When Does a Ball Hit the Ceiling?

(And Why Does an Atomic Physicist Care?)

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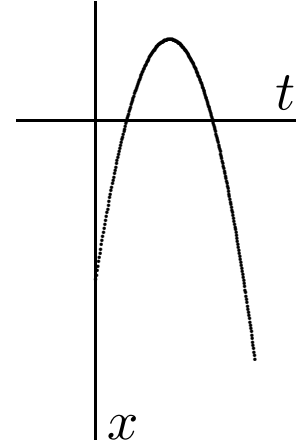
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Falling body (+ direction is *down*).

$$x(t) = x_0 + v_0 t + \tfrac{1}{2} g t^2.$$

$$\frac{d^2 x}{dt^2} = g, \quad x(0) = x_0, \quad \frac{dx}{dt}(0) = v_0.$$

v_0 can be negative!

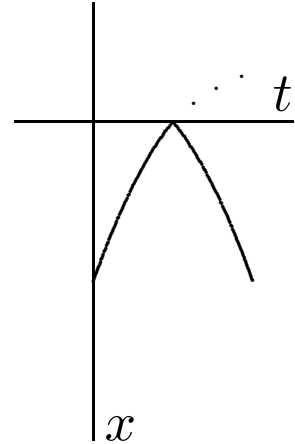$$t_{\max} = -\frac{v_0}{g}, \quad x_{\max} = x_0 - \frac{1}{2g} \, v_0{}^2.$$

Ceiling

Require: x can never be negative. If x hits 0, the ball bounces back with equal but opposite v.

$$0 = x_0 + v_0 t_{\mathrm{b}} + \tfrac{1}{2} g t_{\mathrm{b}}^2.$$

$$t_{\mathrm{b}} = \frac{1}{g}\left(-v_0 - \sqrt{v_0{}^2 - 2gx_0}\right).$$

Later, $x(t) = -v_0(t - t_{\mathrm{b}}) + \tfrac{1}{2} g(t - t_{\mathrm{b}})^2.$

Two-point boundary value problem
(needed for quantum mechanics)

I want to throw the ball so that $x(t_{\mathrm{f}}) = x_{\mathrm{f}}$.
v_0 is at my disposal.

<center>No ceiling</center>

$$x_{\mathrm{f}} = x_0 + v_0 t_{\mathrm{f}} + \tfrac{1}{2} g t_{\mathrm{f}}^2 \;\Rightarrow\; v_0 = \frac{x_{\mathrm{f}} - x_0}{t_{\mathrm{f}}} - \tfrac{1}{2} g t_{\mathrm{f}}\,.$$
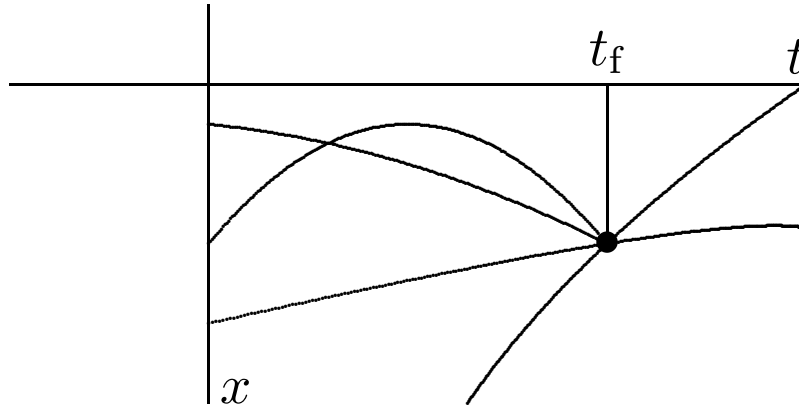
Note: Exactly one solution. *But this is an accident.*

With ceiling

Possibilities:
- *Bounce path:* Ball hits ceiling at $0 < t_{\mathrm{b}} < t_{\mathrm{f}}$.
- *Direct path:* Ball never reaches ceiling (so previous solution is OK).
  - t_{b} is outside $(0, t_{\mathrm{f}})$ (so ball never turns around).
    - $v > 0$ always (falling).
    - $v < 0$ always (rising).
  - $x_{\mathrm{max}} \geq 0$. (Both of these could be true.)

Four kinds of direct paths

Look at direct path with $x(t_{\mathrm{max}}) = x_{\mathrm{max}}$, $v(t_{\mathrm{max}}) = 0$.
Earlier and later, $x(t) = x_{\mathrm{max}} + \frac{1}{2}g(t - t_{\mathrm{max}})^2$.

So $|t - t_{\mathrm{max}}| = \lambda\sqrt{x - x_{\mathrm{max}}}$, $\quad \lambda \equiv \sqrt{\dfrac{2}{g}}$.

So total time of any direct path is

$$(t_{\mathrm{f}} - t_{\mathrm{max}}) + (t_{\mathrm{max}} - 0) = \lambda(\sqrt{x_{\mathrm{f}} - x_{\mathrm{max}}} + \sqrt{x_0 - x_{\mathrm{max}}}) .$$

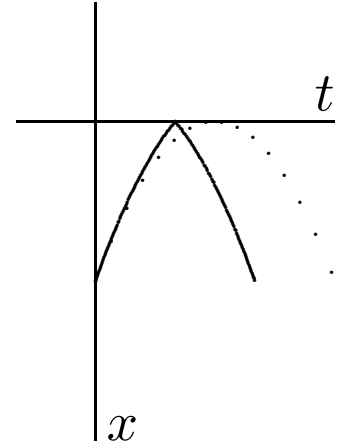$$x_{\mathrm{max}} \geq 0 \;\Rightarrow\; t_{\mathrm{f}} \leq \lambda(\sqrt{x_0} + \sqrt{x_{\mathrm{f}}}).$$

Monotonic trajectories also satisfy this key inequality.

The equality, $t_f = \lambda(\sqrt{x_0} + \sqrt{x_f})$, describes a path that just grazes the ceiling.
A bounce path must start (and end) with *higher* speed, so it will come back *before* the grazing path.
So *bounce paths also satisfy the key inequality*

$$t_f \leq \lambda(\sqrt{x_0} + \sqrt{x_f}).$$

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Conclusion:

- $t_{\mathrm{f}} < \lambda(\sqrt{x_0} + \sqrt{x_{\mathrm{f}}}) \ \Rightarrow \ $ **two** solutions
  (direct and bounce).
- $t_{\mathrm{f}} = \lambda(\sqrt{x_0} + \sqrt{x_{\mathrm{f}}}) \ \Rightarrow \ $ one (grazing) solution.
- $t_{\mathrm{f}} > \lambda(\sqrt{x_0} + \sqrt{x_{\mathrm{f}}}) \ \Rightarrow \ $ **no** solutions!

Solutions of two-point boundary-value problems may not exist and may be nonunique.

THE TIME OF BOUNCE

Suppose bounce occurs at $t = t_{\mathrm{b}}$. For $0 < t < t_{\mathrm{b}}$ we can use our 2-point no-ceiling formula

$$x(t) = x_0 + \left(\frac{x_{\mathrm{f}} - x_0}{t_{\mathrm{f}}} - \tfrac{1}{2}gt_{\mathrm{f}} \right) t + \tfrac{1}{2}gt^2$$

with $t_{\mathrm{f}} = t_{\mathrm{b}}$ and $x_{\mathrm{f}} = 0$:

$$x(t) = x_0 - \left(\frac{x_0}{t_{\mathrm{b}}} + \tfrac{1}{2}gt_{\mathrm{b}} \right) t + \tfrac{1}{2}gt^2.$$

Similarly, for $t_{\mathrm{b}} < t < t_{\mathrm{f}}$ we take $x_0 = 0$ and start the clock at t_{b} :

$$x(t) = \left(\frac{x_{\mathrm{f}}}{t_{\mathrm{f}} - t_{\mathrm{b}}} - \tfrac{1}{2}g(t_{\mathrm{f}} - t_{\mathrm{b}}) \right)(t - t_{\mathrm{b}}) + \tfrac{1}{2}g(t - t_{\mathrm{b}})^2.$$

The velocities at t_{b} must be equal and opposite:

$$0 = x'(t_{\mathrm{b}} - 0) + x'(t_{\mathrm{b}} + 0)$$

$$= \left(-\frac{x_0}{t_{\mathrm{b}}} + \tfrac{1}{2}gt_{\mathrm{b}} \right) + \left(\frac{x_{\mathrm{f}}}{t_{\mathrm{f}} - t_{\mathrm{b}}} - \tfrac{1}{2}g(t_{\mathrm{f}} - t_{\mathrm{b}}) \right),$$

which simplifies to

$$t_{\mathrm{b}}{}^3 - \frac{3t_{\mathrm{f}}}{2}\, t_{\mathrm{b}}{}^2 + \left(\frac{t_{\mathrm{f}}^2}{2} - \frac{x_0 + x_{\mathrm{f}}}{g} \right) t_{\mathrm{b}} + \frac{t_{\mathrm{f}} x_0}{g} = 0,$$

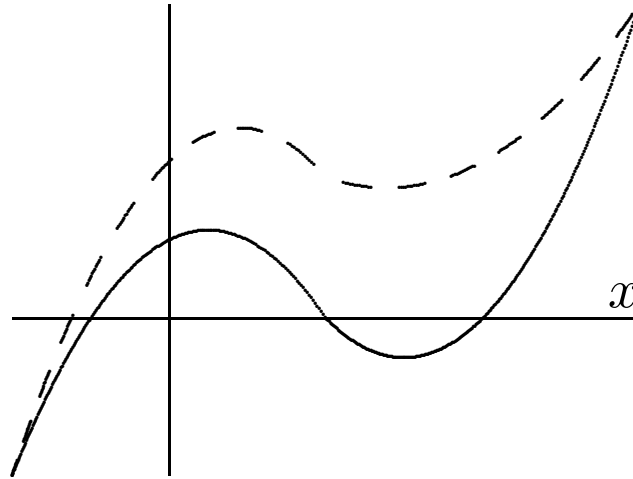a cubic equation to be solved for t_{b} .

Cubic equations

$$x^3 + ax^2 + bx + c = 0.$$

1. *Completing the cube*: Let $x = y - \frac{a}{3}$; the y^2 term cancels. So we can assume the form

$$x^3 + px + q = 0.$$

Henceforth assume p and q are real.

2. *There may be* **three** *real roots, or* **one**. (Or **two**, one of which is double.)

x

3. *Renaissance Italian history*: It is tangled.

- 16th-century mathematicians did not publish journal articles; they saved up theorems for books, often written years later.
- Professional ethics was primitive: plagiarism, charges of plagiarism, secrecy.

Scipione del Ferro (1465–1526):

- Taught his student Antonio Fior how to solve at least some cubic equations.
- Left a notebook to his son-in-law.

Niccolo Fontana "Tartaglia" (1500–1557): Discovered the general solution and won a public competition with del Ferro's student to solve examples.

Girolamo Cardano (1501–1576): Wrote famous treatise *Ars Magna* (1545). He had learned Tartaglia's method or formula by promising not to reveal it.

Lodovico Ferrari (1522–1565): Student of Cardano; used Tartaglia's solution of the general cubic to solve the general quartic equation.

Cardano wanted to cover the solution of the quartic in his book, but it made no sense without also treating the cubic. Cardano and Ferrari learned from del Ferro's son-in-law that del Ferro had discovered more about the general cubic than his inept student had retained. Cardano used this as an excuse to break his promise to Tartaglia: If del Ferro had solved the problem, Tartaglia had no right to embargo the information. (Tartaglia did not agree.)

4. *Tartaglia's formula in modern notation:*
A solution of $x^3 + px + q = 0$ is

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \sqrt[3]{-\frac{q}{2} - \sqrt{R}}, \quad R = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3.$$

When there is only one real root, this formula gives it. That satisfied the Renaissance mathematicians, who didn't know about complex numbers.

In cases with 3 real roots, R is *negative*, so $\sqrt{R}$ is

imaginary. In the end, the imaginary parts cancel. But the formula was gibberish in 1545. (And *Maple* still has trouble graphing it.)

Rafael Bombelli (1526–1573): Was led by this formula to discover complex numbers. If he knew a root, he could prove that the formula was consistent, but he didn't yet know how to use it when the root was unknown. (Note: If you already know one root, you can find the other two by factoring and you don't need the formula.)

Example: (Boyer, *A History of Mathematics*)

$$x^3 = 15x + 4 \implies x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

But it's also easy to see that $x = 4$ is a solution. If formula is right, $\sqrt[3]{2 \pm \sqrt{-121}}$ must equal $2 \pm ib$; but also its cube must equal $2 + 11i$:

$$2^3 - 3 \cdot 2b^2 = 2, \quad 3 \cdot 2^2 b - b^3 = 11.$$

$b = 1$ satisfies both equations!

Unfortunately, expressing $\sqrt[3]{x + iy}$ as $a + ib$ in general requires trigonometric functions. $\sqrt[3]{re^{i\theta}} = \sqrt[3]{r}\, e^{i\theta/3}$.

5. *The trig solution — François Viète (1540–1603):*

- Introduced letters for variables.
- Proved many trig identities such as

$$\cos^3 \theta - \tfrac{3}{4}\cos \theta - \tfrac{1}{4}\cos(3\theta) = 0.$$

- Used that one to solve cubics:

Write general cubic as $y^3 + 3m^2 py + m^3 q = 0$
(m arbitrary). Substitute $y = \cos\theta$ and get

$$3m^2 p = -\tfrac{3}{4}, \quad -\tfrac{1}{4}\cos(3\theta) = m^3 q.$$

Solve the first equation for m, then the second equation for θ (using a table or approximate algorithm for inverse cosines; there is no free lunch). Computing the cosine, we get y. (The triple nature of the root is related to the multivaluedness of the inverse cosine.)

In the ceiling problem, our cubic equation

$$t_{\mathrm{b}}{}^3 - \frac{3t_{\mathrm{f}}}{2}\,t_{\mathrm{b}}{}^2 + \left(\frac{t_{\mathrm{f}}^2}{2} - \frac{x_0 + x_{\mathrm{f}}}{g} \right) t_{\mathrm{b}} + \frac{t_{\mathrm{f}}x_0}{g} = 0$$

always has three real roots; the relevant one is the one in the middle.

Tweaking Viète's method a little, we get

$$t_{\mathrm{b}} = \frac{t_{\mathrm{f}}}{2} + \frac{1}{\sqrt{3}} \sqrt{t_{\mathrm{f}}^2 + 2\lambda^2(x_0 + x_{\mathrm{f}})}$$

$$\times \sin\left[\frac{1}{3}\sin^{-1}\frac{3\sqrt{3}\lambda^2\, t_{\mathrm{f}}(x_0 - x_{\mathrm{f}})}{[t_{\mathrm{f}}^2 + 2\lambda^2(x_0 + x_{\mathrm{f}})]^{3/2}}\right]$$

where $\lambda^2 = \dfrac{2}{g}$.

6. *Numerical methods*: For most practical purposes it is better to solve cubic equations numerically (e.g., Newton's method). That's why we didn't learn Tartaglia's formula (or Viète's) in high school.

But in the quantum mechanics application it is nice to have a *formula* that can be differentiated with respect to parameters (such as x_0).

- Papa Bear's formula (Newton) is too crass and crude.

- Mama Bear's formula (Tartaglia) is elegant but useless.

- Baby Bear's formula (Viète) is just right!

Quantum mechanics
PROPAGATOR

Suppose a particle starts at x_0 at time 0.
Then at time t_f, $|U(t_\mathrm{f}, x_\mathrm{f}, x_0)|^2$ is the probability density that it will be observed at x_f.

$$i\hbar\,\frac{\partial U}{\partial t} = -\,\frac{\hbar^2}{2m}\,\frac{\partial^2 U}{\partial x^2} + V(x)U \qquad (x_0 \text{ fixed}).$$

$U(0, x, x_0)$ is completely concentrated at x_0.

SEMICLASSICAL PRESCRIPTION

Let $x(t)$ be a solution of the *classical* equation of motion ("path").

Lagrangian: $L(t, x, x_0) = \dfrac{m}{2} \left(\dfrac{dx}{dt} \right)^2 - V(x)$.

Action: $S(t_{\mathrm{f}}, x_{\mathrm{f}}, x_0) = \displaystyle\int_0^{t_{\mathrm{f}}} L[x(t)]\, dt$.

Amplitude: $A(t_{\mathrm{f}}, x_{\mathrm{f}}, x_0) = \sqrt{\left| \dfrac{\partial^2 S}{\partial x_{\mathrm{f}}\, \partial x_0} \right|}$.

$$U(t_{\rm f}, x_{\rm f}, x_0) \approx \sum \frac{1}{\sqrt{2\pi i\hbar}} \, A(t_{\rm f}, x_{\rm f}, x_0) e^{iS(t_{\rm f}, x_{\rm f}, x_0)/\hbar}$$

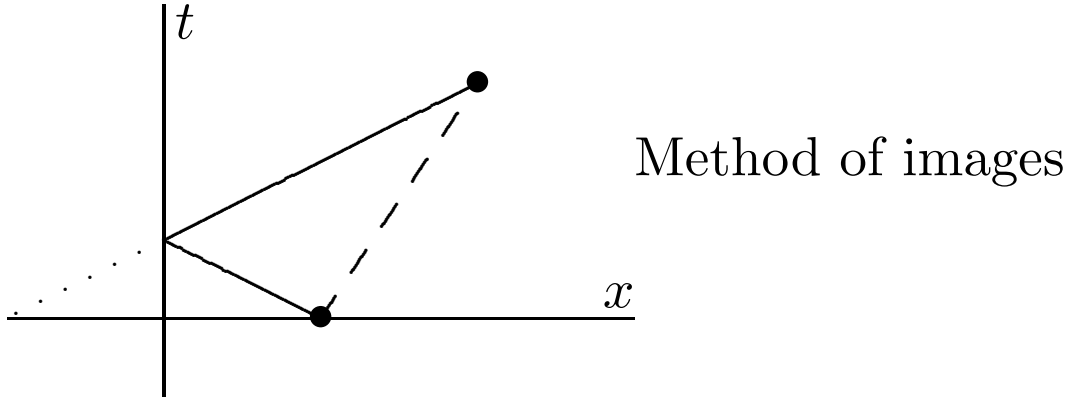summed over all classical paths that start at x_0 at time 0 and arrive at $x_{\rm f}$ at time $t_{\rm f}$.

Free particle

$$x(t) = x_0 + \frac{x_\mathrm{f} - x_0}{t_\mathrm{f}} \, t \text{ is the only such path.}$$

$$L = \frac{m}{2} \left(\frac{x_\mathrm{f} - x_0}{t_\mathrm{f}} \right)^2 , \qquad S = \frac{m}{2} \frac{(x_\mathrm{f} - x_0)^2}{t_\mathrm{f}} ,$$

$$A^2 = \frac{m}{t_\mathrm{f}} , \qquad U_0 = \sqrt{\frac{m}{2\pi i \hbar t_\mathrm{f}}} \, e^{im(x_\mathrm{f} - x_0)^2 / 2\hbar t_\mathrm{f}} \quad (\text{exact}).$$

WALL (reflecting boundary but no gravity)

Method of images

$$U(t_\mathrm{f}, x_\mathrm{f}, x_0) = U_0(t_\mathrm{f}, x_\mathrm{f}, x_0) - U_0(t_\mathrm{f}, x_\mathrm{f}, -x_0) \quad \text{(exact)}.$$

FALLING BODY (no ceiling)

$$U_{\text{direct}}(t_{\text{f}}, x_{\text{f}}, x_0) =$$
$$\sqrt{\frac{m}{2\pi i \hbar t_{\text{f}}}} \, e^{im(x_{\text{f}}-x_0)^2/2\hbar t_{\text{f}}} e^{img(x_0+x_{\text{f}})t_{\text{f}}/2\hbar} e^{-img^2 t_{\text{f}}^3/24\hbar}$$

(exact).

$\lambda(\sqrt{x_0} + \sqrt{x_\mathrm{f}})$

bounce

$t_\mathrm{b}(t, x_\mathrm{f}, x_0)$

direct

t

x
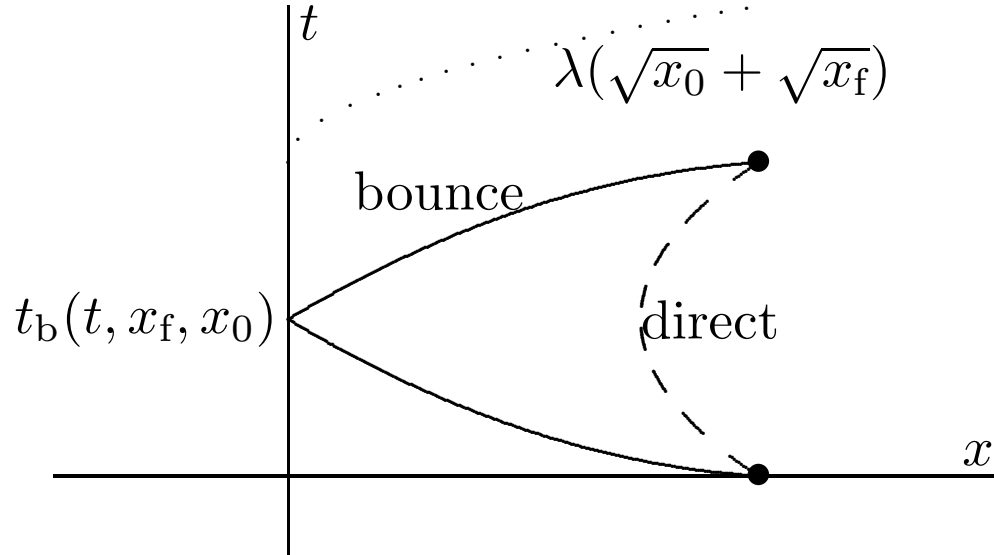
For $t_{\rm f} < \lambda(\sqrt{x_0} + \sqrt{x_{\rm f}})$, $U \approx U_{\rm direct} + U_{\rm bounce}$.

$$A_{\rm bounce}^2 = \left| \frac{\partial^2 S}{\partial x_{\rm f}\, \partial x_0} \right|, \qquad S_{\rm bounce} = \int_0^{t_{\rm b}} L\, dt + \int_{t_{\rm b}}^{t_{\rm f}} L\, dt.$$

$$\frac{\partial}{\partial x_0} = \frac{\partial}{\partial x_{\rm b}} \frac{\partial x_{\rm b}}{\partial x_0} + \cdots, \qquad \text{etc.,}$$

and that is why we need a formula for the root of the cubic.

For $t_{\mathrm{f}} > \lambda(\sqrt{x_0} + \sqrt{x_{\mathrm{f}}})$, there are no classical paths, so this approximation gives $U \approx 0$.

For $t_{\mathrm{f}} \approx \lambda(\sqrt{x_0} + \sqrt{x_{\mathrm{f}}})$, the approximation is poor.

Remedy: Construct a semiclassical approximation for the propagator when the particle starts with a known initial *momentum* (instead of position). This is a simple instance of "Maslov theory". It involves intricate classical mechanics (many cases) but no cubic equations.

Credits

- Berthold-Georg Englert (Visiting Professor of Physics, 2001–2002): final trig formula for the cubic root.
- Todd Zapata (M.S. thesis, 2007; now Ph.D. student in ECE): Semiclassical solution for initial data in momentum space.
- Various history-of-math textbooks.

MORE REFERENCES

- V. Namias, *American Journal of Physics* **53** (1985) 775

- E. W. Weisstein (at *MathWorld – A Wolfram Web Resource*),
`http://mathworld.wolfram.com/CubicFormula.html`