculus where all (except possibly one) Schubert variety is simple. That is, intersections of the form

$$\mathcal{X}_{\lambda} F_{\bullet}^{0} \cap \mathcal{X}_1 F_{\bullet}^{1} \cap \cdots \cap \mathcal{X}_1 F_{\bullet}^{mp-|\lambda|} \,,$$

where the flags are in general position. Since each simple Schubert variety $\mathcal{X}_1 F_{\bullet}$ is a hyperplane section of the Grassmannian, the expected number of points in this intersection is $\deg \mathcal{X}_{\lambda} F_{\bullet}$. This is obtained if the intersection is transverse, by Theorem **Bézout**.

A key step towards understanding the simple Schubert calculus is the following result of Schubert, concerning the intersection of Schubert varieties. Subvarieties $\mathcal{Y}, \mathcal{Z}$ of a smooth variety $\mathcal{X}$ meet generically transversally if every irreducible component $\mathcal{V}$ of $\mathcal{Y} \cap \mathcal{Z}$ has a dense open subset $\mathcal{U}$ of its smooth points such that for $u \in \mathcal{U}$.

$$T_u \mathcal{Y} + T_u \mathcal{Z} \ = \ T_u \mathcal{X} \,,$$

that is, $\mathcal{Y}$ and $\mathcal{Z}$ meet transversally along $\mathcal{U}$.

Theorem 9.27 *Let $\mathbb{F}$ be any field. Then in Grass(p, n)*

$$\Omega_{\alpha} \cap \mathcal{V}(z_{\alpha}) \ = \ \bigcup_{\beta \lessdot \alpha} \Omega_{\beta} \,, \tag{9.16}$$

and this intersection is generically transverse.

[3] Do it!

Proof: The Plücker coordinate z_α does not vanish on the Schubert cell Ω_α°, but it vanishes on each Schubert cell Ω_β° for $\beta < \alpha$. This implies the first equality below.

$$\Omega_\alpha \cap \mathcal{V}(z_\alpha) \;=\; \bigcup_{\gamma<\alpha} \Omega_\gamma \;=\; \bigcup_{\beta \lessdot \alpha} \Omega_\beta\,.$$

The second equality follows as elements of the set $\{\beta \mid \beta \lessdot \alpha\}$ are maximal in the open interval $[\hat{0}, \alpha)$.

To see that this intersection is generically transverse, consider the system $\mathcal{Z}_{\alpha,\beta}$ of local coordinates for Ω_α defined for each $\beta \lessdot \alpha$: Let $\mathcal{Z}_{\alpha,\beta} \subset \mathcal{M}_\alpha$ be the set of $p \times n$ matrices $X = (x_{i,j})$ with

$$\begin{aligned} x_{i,\beta_i} &= 1 \\ x_{j,\beta_i} &= 0 \;\text{ for } j \neq i \\ x_{i,j} &= \quad \text{for } j > \alpha_i \end{aligned}$$

Thus a matrix X in $\mathcal{Z}_{\alpha,\beta}$ has almost the same form as a matrix (9.12) parameterizing the Schubert cell Ω_β°, except that the entry x_{k,α_k} need not vanish, where k is the unique index with $\beta_k < \alpha_k$. For example if $m = 5$, $p = 4$, $\alpha = (3,6,7,9)$ and $\beta = (3,5,7,9)$, then $\mathcal{Z}_{\alpha,\beta}$ is

$$\left[\begin{array}{ccccccccc} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & x_{2,6} & 0 & 0 & 0 \\ * & * & 0 & * & 0 & * & 1 & 0 & 0 \\ * & * & 0 & * & 0 & * & 0 & * & 1 \end{array}\right]$$

A matrix $X \in \mathcal{Z}_{\alpha,\beta}$ represents a unique p-plane in Ω_α, and on $\mathcal{Z}_{\alpha,\beta}$, the Plücker coordinate z_α is x_{k,α_k}. Since $x_{k,\alpha_k} = 0$ defines the Schubert cell Ω_β°, we see that the intersection (9.16) is transverse along Ω_β°, which proves the theorem. $\square$

By the Bézout theorem, Theorem 9.27 implies that

$$\deg \Omega_\alpha \;=\; \sum_{\beta \lessdot \alpha} \deg \Omega_\beta\,.$$

Together with the fact that $\Omega_{\hat{0}}$ is a point, and hence has degree 1, this gives another proof of Corollary 9.22(ii).

For the remainder of this section, let $E_\bullet \subset \mathbb{F}^n$ be a flag whose m-dimensional subspace E_m has no vanishing Plücker coordinates. That is, if E_m is the row space of a $m \times n$ matrix K, then no maximal minor of K vanishes. When $\mathbb{F}$ is infinite, such a choice is possible, as no Plücker coordinate vanishes on the whole Grassmannian. Such a choice may not be possible for finite fields.

Define an action of $\mathbb{F}^\times$ on $\mathbb{F}^n$ by $t.e_i = t^i e_i$, where $t \in \mathbb{F}^\times$ and $e_1, \ldots, e_n$ are the distinguished basis for $\mathbb{F}^n$. This induces an action of $\mathbb{F}^\times$ on the Grassmannian, and we have $t.\mathcal{X}_1 E_\bullet = \mathcal{X}_1(t.E_\bullet)$. Write $\mathcal{X}_1(t)$ for the Schubert variety $\mathcal{X}_1(t.E_\bullet)$.

We find the equation for $\mathcal{X}_1(t)$ by considering the condition for $H \in \mathrm{Grass}(p, m+p)$ to meet $t.E_m$. If H is the row space of a $p \times n$ matrix X and E_m the row space of a $m \times n$ matrix K, then

$$H \cap t.E_m \;\neq\; \{0\} \quad\Longleftrightarrow\quad \det\left[\begin{array}{c} X \\ t.K \end{array}\right] \;=\; 0\,.$$

Laplace expansion along the rows of X gives

$$\det \begin{bmatrix} X \\ t.K \end{bmatrix} \;=\; \sum_{\alpha} z_\alpha(H)(t.K)_\alpha\,,$$

where $z_\alpha(H)$ is the αth maximal minor of X, a Plücker coordinate of H, and $(t.K)_\alpha$ is the complementary maximal minor of $t.K$. Since the ith column of $t.K$ is the scalar multiple t^i times the ith column of K, we have

$$(t.K)_\alpha \;=\; t^{1+2+\cdots+n-\alpha_1-\cdots-\alpha_p} k_\alpha \;=\; t^{\binom{m+1}{2}} t^{mp-|\alpha|} k_\alpha\,,$$

where $k_\alpha := (K)_\alpha$. Dividing by the common non-zero scalar $t^{\binom{m+1}{2}}$, we obtain the equation for $\mathcal{X}_1(t)$:

$$\mathcal{X}_1(t) \;=\; \mathcal{V}\left(\sum_{\alpha\in\binom{[n]}{p}} t^{mp-|\alpha|} k_\alpha \cdot z_\alpha \right)\,.$$

We show that general Schubert varieties of the form $\mathcal{X}_1(t)$ meet properly. As usual, a negative dimension is interpreted to mean that the variety is the empty set.

Lemma 9.28 *Suppose $\mathbb{F}$ is algebraically closed. Let $\alpha \in \binom{[n]}{p}$ and suppose $t_1,\ldots,t_k \in \mathbb{F}^\times$ are distinct. Then*

$$\Omega_\alpha \cap \mathcal{X}_1(t_1) \cap \cdots \cap \mathcal{X}_1(t_k) \tag{9.17}$$

has dimension $|\alpha| - k$.

When $\mathbb{F}$ is not algebraically closed, then we deduce that (9.17) is empty if $k > |\alpha|$.

Proof: Recall that $\dim \Omega_\alpha = |\alpha|$. Let $H \in \Omega_\alpha$. Since the Plücker coordinates $z_\beta(H)$ of H are nonzero only for $\beta \leq \alpha$, we see that H lies in $\mathcal{X}_1(t)$ only if

$$0 \;=\; \sum_{\beta\leq\alpha} t^{mp-|\beta|} z_\beta(H) k_\beta \;=\; t^{mp-|\alpha|} \sum_{\beta\leq\alpha} t^{|\alpha|-|\beta|} z_\beta(H) k_\beta\,.$$

Since the rightmost sum is a polynomial of degree at most $|\alpha|$ in t, we see that $H \in \Omega_\alpha$ lies in at most $|\alpha|$ of the Schubert varieties $\mathcal{X}_1(t)$. Thus (9.17) is empty if $k > |\alpha|$.

Suppose now that $k \leq |\alpha|$. Since each $\mathcal{X}_1(t)$ is a hyperplane section, (9.17) has dimension at least $|\alpha| - k$. If the dimension of (9.17) exceeds $|\alpha| - k$, then for any $t_{k+1},\ldots,t_{|\alpha|+1} \in \mathbb{F}^\times$,

$$\Omega_\alpha \cap \mathcal{X}_1(t) \cap \cdots \cap \mathcal{X}_1(t_{|\alpha|+1})$$

is non-empty (as each $\mathcal{X}_1(t)$ is a hyperplane section), which is a contradiction. $\square$

Let $\mathcal{Z} \subset \Omega_\alpha \times \mathbb{A}$ be the subvariety defined by the polynomial

$$\sum_{|\beta|\leq|\alpha|} t^{|\alpha|-|\beta|} z_\beta k_\beta\,, \tag{9.18}$$

where t is the coordinate of $\mathbb{A}^1$ and z_β the Plücker coordinates on Ω_α. For $t \neq 0$, the fibre $\mathcal{Z}_t$ is just the cycle $\Omega_\alpha \cap \mathcal{X}_1(t)$, and Theorem 9.27 shows that the fibre $\mathcal{Z}_0$ is

$$\mathcal{Z}_0 \;=\; \Omega_\alpha \cap \mathcal{V}(z_\alpha) \;=\; \bigcup_{\beta \lessdot \alpha} \Omega_\beta \,.$$

Let $d(\alpha) := \deg \Omega_\alpha = \#Ч[\hat{0}, \alpha]$, the number of chains in the interval $[\hat{0}, \alpha]$. We prove the main result of this section.

Theorem 9.29 *Suppose $\mathbb{F}$ is algebraically closed. Then there exist $t_1, \ldots, t_{mp} \in \mathbb{F}\times$ such that if $\alpha \in \binom{[n]}{p}$, then*

$$\Omega_\alpha \cap \mathcal{X}_1(t_1) \cap \cdots \cap \mathcal{X}_1(t_{|\alpha|}) \tag{9.19}$$

is transverse and consists of $d(\alpha)$ points.

If $\mathbb{F} = \mathbb{R}$, then then we may choose $t_1, \ldots, t_{mp} \in \mathbb{R}^\times$ such that (9.19) is transverse in the complex Grassmannian and each of its $d(\alpha)$ points is real.

The first statement establishes the Transversality Conjecture for the simple Schubert calculus.

Proof: We induct on the dimension $|\alpha|$ of Ω_α to find the t_i, simultaneously establishing both the case of $\mathbb{F}$ algebraically closed and of $\mathbb{F}$ real. When $\alpha = \hat{0}$, $|\alpha| = 0$ and $\Omega_\alpha = \langle e_1, \ldots, e_p \rangle$, and the conclusion of the theorem holds vacuously. When $\alpha = (1, 2, \ldots, p-1, p+1)$, then Ω_α is the coordinate $\mathbb{P}^1$ spanned by $z_{\hat{0}}$ and z_α in the Plücker embedding. The equation (9.18) for $\mathcal{X}_1(t) \cap \Omega_\alpha$ is non-trivial and has degree 1, when $t \neq 0$. Thus for $t \neq 0$, $\mathcal{X}_1(t) \cap \Omega_\alpha$ consists of a single point, which is real is $\mathbb{F}$ is real. Since $d(\alpha) = 1$, this establishes the theorem in this case.

Suppose now that we have $t_1, \ldots, t_k \in \mathbb{F}^\times$ such that if $|\beta| \leq k$ then

$$\Omega_\beta \cap \mathcal{X}_1(t_1) \cap \cdots \cap \mathcal{X}_1(t_{|\beta|})$$

is transverse and consists of $d(\beta)$ points, with all of them real when $\mathbb{F} = \mathbb{R}$. Let $\alpha \in \binom{[n]}{p}$ with $|\alpha| = k+1$ and consider $\mathcal{Z} \cap \mathcal{X}_1(t_1) \cap \cdots \cap \mathcal{X}_1(t_k)$ viewed as a family over $\mathbb{A}^1$.

The fibre of this family at $t = 0$ is

$$\bigcup_{\beta \lessdot \alpha} \Omega_\beta \cap \mathcal{X}_1(t_1) \cap \cdots \cap \mathcal{X}_1(t_k) \,.$$

By the induction hypothesis, each term

$$\Omega_\beta \cap \mathcal{X}_1(t_1) \cap \cdots \cap \mathcal{X}_1(t_k)$$

is transverse and consists of $d(\beta)$ points (all real if $\mathbb{F} = \mathbb{R}$). Then the fibre will consist of $d(\alpha) = \sum_{\beta \lessdot \alpha} d(\beta)$ points and be a transverse intersection if no two terms of this union have any points in common.

Consider the intersection of the terms indexed by β and γ

$$\Omega_\beta \cap \Omega_\gamma \cap \mathcal{X}_1(t_1) \cap \cdots \cap \mathcal{X}_k(t_k) \,.$$

Since $\Omega_\beta \cap \Omega_\gamma$ is a union of Schubert varieties of dimension strictly less than k, Lemma 9.28 implies that this intersection is empty.

Since the fibre at $t = 0$ of $\mathcal{Z} \cap \mathcal{X}_1(t_1) \cap \cdots \cap \mathcal{X}_1(t_k)$ is transverse and consists of $d(\alpha)$ points, there is an open subset $\mathcal{O}_\alpha \subset \mathbb{A}^1$ such that for each $0 \neq t \in \mathcal{O}_\alpha$, the fibre at t of $\mathcal{Z} \cap \mathcal{X}_1(t_1) \cap \cdots \cap \mathcal{X}_1(t_k)$ is transverse and consists of $d(\alpha)$ points. But this fibre is

$$\Omega_\alpha \cap \mathcal{X}_1(t) \cap \mathcal{X}_1(t_1) \cap \cdots \cap \mathcal{X}_1(t_k)\,.$$

We let $0 \neq t_{k+1}$ be any point common to the sets $\mathcal{O}_\alpha$ for $|\alpha| = k+1$.

When $\mathbb{F} = \mathbb{R}$, the set $\mathcal{O} = \bigcap\{\mathcal{O}_\alpha \mid |\alpha| = k+1\}$ contains 0 and hence it contains some interval of the form $(-a, a)$. Since for all $-a < t < a$ and α with $|\alpha| = k+1$, the intersection $\Omega_\alpha \cap \mathcal{X}_1(t_1) \cap \cdots \cap \mathcal{X}_1(t_k)$ is transverse, and when $t = 0$ it consists of $d(\alpha)$ real points, it consists of $d(\alpha)$ real points for all $t \in (-a, a)$. We let $0 \neq t_{k+1}$ be any point in the interval $(-a, a)$. □

Remark 9.30 We describe a special case of the Schubert varieties $\mathcal{X}_1(t)$ that (conjecturally at least) have very remarkable properties when $\mathbb{F} = \mathbb{R}$. Suppose $\mathbb{F} = \mathbb{R}$ and consider the map

$$\begin{array}{rcl} v \;:\; \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ t & \longmapsto & (1, t, t^2, \ldots, t^{n-1})\,. \end{array}$$

For each $i = 1, \ldots, n$ define

$$E_i(t) \;:=\; \langle v(t), \frac{d}{dt}v(t), \frac{d^2}{dt^2}v(t), \ldots, \frac{d^{i-1}}{dt^{i-1}}v(t)\rangle\,.$$

This defines a flag $E_\bullet(t)$ for every $t \in \mathbb{F}$ with $E_\bullet(0) = F_\bullet$, the standard flag. If we consider the flag $E_\bullet(t)$ in the projective space $\mathbb{P}^{n-1}$, then $v(t) = E_1(t)$ is the rational normal curve and $E_{i+1}(t)$ is the i-plane osculating the rational normal curve at the point $v(t)$. The flag $E_\bullet(t)$ is the flag of subspaces osculating the rational normal curve.

For instance, when $m = 4$ and $p = 3$, we have

$$E_4(t) \;=\; \text{row space} \begin{bmatrix} 1 & t & t^2 & t^3 & t^4 & t^5 & t^6 \\ 0 & 1 & 2t & 3t^2 & 4t^3 & 5t^4 & 6t^5 \\ 0 & 0 & 2 & 6t & 12t^2 & 20t^3 & 30t^4 \\ 0 & 0 & 0 & 6 & 24t & 60t^2 & 120t^3 \end{bmatrix}\,.$$

When $t \neq 0$, $E_i(t)$ is also $\langle tv(t), t^2v'(t), t^3v''(t), \ldots, t^iv^{(i-1)}(t)\rangle$. The $i \times n$ matrix with these row vectors has entry in position k, l:

$$t^l \frac{(l-1)!}{(l-k)!}\,.$$

Thus $E_i(t) = t.E_i(1)$. If we let $\gamma = \gamma_1 < \cdots < \gamma_m$, then the γth Plücker coordinate of $E_m(1)$ is

$$\det\begin{vmatrix} 1 & \cdots & 1 \\ \gamma_1-1 & & \gamma_m-1 \\ \vdots & \dfrac{(\gamma_j-1)!}{(\gamma_j-i)!} & \vdots \\ \dfrac{(\gamma_1-1)!}{(\gamma_1-m)!} & \cdots & \dfrac{(\gamma_m-1)!}{(\gamma_m-m)!} \end{vmatrix} = \prod_{i<j}(\gamma_j-\gamma_i)\,,$$

which is non-zero. (See Exercise 9.11 for the computation.) Thus the second statement of Theorem 9.29 applies and we deduce that

There exist $t_1, t_2, \ldots, t_{mp} \in \mathbb{R}$ such that if $\alpha \in \binom{[n]}{p}$, then

$$\Omega_\alpha \cap \mathcal{X}_1 E_{\bullet}(t_1) \cap \cdots \cap \mathcal{X}_1 E_{\bullet}(t_{|\alpha|})$$

is transverse in the complex Grassmannian and consists of $d(\alpha)$ real points.

There is a far-reaching conjecture of Boris Shapiro and Michael Shapiro related to this result.

Corollary 9.31 (Shapiro-Shapiro) *Let $\lambda^1, \ldots, \lambda^s$ be Schubert data. If $t_1, \ldots, t_s \in \mathbb{R}$ are distinct, then*

$$\mathcal{X}_{\lambda^1} E_{\bullet}(t_1) \cap \cdots \cap \mathcal{X}_{\lambda^s} E_{\bullet}(t_s)$$

is transverse and consists of $d(\lambda^1, \ldots, \lambda^s)$ real points.

At this time, the general case of this conjecture remains open. The strongest partial result is due to Eremenko and Gabrielov, who proved it when $\min\{m,p\} = 2$.

Exercise 9.10 Suppose $e_1, \ldots, e_n$ is an ordered basis for $\mathbb{F}^n$ and let $F_{\bullet}$ be the standard flag with respect to this basis. For $\alpha \in \binom{[n]}{p}$, let $K_\alpha := \langle e_j \mid j \notin \alpha\rangle$ and suppose that $E_{\bullet}$ is a flag whose m-plane is K_α so that $\mathcal{X}_1 E_{\bullet} = \{H \mid H \cap K_\alpha \neq \{0\}\}$.

(i) Argue directly using Schubert conditions that

$$\Omega_\alpha \cap \mathcal{X}_1 E_{\bullet} = \bigcup_{\beta \lessdot \alpha} \Omega_\beta\,.$$

(ii) Using the description of the tangent space to a Schubert variety given at the end of Section 9.3, show that this intersection is transverse.

Hint: For (i) consider the sets $\{H \in \Omega_\alpha \mid H \cap K_\alpha \cap F_{\alpha_j} \neq \{0\}\}$ for $j = 1, \ldots, p$.

Exercise 9.11 Let $f_1(x), \ldots, f_n(x)$ be polynomials of degree at most $n-1$. Consider the matrix X whose i,j-entry is $f_i(a_j)$, where $a_1, \ldots, a_n$ are indeterminants. Show that

$$\det X \;=\; \det A \times \prod_{i<j}(a_j - a_i)\,,$$

where A is the coefficient matrix of the polynomials f_i, the matrix that transforms the monomial basis $1, x, x^2, \ldots, x^{n-1}$ into the polynomials $f_1(x), \ldots, f_n(x)$. (Hint: Factor X as $A \times B$, where B is the Van der Monde matrix (a_j^{i-1}).)

In particular, when each f_i is monic of degree $i-1$, then A is lower triangular and so has determinant 1.

9.5 Enumerative Geometry and the Schubert Calculus II

Intersection rings give methods for determining the number of solutions to the Schubert-type enumerative problems of Section 9.4. It is very instructive to review a general framework for enumerative geometry, first proposed in the 19th century. For our purposes, enumerative geometry is concerned with all questions of the following form: How many goemetric figures of some type have a specified position with respect to certain general fixed figures?

Let $\mathcal{X}$ be the space of the geometric figures we wish to count. Then the set of figures having specified position $\mathbf{a}$ with respect to a fixed figure x is a subvariety $\mathcal{Y}_{\mathbf{a}}(x)$ of $\mathcal{X}$. Here, $\mathbf{a}$ encodes the type of condition imposed by the figure x. Our problem becomes: given conditions $\mathbf{a}, \mathbf{b}, \dots, \mathbf{c}$ and general fixed figures $x, y, \dots, z$, determine the number of points in $\mathcal{X}$ in the intersection

$$\mathcal{Y}_{\mathbf{a}}(x) \cap \mathcal{Y}_{\mathbf{b}}(y) \cap \cdots \cap \mathcal{Y}_{\mathbf{c}}(z)\,.$$

For instance, in the Schubert calculus of enumerative geometry, the Grassmannian is the ambient space $\mathcal{X}$, flags $F_{\bullet}$ are the fixed figures, partitions λ encode the conditions, and the Schubert variety $\mathcal{X}_\lambda F_{\bullet}$ is the set of figures having specified position λ with respect to the flag $F_{\bullet}$.

In the 19th century, a formal calculus was developed to solve such problems. Geometric conditions $\mathbf{a}, \mathbf{b}, \mathbf{c}$ were treated as formal symbols with $\mathbf{a}+\mathbf{b}$ interpreted as the condition that either $\mathbf{a}$ or $\mathbf{b}$ holds, and $\mathbf{a}\cdot\mathbf{b}$ as the condition that both $\mathbf{a}$ and $\mathbf{b}$ hold, when $\mathbf{a}$ and $\mathbf{b}$ are independent. Conditions $\mathbf{a}$ and $\mathbf{b}$ are *numerically equivalent* if, whenever $\mathbf{c}$ is a geometric condition such that only finitely many figures satisfy both $\mathbf{a}$ and $\mathbf{c}$ and also finitely many satisfy both $\mathbf{b}$ and $\mathbf{c}$, then these two numbers are equal. This formal calculus was used to great effect in solving many enumerative geometric problems.

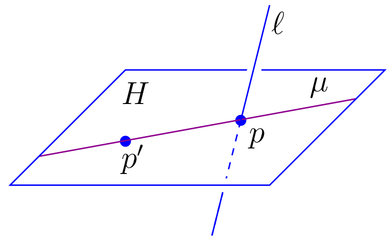
Example 9.32 We use this formal calculus to determine how many lines meet four fixed lines in $\mathbb{P}^3$. Our ambient space will be the Grassmannian of 2-planes in $\mathbb{F}^4$, which is also called the Grassmannian of lines in $\mathbb{P}^3$. Let p, ℓ, H be respectively a point, line, and plane in $\mathbb{P}^3$. Consider the following conditions on lines μ in $\mathbb{P}^3$:

$$\begin{aligned}
[\ell] &:= \text{The line } \mu \text{ meets } \ell\\
[p] &:= \text{The line } \mu \text{ contains } p\\
[H] &:= \text{The line } \mu \text{ lies in } H\\
[p\in H] &:= \text{The line } \mu \text{ contains } p \text{ and lies in } H
\end{aligned}$$

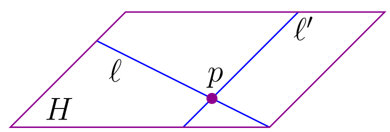
(For this last condition, we must have $p \in H$.) Suppose now that p' and H' are a point and a plane in general position with respect to p, ℓ, H. Then, we have

$$\begin{aligned}
[p]\cdot[H'] &= 0 && \text{(a general point does not lie on a plane)}\\
[p]\cdot[p'] &= [pt] && \text{(two points determine a line)}\\
[H]\cdot[H'] &= [pt] && \text{(two planes meet in a line)}\\
[\ell]\cdot[p'\in H'] &= [pt]
\end{aligned}$$

For the last product, let $p := \ell \cap H$, then the line μ spanned by p and p' is the unique line meeting both p' and ℓ, and lying in H':



Similar considerations show that $[\ell] \cdot [H] = [\ell] \cdot [p] = [p \in H]$. To compute $[\ell] \cdot [\ell']$, we suppose that ℓ and ℓ' meet in a point p and thus span a plane H:



Thus if a line μ meets both ℓ and ℓ', then either μ contains p or else μ lies in H. This shows that

$$[\ell] \cdot [\ell'] \;=\; [p] + [H]\,.$$

To determine the number of lines μ in $\mathbb{P}^3$ that meet each of four general lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4, we compute the product $[\ell_1] \cdot [\ell_2] \cdot [\ell_3] \cdot [\ell_4]$:

$$\begin{aligned}
[\ell_1] \cdot [\ell_2] \cdot [\ell_3] \cdot [\ell_4] &= ([p] + [H]) \cdot ([p'] + [H']) \\
&= [p][p'] + [p][H] + [H][p'] + [H][H'] \\
&= [pt] + 0 + 0 + [pt] \;=\; 2[pt]\,.
\end{aligned}$$

(The first and third equalities follow from the calculations above.) Since 2 is the coefficient of $[pt]$ in this product, we conclude that 2 lines meet four given lines in $\mathbb{P}^3$, which agrees with the calculations in Chapter 4.

The great flaw in this formal calculus was that there was inadequate justification that it reliably computed the number of solutions to these enumerative problems. During the 20th century, adequate justification for this formal calculus was provided by Algebraic Topology and Intersection Theory. Presently the main tool for solving problems in enumerative geometry is the Chow ring $A^*\mathcal{X}$ of a smooth projective variety $\mathcal{X}$. For complex varieties, the cohomology ring $H^*\mathcal{X}$ may be used in place of the Chow ring. Appendix???? summarizes the main properties of cohomology and Chow rings that we use.

Briefly, the Chow ring $A^*\mathcal{X}$ is graded with degree k component $A^k\mathcal{X}$ generated by classes $[\mathcal{Y}]$ associated to subvarieties $\mathcal{Y}$ of $\mathcal{X}$ having codimension k. More specifically, let $Z^k\mathcal{X}$ be the group of codimension k algebraic cycles on $\mathcal{X}$, the free abelian group generated by symbols $[\mathcal{Y}]$ for each irreducible subvariety $\mathcal{Y}$ of $\mathcal{X}$ of codimension k. Then $A^k\mathcal{X}$ is the quotient of the group $Z^k\mathcal{X}$ of cycles by a subgroup generated by an equivalence relation that refines numerical equivalence. Different equivalence relations give different theories, but when $\mathcal{X}$ is the Grassmannian, they all coincide.

The formal properties of the Chow ring that we use are the following.

1. Each irreducible subvariety $\mathcal{Y}$ with codimension k in $\mathcal{X}$ determines a fundamental cycle $[\mathcal{Y}] \in A^k\mathcal{X}$. This extends to possibly reducible subvarieties $\mathcal{Y}, \mathcal{Z}$ of $\mathcal{X}$ so that if $\mathcal{Y} \cap \mathcal{Z}$ shares no ireducible components with either of $\mathcal{Y}$ or $\mathcal{Z}$, then
$$[\mathcal{Y} \cup \mathcal{Z}] \;=\; [\mathcal{Y}] + [\mathcal{Z}]\,.$$

2. If $\mathcal{Y}, \mathcal{Z}$ are irreducible subvarieties of $\mathcal{X}$ whose intersection is proper, then there is a positive interger a_i associated to each irreducible component $\mathcal{V}_i$ of $\mathcal{Y} \cap \mathcal{Z}$, such that
$$[\mathcal{Y}] \cdot [\mathcal{Z}] \;=\; \sum_i a_i [\mathcal{V}_i]\,.$$
This integer a_i is called the intersection multiplicity of $\mathcal{Y}$ and $\mathcal{Z}$ along $\mathcal{V}_i$. The intersection multiplicity equals 1 if $\mathcal{Y}$ and $\mathcal{Z}$ meet generically transversally along $\mathcal{V}_i$.

3. The association $pt \mapsto 1 \in \mathbb{Z}$, where pt is any point of X induces the degree map $\deg\colon A^{\dim \mathcal{X}}\mathcal{X} \to \mathbb{Z}$ (sometimes written $\int_{\mathcal{X}}$). Together with the product, this induces the intersection pairing
$$\begin{array}{rcl} A^k\mathcal{X} \times A^{\dim \mathcal{X} - k}\mathcal{X} & \longrightarrow & \mathbb{Z} \\ (\alpha, \beta) & \longmapsto & \deg(\alpha \cdot \beta)\,. \end{array}$$

4. If $\mathcal{Z} \subset \mathcal{X} \times \mathbb{A}^1$ is an irreducible subvariety whose fibres $\mathcal{Z}_t$ over points $t \in \mathbb{A}^1$ are equidimensional, then, for any $t, t' \in \mathbb{A}^1$,
$$[\mathcal{Z}_t] \;=\; [\mathcal{Z}_{t'}]\,.$$

5. If $\mathcal{X} = \coprod_{\mathbf{a} \in I} \mathcal{X}_{\mathbf{a}}$ is the union of locally closed subvarieties $\mathcal{X}_{\mathbf{a}}$ with each $\mathcal{X}_{\mathbf{a}}$ isomorphic to an affine space, then $A^*\mathcal{X}$ has an integral basis given by the fundamental cycles $[\overline{\mathcal{X}_{\mathbf{a}}}]$ of the closures of the strata.

If we associate the fundamental cycle $[\mathcal{Y}_{\mathbf{a}}(x)] \in A^*\mathcal{X}$ to the formal condition $\mathbf{a}$, then the properties of the Chow rings $A^*\mathcal{X}$ provide a justification for the formal calculus used in the 19th century.

We use these formal properties to compute the Chow rings of Grassmannians. Some combinatorics of partitions are helpful to state these results. When writing a partition, trailing zeroes are often omitted so that $(k, 0, \ldots, 0)$ is written as (k) or simply k. The length of a partition λ is its number of non zero parts. Repeated parts may be indicated by an exponent, thus $(m^p) = (m, \ldots, m)$, the partition of mp with p parts, each of size m. The Young diagram of a partition λ is a left justified array of boxes with λ_i boxes in row i. We write λ for the Young diagram of a partition λ, ignoring the distinction between Young diagrams and partitions. Thus
$$(4,2,1) \;\longleftrightarrow\; \ydiagram{4,2,1}\,.$$

Partitions are partially ordered by componentwise comparison, equivalently by inclusion of Young diagrams with $(0) = \hat{0}$ the minimal partition and $(m^p) = \hat{1}$ the maximal partition. The Bruhat order on Schubert varieties is given by reverse inclusion of partitions

$$\mathcal{X}_\mu F_\bullet \subset \mathcal{X}_\lambda F_\bullet \iff \mu \supset \lambda\,.$$

Lastly, given a partition $\lambda \subset (m^p)$, define $\lambda^\vee$ by $\lambda^\vee_i := \lambda_{p+1-i}$ for $i = 1, 2, \ldots, p$. Flags $F_\bullet, F'_\bullet$ are opposite if they are in linear general position. That is, if

$$F_i \cap F'_j \;=\; \{0\}$$

unless $i + j > n$, and in that case the intersection has dimension $i + j - n$.

Define the Schubert class $\sigma_\lambda \in A^{|\lambda|}\mathrm{Grass}(p, V)$ to be the fundamental cycle of a Schubert variety $\mathcal{X}_\lambda F_\bullet$. This class is independent of the choice of flag $F_\bullet$ as the Schubert varieties are fibres of the universal family of Schubert varieties over $GL(V)$, and $GL(V)$ is a rational variety. The initial result about Chow rings of Grassmannians is the Basis Theorem, which is a consequence of the Schubert decomposition and Property 4 of Chow rings.

Theorem 9.33 (Basis Theorem) *The Schubert classes form an integral basis of the Chow ring of a Grassmannian,*

$$A^*\mathrm{Grass}(p, V) \;=\; \bigoplus_{\lambda \subset (m^p)} \sigma_\lambda \cdot \mathbb{Z}\,.$$

The next result concerns the intersection pairing.

Theorem 9.34 (Duality Theorem) *The intersection pairing*

$$A^*\mathrm{Grass}(p, V) \times A^*\mathrm{Grass}(p', V') \;\to\; \mathbb{Z}$$

is a perfect pairing with the classes σ_λ and $\sigma_{\lambda^\vee}$ dual classes. Specifically, if $|\lambda| + |\mu| = mp$, then

$$\sigma_\lambda \cdot \sigma_\mu \;=\; \delta_{\lambda,\mu^\vee}\,.$$

The Duality Theorem follows from Lemma 9.37 below. It allows us to compute fudamental cycles of many subvarieties of $\mathrm{Grass}(p, V)$.

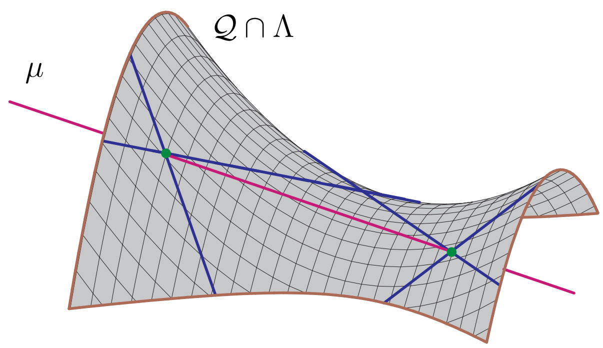
Corollary 9.35 *If $Y \subset \mathrm{Grass}_{m,p}$, then $[Y] = \sum_\lambda c^\lambda_{[Y]}\, \sigma_\lambda$, where $c^\lambda_{[Y]} = \#(Y \cap X_{\lambda^\vee} F_\bullet)$, when this intersection is transverse.*

Example 9.36 The basis Theorem also allows us to solve some enumerative geometric problems. We determine the number of lines that lie on the intersection of two quardrics $\mathcal{Q}, \mathcal{Q}'$ in $\mathbb{P}^4$.

Let $\mathcal{Y}_\mathcal{Q} \subset \mathrm{Grass}(1, \mathbb{P}^4)$ be the set of lines lying on a quadric $\mathcal{Q}$. We determine its fundamental cycle $[\mathcal{Y}_\mathcal{Q}]$ and then compute $\deg([\mathcal{Y}_\mathcal{Q}]^2)$ to answer the question. A given line ℓ lies on the quadric $\mathcal{Q} := \mathcal{V}(q)$ if the quadratic form $q|_\ell$ is identically zero. Since $\ell \simeq \mathbb{P}^1$, this is a quadratic polynomial on $\mathbb{P}^1$, determined by 3 coefficients. Thus the vanishing of this quadratic form gives three equations (one for coefficient) in the coordinates of ℓ, and so we expect that $\mathcal{Y}_{\mathcal{Q}}$ has codimension 3 in $\mathrm{Grass}(1,\mathbb{P}^4)$. Since $\dim \mathrm{Grass}(1,\mathbb{P}^4) = 6$, we find $[\mathcal{Y}_{\mathcal{Q}}]$ by finding the intersection of $\mathcal{Y}_{\mathcal{Q}}$ with each of the two codimension 3 Schubert varieties:

$$\begin{aligned} \mathcal{X}_3 &:= \{\text{lines } \ell \mid \ell \text{ contains a point } p\} \\ \mathcal{X}_{2,1} &:= \{\text{lines } \ell \mid \ell \text{ meets a line } \mu \text{ and lies in a hyperplane } \Lambda\} \end{aligned}$$

Since a general point p will not meet $\mathcal{Q}$, we see that $\mathcal{Y}_{\mathcal{Q}} \cap \mathcal{X}_3 = \emptyset$. Next, fix a general line $\mu \subset \Lambda \simeq \mathbb{P}^3$, a general hyperplane in $\mathbb{P}^4$. Then lines $\ell \subset \mathcal{Q}$ that also meet μ and lie in Λ are those lines in $\Lambda \simeq \mathbb{P}^3$ that lie on the quadric $\mathcal{Q} \cap \Lambda$ and also meet the line μ. The quadric $\mathcal{Q}$ meets the line μ in 2 points, and there are 2 lines on $\mathcal{Q} \cap \Lambda$, one in each family, that meet each of these points, for 4 lines in all. This is displayed in the picture below.



Thus we have $\#\mathcal{Y}_{\mathcal{Q}} \cap \mathcal{X}_{2,1} = 4$. Since $(2,1)^\vee = (2,1)$, we see that $[\mathcal{Y}_{\mathcal{Q}}] = 4\sigma_{2,1}$. and so we calculate

$$\deg([\mathcal{Y}_{\mathcal{Q}}]^2) \;=\; \deg((4\sigma_{2,1})^2) \;=\; 16\,,$$

as $\sigma_{2,1}$ is self dual.

This there are 16 lines common to 2 quadrics in $\mathbb{P}^4$.

The Duality Theorem is a consequence of the following lemma.

Lemma 9.37 *Let $\lambda, \mu \subset (m^p)$ be partitions and $F_{\bullet}, F'_{\bullet}$ be opposite flags. Then*

$$\mathcal{X}_\lambda F_{\bullet} \cap \mathcal{X}_\mu F'_{\bullet} \;\neq\; \emptyset$$

if and only if $\lambda \leq \mu^\vee$. When $\lambda \leq \mu^\vee$, the intersection is generically transverse and irreducible of dimension $|\mu^\vee| - |\lambda|$. In particular,

$$\mathcal{X}_\lambda F_{\bullet} \cap \mathcal{X}_{\lambda^\vee} F'_{\bullet}$$

is a transverse intersection consisting of a single point.

Proof: Suppose that $H \in \mathcal{X}_\lambda F_{\bullet} \cap \mathcal{X}_\mu F'_{\bullet}$, so that the intersection is non empty. Then, for each $i, j = 1, 2, \ldots, p$, the conditions defining the Schubert varieties give us

$$\dim H \cap F_{m+i-\lambda_i} \;\geq\; i \qquad \text{and} \qquad \dim H \cap F'_{m+j-\mu_j} \;\geq\; j\,. \tag{9.20}$$

When $j = p+1-i$, these two subspaces of H must have a non-trivial intersection. This forces the corresponding subspaces in the flags to meet non-trivially, so that

$$1 \ \leq\ \dim F_{m+i-\lambda_i} \cap F'_{m+(p+1-i)-\mu_{p+1-i}} \ =\ m+1-\lambda_i-\mu_{p+1-i}\,,$$

as the linear subspaces in the two flags meet properly. Thus we have $\lambda_i \leq m-\mu_{p+1-i} = \mu_i^\vee$ for $i = 1, 2, \ldots, p$, which implies that $\lambda \leq \mu^\vee$.

□

1. Opposite flags imply irreducible and generically transverse intersection.
2. Do Pieri formula via triple intersections
3. Deduce Pieri formula
4. Show that special Schubert classes generate Chow ring
5. Deduce Giambelli formula
6. Describe algorithms
7. Describe Littlewood-Richardson rule

Exercise 9.12 Show that the product in Example 9.32 is associative; in particular we get the same answer computing $((([\ell]\cdot[\ell])\cdot[\ell])\cdot[\ell]$, as for $([\ell]\cdot[\ell])\cdot([\ell]\cdot[\ell])$.

9.6 Variants

1. Flag variety, $\mathbb{F}\ell_n$.
2. Its Schubert decomposition
3. Describe its Chow ring
4. Schuberet polynomials, Monk formula, Pieri-type formula

5. Lagrangian Grassmannian
6. Schubert decomposition
7. Chow ring
8. Pieri formula

Part III

Applications

Chapter 10

The Class of Linear Systems

From Joachim
I plan to rewrite this chapter completely

10.1 Coprime factorization and realization theory

Consider the transfer function of a system Σ_n

$$H(s) = C(sI - A)^{-1}B \ + \ D$$

which is the Laplace transform of the state space representation. $H(s)$ is a proper matrix, $H(\infty) = D$ and $G(s) = C(sI - A)^{-1}B$ is a strictly proper matrix.

Definition 10.1

1. A left matrix factorization of a rational matrix R(s) is a factorization of the form $R(s) = D^{-1}(s) \cdot N(s)$ where $D(s)$ and $N(s)$ are polynomial matrices.

2. A left matrix factorization is called coprime, if there are polynomial matrices U(s),V(s) such that
$$N(s)U(s) + D(s)V(s) = I$$

3. A right matrix (coprime) factorization of $R(s)$ is defined as a left matrix (coprime) factorization of $(R(s))^t$

In the following theorem, some important facts about coprime factorizations are summarized. All proofs can be found in [10].

Theorem 10.2

1. *If $H(s)$ is a proper rational matrix, a left (right) coprime factorization exists.*

2. *A left (right) coprime factorization is unique up to multiplication by a unimodular matrix. In other words, if $H(s) = D^{-1}(s) \cdot N(s)$ is coprime, then any other coprime factorization is of the form $H(s) = (U(s)D(s))^{-1}U(s)N(s)$ where U(s) is a unimodular matrix.*

3. *If $H(s) = D^{-1}(s) \cdot N(s)$ is a coprime factorization and (A,B,C,D) is a minimal realization of $H(s)$, then*

$$\det D(s) \ = \ const. \cdot \det(sI - A)$$

This theorem enables one to identify a system Σ_n with a pair of coprime polynomial matrices

$$(N(s)\ ,\ D(s))$$

In this representation, two pairs of polynomial matrices $(N(s)\ ,\ D(s))$ and $(\tilde{N}(s)\ ,\ \tilde{D}(s))$ represent the same system, if there is a unimodular matrix U(s) such that $\tilde{N}(s) \ = U(s)N(s)$ and $\tilde{D}(s) = U(s)D(s)$

Definition 10.3 Given a system Σ represented by a coprime factorization $D^{-1}(s)N(s)$ of its transfer function. The map

$$\begin{array}{rrcl} \phi_\Sigma \ : & \mathbb{CP}^1 & \longrightarrow & \mathrm{Grass}(p, m+p) \\ & s & \longmapsto & [N(s)\ D(s)] \end{array} \qquad (10.1)$$

is called the Hermann Martin curve of the system Σ.

Remark 10.4 1. From coprimeness it follows that $[N(s)\ D(s)]$ is always of full rank.

2. ϕ_Σ depends only on Σ by part 2 of Theorem 10.2.

The Hermann Martin curve is a rational curve. Every system can therefore be identified with an algebro-geometric object. Moreover it turns out that every algebraic (or holomorphic) map from $\mathbf{CP}^1 \to \mathrm{Grass}(p, m+p)$ represents a m–input, p–output linear system. The following theorem shows that this identification is even much deeper.

Theorem 10.5 (Martin and Hermann[31]) *1. The degree of ϕ_Σ as an algebraic curve is equal to the McMillan degree.*

2. *The poles of Σ are those points in $\mathbf{CP}^1$ which are mapped into the Schubert variety $S(m, m+2, \ldots, m+p)$.*

3. *Assume Σ is strictly proper with Kronecker indices $k_1 \leq \ldots \leq k_r$ then the pull back of the universal p–bundle is isomorphic to a sum of line bundles:*

$$\phi^*(\xi^*_{p,n}) \ = \ \mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_r)$$

where $k_1, \ldots, k_r$ are the Grothendieck invariants of this bundle.