Testing a Series $\sum^{\infty} a_n$ for Convergence or Divergence

I. Do the terms tend to zero? [That is, $a_n \to 0$ as $n \to \infty$.] If *not*, the series *diverges*.

II. Is the series recognizable as one for which the answer is known? Examples:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3}\sum_{n=0}^{\infty} \frac{1}{3^n}$ converges to $\displaystyle\frac{2}{3}\,\frac{1}{1-\frac{1}{3}} = 1$ (*geometric series* with $|r| < 1$).

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{10^n}{n!} x^{2n}$ converges to e^{10x^2} (*Maclaurin (Taylor) series of* e^z with $z = 10x^2$).

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2}$ converges (*p-series* with $p > 1$).

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+2} = \sum_{m=2}^{\infty} \frac{1}{m}$ [let $m = n+2$] *diverges* (*harmonic series*).

III. Are all the terms positive? If so, go to IV. If not, try the following strategies:

A. Are all the hypotheses of the *alternating series* theorem satisfied?

$$a_n = (-1)^n |a_n|\,; \qquad |a_n| \text{ is decreasing}\,; \qquad |a_n| \to 0\,.$$

If so, the series converges (and the error is bounded by the magnitude of the first term omitted).

B. Try to show that the series is *absolutely convergent* [that is, $\sum |a_n|$ converges] by one of the methods under IV.

IV. If $a_n \geq 0$ (at least for all sufficiently large n), try the following strategies:

A. If the terms contain factorials, or n in an exponent, try the ratio or root test. (In particular, these methods are used to find the *radius of convergence of a power series*.)

1. If a_n contains factorials, the *ratio test* works best. Example: $a_n = \dfrac{3^n n^2}{n!}$.
2. If n appears in both an exponent and its base, the *root test* works best. Example: $a_n = \dfrac{2^n}{n^n}$.

B. If a_n is a rational function [for example, $a_n = \frac{n^2+3}{n^3-4}$], the ratio and root tests *won't work*, because the limit involved turns out to be 1. Try the following strategies:

1. Use the comparison test or limit comparison test to replace the series by one for which you know the answer (see II) or can find the answer by the integral test (see below). In choosing a comparison series $\sum b_n$, keep (at least as a first step) from each *factor* in a_n the *term* that grows fastest as $n \to \infty$. Examples:

   (a) If $a_n = \frac{1}{n^2+1}$, take $b_n = \frac{1}{n^2}$. Since $a_n < b_n$ and $\sum b_n$ is a convergent p-series, $\sum a_n$ converges by the *comparison test*. Alternatively, since $a_n/b_n \to 1$ (a finite, nonzero limit) as $n \to \infty$, and $\sum b_n$ converges, the *limit comparison test* shows that $\sum a_n$ converges.

   (b) If $a_n = \frac{\sqrt{n}}{n-\frac{1}{2}}$, take (as a first step) $b_n = \frac{\sqrt{n}}{n}$. We have $a_n > b_n$. Furthermore, $b_n \geq c_n$, where $c_n = \frac{1}{n}$, and the harmonic series $\sum c_n$ diverges. Therefore, $\sum a_n$ *diverges*, by the *comparison test*. Alternatively, try the *limit comparison test*: $a_n/c_n = \sqrt{n}\,\frac{n}{n-\frac{1}{2}}$ approaches infinity, and the harmonic series diverges, so $\sum a_n$ *diverges*.

   (c) The signs of the constant terms in the two previous examples were chosen to make the comparison test easy to apply. For $a_n = \frac{1}{n^2-1}$ or $a_n = \frac{\sqrt{n}}{n+\frac{1}{2}}$, a suitable comparison series, with the inequalities running in the right direction, is not so obvious; therefore, the *limit comparison test* is more convenient in those cases.

   (d) If $a_n = (2 + \cos n)/n^2$, neither term in the numerator dominates as $n \to \infty$, so the limit comparison test is hard to use. But $a_n \leq b_n = 3/n^2$, so the *comparison test* shows that $\sum a_n$ converges.

2. Use the *integral test* if $a_n = f(n)$, where $f(x)$ is *positive and decreasing* and $\int_1^\infty f(x)\,dx$ can be evaluated. Example:

$$\int_1^a \frac{\ln x}{x^p}\,dx$$

   can be evaluated by the substitution $u = \ln x$. (This integrand isn't decreasing at first, but it is for sufficiently large x, which is all that matters.) The integral converges if and only if $p > 1$, so

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$$

   converges or diverges just like a p-series without the logarithmic factor. (This conclusion can also be reached by comparison or limit comparison arguments.)