Functions and Distributions in Spaces with Thick Points

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ABSTRACT

Euler's vision of a generalized concept of function was a forerunner of the modern concept of distribution, and his efforts to give meaning to divergent series eventually led to the concepts of asymptotic series, summability, and distributional convergence. The introduction of such suitable abstract concepts does not automatically prevent mistakes or inconsistencies resulting from careless formal reasoning. We deal with a cluster of such issues associated with the occurrence of a distributional singularity on the boundary of a domain of integration. Apparent paradoxes are resolved by introducing new classes of test functions and distributions adapted to the problems at hand; one can regard the construction as attributing internal structure to boundary points.

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1 Introduction

It is clear from the articles in the *Mathematics Magazine* special issue devoted to Leonhard Euler, especially those of Lützen [13] and Kline [12], that in some ways Euler's sensibilities and talents were closer to those of a creative theoretical physicist such as Richard Feynman or Paul Dirac than those of a modern rigorous mathematician. If it had not been so, many of his most important contributions would not have seen the light of day.

Like most of his contemporaries, Euler "trusted the symbols far more than logic" [12]. If an infinite series did not converge for some values of a variable, the response was not to reject it as meaningless; rather, it was taken for granted that the series had a meaning and the task was to find it. Often, formal manipulations gave useful results, whose justifications were achieved only in the twentieth-century theories of asymptotics, summability (in the sense of Cesàro and Riesz), and convergence in spaces of distributions [12].

Euler worked during the period of transition from the naive notion of a function as an algebraic formula to the modern concept of a function as an arbitrary association of dependent and independent variables. His writings can be cited on both sides of the debate [13]. Because of the limitations of nineteenth-century analysis, the new definition was a restriction as much as an extension. Euler's intuition told him that generalized functions that do not satisfy draconian requirements of smoothness or even pointwise definability are too important to be left out. His vision of a generalized calculus was vindicated, as closely as it could be, by the modern theory of distributions [13].

Powerful ideas are dangerous. To paraphrase a remark of Valentine Bargmann, it is not correct to say that the work of Laurent Schwartz justifies everything that physicists do with the Dirac delta function, because sometimes they do things that are clearly wrong. There is a spectrum of responses to this situation. The first (chosen by too many mathematicians) is to dismiss distributions as untrustworthy, a kind of pornography that should be kept out of the hands of engineering and science students. Another (adopted by many practitioners) is to rationalize after the fact whatever interpretation of the symbols gives the right answer in the problem at hand; as we shall see below, sometimes this is done in blatant contradiction to interpretations adopted in other contexts. A safer approach is to regard the delta function as a heuristic device that leads rapidly to formulas whose correctness must then be rigorously verified (e.g., by substituting a putative solution back into a differential equation). But one cannot be satisfied just with this; if distributions are unambiguously defined as linear functionals on spaces of test functions, then their properties must be unambiguous, and the mathematician should determine which formulas and calculational rules are true and why — tightening up the definitions when necessary.

2 Some puzzles

We begin with a problem that surely would have delighted Euler: Evaluate the integral

$$\int_0^\infty \cos(2kx)\,dx. \tag{2.1}$$

In the classical sense it does not converge, but nevertheless it arises naturally in the spectral theory of simple differential operators and in related applications to, for example, quantum field theory. (It is a simple analogue of integrals that arose in [3].) One expects (2.1) to make sense as a distribution in k, with $k \geq 0$. (It is essentially the orthogonality relation for the Fourier cosine transform, in which k is inherently nonnegative.) We now evaluate the integral in two very plausible ways, getting two different answers.

First, we argue that

$$\begin{aligned}\int_0^\infty \cos(2kx)\,dx &= \frac{1}{2}\int_{-\infty}^\infty \cos(2kx)\,dx \\ &= \frac{1}{2}\int_{-\infty}^\infty e^{2ikx}\,dx \\ &= \pi\delta(2k) \\ &= \frac{\pi}{2}\,\delta(k).\end{aligned} \tag{2.2}$$

Here the last two steps are well-known distributional identities, and the rest is elementary complex analysis.

On the other hand, we calculate

$$\begin{aligned}\int_0^\infty \cos(2kx)\,dx &= \left.\frac{\sin(2kx)}{2k}\right|_{x=0}^\infty \\ &= \lim_{x\to\infty}\frac{\sin(2kx)}{2k}\,.\end{aligned}$$

By definition of a distributional integral, we must evaluate this limit after integrating over a test function, $f(k)$, with support in $[0,\infty)$:

$$\begin{aligned}\lim_{x\to\infty}\int_0^\infty \frac{\sin(2kx)}{2k}\,f(k)\,dk &= \lim_{x\to\infty}\int_0^\infty \frac{\sin u}{2u}\,f\left(\frac{u}{2x}\right)\,du \\ &= \frac{1}{2}\,f(0)\int_0^\infty \frac{\sin u}{u}\,du \\ &= \frac{\pi}{4}\,f(0),\end{aligned} \tag{2.3}$$

where the last step uses a well-known integral. That is,

$$\int_0^\infty \cos(2kx)\,dx = \frac{\pi}{4}\,\delta(k). \tag{2.4}$$

One way to resolve the conflict between (2.2) and (2.4) is to conclude *a posteriori* that the (standard!) interpretation of δ used to pass from (2.3) to (2.4) is incorrect: Instead, one should postulate that

$$\int_0^\infty f(k)\delta(k)\,dk = \frac{1}{2}\,f(0), \tag{2.5}$$

or, equivalently,

$$\delta(k)H(k) = \frac{1}{2}\,\delta(k). \tag{2.6}$$

Here H is the Heaviside step function; other common notations for it are u and θ.

Formulas (2.5) and (2.6) are not new to the literature [16, 2]. Vibet [16] offers a proof: Since $\frac{1}{2}\frac{d}{dk}H(k)^2 = H(k)\frac{dH}{dk} = H(k)\delta(k)$ and $H(k)^2 = H(k)$, (2.6) follows. He does not note that multiplying (2.6) by $H(k)$ and using $H(k)^2 = H(k)$ again yields the surprising equation $H(k)\delta(k) = \frac{1}{2}H(k)\delta(k)$ (and by iteration $\ldots = \frac{1}{4}H(k)\delta(k) = \ldots$); but he does assert that $H(k)^{n-1}\delta(k) = \frac{1}{n}\delta(k)$ and that this relationship is needed to solve certain engineering problems properly. (Paskusz [14] criticizes [16] similarly by concluding from (2.6) that $H(k) = \frac{1}{2}$. It may be objected (correctly) that this conclusion holds only for $k = 0$, but even that is inconsistent with a literal interpretation of $H(k)^2 = H(k)$. The real point, of course, is that the value of a distribution at one point is not well-defined, in general. It is noteworthy, however, that such pointwise definitions of $H(0)$ are used even in relatively sophisticated and accurate papers such as [2] to motivate, or at least to parametrize, rival definitions of $\delta(k)$ when 0 is an endpoint of the interval of integration. We thank R. Nevels for pointing out references [16, 14, 2].)

Clearly, a more careful analysis of definitions is needed to determine whether or not the factor $\frac{1}{2}$ does belong in (2.4), (2.5), (2.6). It will be necessary to redefine distributions, treating the point $k = 0$ in a special way.

3 Spaces with thick points

Let $a \in \mathbb{R}$. We shall define $\mathcal{D}_{*,a}$, the space of test functions with a *thick point* located at $x = a$, and $\mathcal{D}'_{*,a}$, the corresponding space of distributions. A function ϕ with domain $\mathbb{R}$ belongs to $\mathcal{D}_{*,a}$ if it has compact support, it is smooth in $\mathbb{R}\setminus\{a\}$, and at $x = a$ all its one-sided derivatives,

$$\phi^{(n)}(a \pm 0) = \lim_{x\to a^\pm} \phi^{(n)}(x), \quad \forall n \in \mathbb{N}, \tag{3.1}$$

exist. $\mathcal{D}_{*,a}$ has a natural topology, in which $\mathcal{D}(\mathbb{R})$ is the closed subspace where $\phi^{(n)}(a+0) = \phi^{(n)}(a-0)$, $\forall n \in \mathbb{N}$. The elements of $\mathcal{D}'_{*,a}$ are the distributions defined in the standard way as the linear functionals on this enlarged space of test functions.

One can also define in a similar way the spaces $\mathcal{A}_{*,a}$ and $\mathcal{A}'_{*,a}$ for any of the usual spaces of test functions and distributions. For instance, $\mathcal{E}'_{*,a}$ is the space of compactly supported distributions with a thick point at $x = a$, and $\mathcal{S}'_{*,a}$ the corresponding space of tempered distributions. Without loss of generality we shall take $a = 0$ and use the simpler notations $\mathcal{A}_*$ and $\mathcal{A}'_*$. It is clear that instead of one thick point one could consider a space with a finite number of thick points, or even an infinite (but discrete) set of them. Somewhat less trivial, and beyond the scope of this paper, would be the extension to distributions in several variables. In fact, the idea of considering functions and generalized functions in spaces with thick points was apparently first proposed by Blanchet and Faye [1] in the context of finite parts, pseudo-functions and Hadamard regularization studied by Sellier [15]; their analysis is aimed at the study of the dynamics of point particles in high post-Newtonian approximations of general relativity, and it thus developed in dimension 3 (which is also the natural arena for a precise reformulation of the work of Blinder [2]).

If X and Y are topological vector spaces with $X \subset Y$, the inclusion, i, being continuous, we shall denote by π the adjoint operator, $\pi = i'$, which is a projection from Y' to X'. In the case of spaces with thick points, one has $\mathcal{A} \subset \mathcal{A}_{*,a}$, and thus we have a projection $\pi\colon \mathcal{A}'_{*,a} \to \mathcal{A}'$, given explicitly as

$$\langle \pi(f), \phi \rangle_{\mathcal{A}' \times \mathcal{A}} = \langle f, \phi \rangle_{\mathcal{A}'_{*,a} \times \mathcal{A}_{*,a}} .$$

Every distribution $g \in \mathcal{A}'$ can be extended to $\mathcal{A}'_{*,a}$; that is, there exist distributions $f \in \mathcal{A}'_{*,a}$ such that $\pi(f) = g$. If f_0 is any extension, then the most general extension is given as

$$f = f_0 + \sum_{j=0}^{n} \alpha_j s_j \, , \tag{3.2}$$

where $s_j = s_{j,a}$ are the distributions that give the *saltus* (jump) of the jth derivative across $x = a$,

$$\langle s_j, \phi \rangle = \phi^{(j)}(a+0) - \phi^{(j)}(a-0), \tag{3.3}$$

where $n \in \mathbb{N}$, and where $\alpha_0, \ldots, \alpha_n$ are arbitrary constants.

The derivative of $\phi \in \mathcal{A}_{*,a}$ is defined classically. (In particular, a saltus in ϕ does *not* generate a δ term in ϕ'.) Then we may define the derivatives of the distributions of $\mathcal{A}'_{*,a}$ by the usual duality process, $\langle f', \phi \rangle = -\langle f, \phi' \rangle$. Clearly, $\pi(f') = \pi(f)'$. Also, $s_j = (-1)^j s_0^{(j)}$.

We shall consider the one-sided delta functions at the thick point, $\delta_\pm(x) = \delta\big(x - (a \pm 0)\big)$, defined as

$$\big\langle \delta\big(x - (a \pm 0)\big), \phi(x) \big\rangle = \phi(a \pm 0). \tag{3.4}$$

Observe that $s_0(x) = \delta\big(x-(a+0)\big) - \delta\big(x-(a-0)\big)$, and more generally $(-1)^j s_j(x) = \delta^{(j)}\big(x - (a+0)\big) - \delta^{(j)}\big(x - (a-0)\big)$.

It is important to observe that the derivative formulas in the space $\mathcal{A}'_{*,a}$ can be somewhat different from the usual derivative formulas. Indeed, suppose that $f \in \mathcal{A}'_{*,a}$ is a regular distribution generated by a function that is of class C^1 in both $(-\infty, a]$ and $[a, \infty)$ but that may have a jump $[f] = f(a+0) - f(a-0)$ across the thick point. Then f can also be considered an element of the usual space of distributions $\mathcal{A}'$, and we have the well-known formula [11]

$$\frac{\overline{d}f}{dx} = \frac{df}{dx} + [f]\delta(x-a), \tag{3.5}$$

where the overbar denotes the distributional derivative and df/dx is the ordinary (classical) derivative. However, the derivative in the space $\mathcal{A}'_{*,a}$, denoted d^*f/dx, is given by the relation

$$\frac{d^*f}{dx} = \frac{df}{dx} + f(a+0)\delta_+(x) - f(a-0)\delta_-(x). \tag{3.6}$$

Naturally, (3.5) and (3.6) satisfy $\pi(d^* f/dx) = \overline{d} f/dx$. Nevertheless, if f is continuous at $x = a$, then the $\overline{d}$ derivative coincides with the ordinary derivative, but in the space $\mathcal{A}'_{*,a}$ we have

$$\frac{d^* f}{dx} = \frac{df}{dx} + f(a) s_0(x). \tag{3.7}$$

Observe, in particular, that if c is a constant then

$$\frac{d^* c}{dx} = c s_0(x). \tag{3.8}$$

In the space $\mathcal{A}'_{*,a}$ the homogenous differential equation $d^* f/dx = 0$ has only the trivial solution, while if $g \in \mathcal{A}'_{*,a}$ the equation $d^* f/dx = g$ has at most one solution. (Actually, in the spaces $\mathcal{D}'_{*,a}$ or $\mathcal{S}'_{*,a}$ the equation $d^* f/dx = g$ has exactly one solution, but in other spaces, such as $\mathcal{E}'_{*,a}$, existence requires the extra condition $\langle g, 1 \rangle = 0$.)
The general form of the extensions of the Dirac delta function $\delta(x-a)$ to the thick-point space that are of order 0, that is, that do not contain derivatives of the deltas, is

$$\delta_{*,a,\lambda}(x) = \lambda \delta\big(x - (a+0)\big) + (1-\lambda)\delta\big(x - (a-0)\big), \tag{3.9}$$

where λ is any constant. The case when $\lambda = \frac{1}{2}$ gives us the only such extension,

$$\widetilde{\delta}(x-a) = \delta_{*,a,1/2}(x) = \tfrac{1}{2}\big[\delta\big(x-(a+0)\big) + \delta\big(x-(a-0)\big)\big], \tag{3.10}$$

that is symmetric with respect to $x = a$.
Let us now consider multiplication in the spaces $\mathcal{A}'_{*,a}$. Any space of distributions $\mathcal{A}'$ has a corresponding Moyal algebra $\mathcal{B}$, the space of multipliers of $\mathcal{A}$ and of $\mathcal{A}'$, i.e., those smooth functions ρ that satisfy $\rho\phi \in \mathcal{A},\ \forall \phi \in \mathcal{A}$. If $\mathcal{A} = \mathcal{D}$ then $\mathcal{B} = \mathcal{E}$; if $\mathcal{A} = \mathcal{E}$ then $\mathcal{B} = \mathcal{E}$; if $\mathcal{A} = \mathcal{S}$ then $\mathcal{B} = \mathcal{O}_M$. (For more on $\mathcal{O}_M$ and the other spaces see [10] or [8].) In the spaces with thick points, if $\rho \in \mathcal{B}_{*,a}$, then $\rho\phi \in \mathcal{A}_{*,a},\ \forall \phi \in \mathcal{A}_{*,a}$, and thus we may define the multiplication $\rho f \in \mathcal{A}'_{*,a}$ whenever $f \in \mathcal{A}'_{*,a}$ by the formula

$$\big\langle \rho(x) f(x), \phi(x) \big\rangle = \big\langle f(x), \rho(x)\phi(x) \big\rangle. \tag{3.11}$$

On the other hand, if $\rho \in \mathcal{B}_{*,a}$ then the multiplication $\rho\phi$ belongs to $\mathcal{A}_{*,a}$ for any $\phi \in \mathcal{A}$, and thus we can define an operator of multiplication $M_\rho\colon \mathcal{A} \to \mathcal{A}_{*,a}$, and, by duality, a corresponding multiplication operator $M_\rho\colon \mathcal{A}'_{*,a} \to \mathcal{A}'$. Observe that

$$\pi(\rho f) = M_\rho(f). \tag{3.12}$$

Notice too that if $\rho_1, \rho_2 \in \mathcal{B}_{*,a}$ then we can perform the operation $\rho_1(\rho_2 f)$, which, naturally, turns out to be $(\rho_1\rho_2)f$. However, the product $M_{\rho_1} M_{\rho_2}$ is not defined. This fact is at the root of the H^2 paradoxes in Sec. 2 (see Sec. 6).
If $\rho \in \mathcal{E}$, then $\rho(x)\delta(x-a) = \rho(a)\delta(x-a)$. The corresponding formula when there are thick points is as follows:

$$\begin{aligned} \rho(x)\delta_{*,a,\lambda}(x-a) = {} & \lambda\rho(a+0)\delta\big(x-(a+0)\big) \\ & + (1-\lambda)\rho(a-0)\delta\big(x-(a-0)\big). \end{aligned} \tag{3.13}$$

Thus $M_\rho\big(\delta_{*,a,\lambda}(x-a)\big) = \big[\lambda\rho(a+0) + (1-\lambda)\rho(a-0)\big]\delta(x)$, and in particular $M_\rho\big(\widetilde{\delta}(x-a)\big) = \{\rho\}\delta(x-a)$, where $\{\rho\} = \frac{1}{2}\big(\rho(a+0) + \rho(a-0)\big)$ is the average value at the thick point.

4 The Fourier transform in spaces with thick points

The Fourier transform of tempered distributions is a much studied and well-known operator. One of the properties of the Fourier transform operator, $\mathcal{F}$, is that it is an isomorphism of the space of test functions, $\mathcal{S}$, to itself as well as an isomorphism of the space of distributions, $\mathcal{S}'$, to itself. When one considers the operator $\mathcal{F}$ in spaces of distributions that are contained in $\mathcal{S}'$, say $\mathcal{A}' \subset \mathcal{S}'$, then $\widehat{f} = \mathcal{F}(f)$ is a tempered distribution whenever $f \in \mathcal{A}'$. However, when $\mathcal{A}'$ is not a space of tempered distributions, the image $\mathcal{F}(\mathcal{A}')$ will not be a space of tempered distributions either. This is the situation in spaces with thick points, since $\mathcal{S}'_*$ is not a subspace of $\mathcal{S}'$. Another example of this situation is the study of the Fourier transform in the space $\mathcal{D}'$ done by Gel'fand and Shilov [9] ; see also [17] for a similar analysis of other integral transforms. In this article we adopt the simplest definition of the Fourier transform: $\widehat{f}(u) = \mathcal{F}\{f(x); u\}$ is given by the integral $\int_{-\infty}^{\infty} f(x)e^{ixu}\,dx$ when the integral exists and defined by duality or other methods when the integral diverges. Naturally, our results will remain valid, modulo trivial modifications, for all the variant conventions, and hence, in particular, for the inverse Fourier transform,

$$\mathcal{F}^{-1}\{f(x); u\} = (2\pi)^{-1}\mathcal{F}\{f(x); -u\}.$$

If $\phi \in \mathcal{S}_*$ then its Fourier transform $\widehat{\phi}$ is a smooth function, but it will not be of rapid decay at infinity, in general. The behavior of $\widehat{\phi}(u)$ as $|u| \to \infty$ follows from the Erdélyi asymptotic formula [4], [8, Example 79],

$$\int_{-\infty}^{\infty} \phi(x)e^{ixu}\,dx \sim \frac{c_1}{u} + \frac{c_2}{u^2} + \frac{c_3}{u^3} + \cdots\,, \quad |u| \to \infty, \tag{4.1}$$

where $c_{n+1} = e^{\pi i(n+1)/2}\left[\phi^{(n)}\right]$. In fact [7, Thm. 8.4.1], a smooth function ψ belongs to $\mathcal{F}(\mathcal{S}_*)$ if and only if there exist constants $c_1, c_2, c_3, \ldots$ such that $\psi(x) \sim \sum_{n=1}^{\infty} c_n x^{-n}$ as $|x| \to \infty$. Therefore, following [7, Chp. 6] we introduce the space $\mathcal{W}$ as follows.

Definition. The test-function space $\mathcal{W}$ consists of those functions $\psi \in C^{\infty}(\mathbb{R})$ that admit an asymptotic expansion of the type

$$\psi(x) \sim \sum_{n=1}^{\infty} c_n x^{-n} \quad \text{as } |x| \to \infty \tag{4.2}$$

for some constants $c_1, c_2, c_3, \ldots$. The space of distributions $\mathcal{W}'$ is the corresponding dual space.

We can now define the Fourier transform of the distributions of the space $\mathcal{S}'_*$.

Definition. If $f \in \mathcal{S}'_*$ then its Fourier transform $\widehat{f} = \mathcal{F}(f)$ is the element of the space $\mathcal{W}'$ defined by

$$\left\langle \widehat{f}(u), \psi(u) \right\rangle = \left\langle f(x), \widehat{\psi}(x) \right\rangle, \quad \psi \in \mathcal{W}. \tag{4.3}$$

Similarly, if $g \in \mathcal{W}'$ then its Fourier transform $\widehat{g} = \mathcal{F}(g)$ is the element of the space $\mathcal{S}'_*$ defined by

$$\left\langle \widehat{g}(x), \phi(x) \right\rangle = \left\langle g(u), \widehat{\phi}(u) \right\rangle, \quad \phi \in \mathcal{S}_*\,. \tag{4.4}$$

The Fourier transform is an isomorphism between the spaces $\mathcal{S}'_*$ and $\mathcal{W}'$, and between the spaces $\mathcal{W}'$ and $\mathcal{S}'_*$.

In order to understand the Fourier transform in these spaces, it is convenient to note several properties of the space $\mathcal{W}'$. This space of generalized functions was introduced in [7] to study the Hilbert transform of distributions. One of the most important characteristics of $\mathcal{W}'$ is that its elements are not distributions over $\mathbb{R}$ but rather distributions over the one-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. We denote by $\delta_{\infty,j}$ the element of $\mathcal{W}'$ given by

$$\langle \delta_{\infty,j}(u), \psi(u) \rangle = c_j \tag{4.5}$$

when $\psi \in \mathcal{W}$ has the development (4.2). Any element $g \in \mathcal{W}'$ admits a "restriction" $\pi g \in \mathcal{S}'$, but that restriction might vanish even if g does not, namely if g is "concentrated at ∞", that is, if it has the form

$$g(u) = \sum_{j=1}^{n} b_j \delta_{\infty,j}(u). \tag{4.6}$$

Each $g \in \mathcal{S}'$ admits "extensions" $\widetilde{g} \in \mathcal{W}'$, but such extensions are not unique, since we could always add a distribution of the form (4.6). Some tempered distributions admit *canonical* extensions to $\mathcal{W}'$, but there is no canonical way to extend *all* elements of $\mathcal{S}'$ to $\mathcal{W}'$. The extension problem is rather similar to the regularization problem studied in [6].

Observe that when a tempered distribution g admits a canonical extension $\widetilde{g} \in \mathcal{W}'$, then its Fourier transform $\mathcal{F}(g)$, which is an element of $\mathcal{S}'$, admits a canonical extension to the space $\mathcal{S}'_*$ of distributions over the line with a thick point at $x = 0$, and this extension is precisely $\mathcal{F}(\widetilde{g})$.

If g is a distribution of compact support, $g \in \mathcal{E}'(\mathbb{R})$, then the equation

$$\langle \widetilde{g}, \psi \rangle_{\mathcal{W}' \times \mathcal{W}} = \langle g, \psi \rangle_{\mathcal{E}' \times \mathcal{E}} \tag{4.7}$$

defines a canonical extension. On the other hand, if $g \in \mathcal{S}'$ satisfies the estimate

$$g(u) = O\left(|u|^{\alpha}\right) \ \ \text{(C)}, \ \text{ as } \ |u| \to \infty, \tag{4.8}$$

in the Cesàro sense [5, 8], and $\alpha < 0$, then g admits a canonical extension given by the Cesàro evaluation

$$\langle \widetilde{g}, \psi \rangle_{\mathcal{W}' \times \mathcal{W}} = \langle g, \psi \rangle \ \ \text{(C)}, \tag{4.9}$$

which exists because $g(u)\psi(u) = O\left(|u|^{\alpha - 1}\right)$ (C). Any tempered distribution g satisfies (4.8) for some $\alpha \in \mathbb{R}$ [5, 8], but if $\alpha > 0$ the extension to $\mathcal{W}'$ is not canonical but depends on k arbitrary constants if $k - 1 \leq \alpha < k$ for some $k \in \{1, 2, 3, \ldots\}$, much as a primitive of order k depends on k arbitrary constants of integration.

Other tempered distributions that admit canonical extensions to $\mathcal{W}'$, obtained by analytic continuation, are the distributions u_+^{α} and u_-^{α} for $\alpha \notin \mathbb{Z}$, the combination $|\widetilde{u}|^{\alpha} = \widetilde{u}_+^{\alpha} + \widetilde{u}_-^{\alpha}$ for $\alpha = 0, \pm 2, \pm 4, \ldots$, and the combination $\operatorname{sgn} u \, |\widetilde{u}|^{\alpha} = \widetilde{u}_+^{\alpha} - \widetilde{u}_-^{\alpha}$ for $\alpha \in \mathbb{C} \setminus 2\mathbb{Z}$ [7, Sec. 6.3]. Therefore the distribution $\widetilde{u}^n$ is defined for *all* integers. In particular, the tempered distribution $1 = |u|^{\alpha}|_{\alpha=0}$ admits a canonical extension $\widetilde{1} = |\widetilde{u}|^{\alpha}|_{\alpha=0}$; this canonical extension is given by the formula

$$\left\langle \widetilde{1}, \psi\left(u\right) \right\rangle = \text{p.v.} \int_{-\infty}^{\infty} \psi\left(u\right) \, du \ , \tag{4.10}$$

the principal value being taken at infinity: $\text{p.v.}\int_{-\infty}^{\infty} = \lim_{A\to\infty}\int_{-A}^{A}$. Alternatively,

$$\left\langle \widetilde{1}, \psi(u) \right\rangle = \int_{-1}^{1} \psi(u)\, du + \int_{|u|>1} \left(\psi(u) - \frac{c_1}{u} \right) du. \tag{4.11}$$

5 Some Fourier transforms

We shall now give the Fourier transform of several distributions of the spaces $\mathcal{W}'$ and $\mathcal{S}'_*$. Observe that if a distribution f_0 of $\mathcal{W}'$ is an extension of a tempered distribution f of the space $\mathcal{S}'$, then the Fourier transform $\widehat{f_0}$ is an element of the space $\mathcal{S}'_*$ that extends the tempered distribution $\widehat{f}$. Similar remarks apply to the Fourier transform of the distributions of the space $\mathcal{S}'_*$.

Let us start with the computation of $\mathcal{F}\{\widetilde{\delta}(x);u\} \in \mathcal{W}'$. Observe that the equation $\left\langle \widetilde{\delta}(x), e^{ixu} \right\rangle = 1$, while correct, just tells us that $\mathcal{F}\{\widetilde{\delta}(x);u\}$ is a regularization in the space $\mathcal{W}'$ of the tempered distribution 1. Therefore, we proceed as follows:

$$\begin{aligned}
\left\langle \mathcal{F}\{\widetilde{\delta}(x);u\}, \psi(u) \right\rangle &= \frac{1}{2}\left(\widehat{\psi}\left(0^+\right) + \widehat{\psi}\left(0^-\right) \right) \\
&= \frac{1}{2}\lim_{x\to 0}\left(\widehat{\psi}\left(x\right) + \widehat{\psi}\left(-x\right) \right) \\
&= \lim_{x\to 0}\int_{-\infty}^{\infty} \cos xu\, \psi(u)\, du.
\end{aligned}$$

We cannot set $x = 0$ in the last integral since that would produce a divergent integral. However, we observe that $\int_{|u|>1} \cos xu\, du/u = 0$ for $x > 0$ and thus obtain with (4.11)

$$\begin{aligned}
\left\langle \mathcal{F}\left\{\widetilde{\delta}(x);u\right\}, \psi(u) \right\rangle &= \lim_{x\to 0}\left\{ \int_{-1}^{1} \cos xu\, \psi(u)\, du \right. \\
&\qquad \left. + \int_{|u|>1} \cos xu \left(\psi(u) - \frac{c_1}{u} \right) du \right\} \\
&= \int_{-1}^{1} \psi(u)\, du + \int_{|u|>1} \left(\psi(u) - \frac{c_1}{u} \right) du \\
&= \left\langle \widetilde{1}, \psi(u) \right\rangle,
\end{aligned}$$

so that

$$\mathcal{F}\left\{\widetilde{\delta}(x);u\right\} = \widetilde{1}\,. \tag{5.1}$$

We can compute $\mathcal{F}\left\{s_0(x),u\right\}$ in a similar fashion,

$$\begin{aligned}
\left\langle \mathcal{F}\left\{s_0(x);u\right\}, \psi(u) \right\rangle &= \lim_{x\to 0^+}\left(\widehat{\psi}(x) - \widehat{\psi}(-x) \right) \\
&= 2i\lim_{x\to 0^+}\int_{-\infty}^{\infty} \sin xu\, \psi(u)\, du \\
&= 2i\lim_{x\to 0^+}\left\{ \int_{-\infty}^{\infty} \sin xu \left(\psi(u) - \frac{c_1}{u} \right) du \right. \\
&\qquad \left. + c_1\int_{-\infty}^{\infty} \frac{\sin xu}{u}\, du \right\} \\
&= 2\pi i c_1\,,
\end{aligned}$$

so that

$$\mathcal{F}\{s_0(x);u\} = 2\pi i\delta_{\infty,1}(u). \tag{5.2}$$

Formulas (5.1) and (5.2) immediately give

$$\mathcal{F}\{\delta_{\pm}(x);u\} = \widetilde{1} \pm \pi i\delta_{\infty,1}(u), \tag{5.3}$$

where $\delta_{\pm}(x) = \delta(x-(0\pm 0))$. Formulas (5.3), in turn, yield the following limits in the space $\mathcal{W}'$:

$$e^{iu0^{\pm}} = \lim_{x\to 0^{\pm}} e^{iux} = \widetilde{1} \pm \pi i\delta_{\infty,1}(u). \tag{5.4}$$

If we now use the fact that $\mathcal{F}^{-1}\{f(u);x\} = (2\pi)^{-1}\mathcal{F}\{f(u);-x\}$, we obtain the formulas

$$\mathcal{F}\{\widetilde{1};x\} = 2\pi\widetilde{\delta}(x), \tag{5.5}$$

$$\mathcal{F}\{\delta_{\infty,1}(u);x\} = is_0(x). \tag{5.6}$$

Relation (5.6) can also be obtained from the Erdélyi asymptotic formula (4.1): If $\psi = \widehat{\phi}$, $\phi \in \mathcal{S}_*$, then

$$\begin{aligned} c_1 &= \left\langle \delta_{\infty,1}(u), \psi(u)\right\rangle = \left\langle \delta_{\infty,1}(u), \widehat{\phi}(u)\right\rangle \\ &= \left\langle \mathcal{F}\{\delta_{\infty,1}(u);x\}, \phi(x)\right\rangle, \end{aligned}$$

but according to Erdélyi's formula $c_1 = i\langle s_0(x), \phi(x)\rangle$, so (5.6) follows.

The usual formulas for the computation of the Fourier transforms of derivatives need to be modified in our context, since the product of a function $\psi(u)$ of the space $\mathcal{W}$ by the function u does not belong to $\mathcal{W}$, in general. Therefore, we introduce the modified multiplication operator $M_u\colon \mathcal{W} \to \mathcal{W}$ and its adjoint $M_u'\colon \mathcal{W}' \to \mathcal{W}'$ as

$$M_u(\psi) = u\psi(u) - c_1 \tag{5.7}$$

and, of course, $\langle M_u'(g), \psi\rangle = \langle g, M_u(\psi)\rangle$. Then we shall see that if $f \in \mathcal{S}_*'$, then

$$\mathcal{F}\{f'(x);u\} = -iM_u'\mathcal{F}\{f(x);u\}. \tag{5.8}$$

Indeed, if $\psi \in \mathcal{W}$, then denoting with a bar the distributional derivative in the space $\mathcal{S}'$ of tempered distributions, and by $[\phi]$ the jump of ϕ at the origin, we have by (3.5)

$$\begin{aligned} \frac{d}{dx}\widehat{\psi}(x) &= \frac{\overline{d}}{dx}\widehat{\psi}(x) - [\widehat{\psi}]\delta(x) \\ &= i\mathcal{F}\{u\psi(u);xt\} - [\widehat{\psi}]\delta(x) \\ &= \mathcal{F}\{iu\psi(u) - [\widehat{\psi}]/2\pi;x\} \\ &= \mathcal{F}\{iM_u(\psi);x\}, \end{aligned}$$

and (5.8) follows by duality.

Similarly, if $g \in \mathcal{W}'$ then

$$\mathcal{F}\{M_u'g(u);x\} = -i\,\frac{d^*}{dx}\mathcal{F}\{g(u);x\}. \tag{5.9}$$

Observe that $M_u'(\delta_{\infty,j}(u)) = \delta_{\infty,j+1}(u)$. Hence by (5.6)

$$\mathcal{F}\{s_j(x);u\} = (-1)^j\mathcal{F}\{s_0^{(j)}(x);u\} = 2\pi i^{j+1}\delta_{\infty,j+1}(u), \tag{5.10}$$

$$\mathcal{F}\{\delta_{\infty,j}(u);x\} = (-i)^{j-1} s_{j-1}(x) = i^{j-1} s_0^{(j-1)}(x). \tag{5.11}$$

Notice that $M'_u(f)$ is related to the multiplication $uf(u)$, but it is not the same, even if the product is well-defined. For instance, $u\delta(u)$ vanishes, but $M'_u\big(\delta(u)\big) = -\delta_{\infty,1}(u)$ since

$$\big\langle M'_u\delta(u), \psi(u)\big\rangle = \big\langle \delta(u), M_u\psi(u)\big\rangle = \big\langle \delta(u), u\psi(u) - c_1\big\rangle = -c_1\,.$$

This gives us yet another proof of (5.6), since it yields by (3.8) that

$$\widehat{\delta_{\infty,1}}(x) = i\,\frac{d^*}{dx}\mathcal{F}\{\delta(u);x\} = i\,\frac{d^*}{dx}1 = is_0(x).$$

It is interesting to observe that if g is a tempered distribution that satisfies the estimate $g(u) = |u|^\alpha$ (C), as $|u| \to \infty$, for some $\alpha < 0$, then g can be considered as an element of $\mathcal{W}'$ in a canonical way, and its Fourier transform is the canonical extension from $\mathcal{S}'$ to $\mathcal{S}'_*$ of the usual Fourier transform $\widehat{g} \in \mathcal{S}'$. However, considering the transform in the spaces $\mathcal{W}'$ and $\mathcal{S}'_*$ may prove to be useful. For instance, let us consider the distribution $f(x) = \mathcal{F}\{\widetilde{u}^{-1};x\}$. Using (5.9) it follows that f satisfies the differential equation $d^*f/dx = i\mathcal{F}\big\{\widetilde{1};x\big\} = 2\pi i\widetilde{\delta}(x)$, which has a *unique* solution in $\mathcal{S}'_*$, given by $f(x) = \pi i \operatorname{sgn} x$; this is the usual Fourier transform of u^{-1}, of course.

6 Some answers

We can now address the puzzles in Section 2.

In the space $\mathcal{D}'_*$ the multiplication by H is always defined, and if $f \in \mathcal{D}'_*$ then $Hf \in \mathcal{D}'_*$ too. Observe in particular the formulas

$$H(x)\delta_{*,a,\lambda}(x) = \lambda\delta_+(x), \tag{6.1}$$

$$H(x)\delta_+(x) = \delta_+(x), \tag{6.2}$$

$$H(x)\delta_-(x) = 0, \tag{6.3}$$

$$H(x)\widetilde{\delta}(x) = \tfrac{1}{2}\delta_+(x). \tag{6.4}$$

Observe also that

$$H(x)(H(x)f(x)) = H(x)f(x), \tag{6.5}$$

since $H^2 = H$ in the space $\mathcal{E}_*$. We can also consider the multiplication projection operator $M_H\colon \mathcal{D}'_* \to \mathcal{D}'$. The formula

$$M_H\big(\widetilde{\delta}(x)\big) = \tfrac{1}{2}\delta(x) \tag{6.6}$$

is well-defined and correct.

The often used but ill-defined formulas (2.5) and (2.6) are loose formulations of (6.6). Indeed, in many contexts one deals with a Dirac delta function (call it $f(x)$) that is a distribution in the space $\mathcal{D}'_*$, whose projection onto $\mathcal{D}'$ is the usual Dirac delta function, and, very importantly, that is symmetric with respect to the origin, $f(-x) = f(x)$. Then if f is of the first order, we should have $f(x) = \widetilde{\delta}(x)$. Formula (6.6) then follows, and in that sense (2.5) and (2.6) are vindicated. However, the symmetry of f is not true *a priori*, and if f turns out to be $\delta_{*,a,\lambda}(x)$, then

$$M_H(\delta_{*,a,\lambda}(x)) = \lambda\delta(x), \tag{6.7}$$

which can be translated loosely as

$$\text{“}\, H(x)\delta(x) = \lambda\delta(x). \text{”} \tag{6.8}$$

Of course, one really should always use (6.6) or (6.7), not (2.6) or (6.8).

Let us reappraise the alleged proof of (2.6) in Sec. 2. The problem with it is that one must be precise and consistent in saying in which space of functions or distributions one is working. Indeed, if we understand (6.5),

$$H(x)H(x) = H(x), \tag{6.9}$$

as an equation in the test-function space $\mathcal{E}_*$, then we obtain

$$2H(x)\,\frac{dH(x)}{dx} = \frac{dH(x)}{dx}\,, \tag{6.10}$$

where $dH(x)/dx$ is the derivative in the space $\mathcal{E}_*$ — that is, the *ordinary* derivative of H. But that ordinary derivative is $dH(x)/dx = 0$, and thus (6.10) is true trivially because it says that "$0 = 0$". On the other hand we may consider (6.9) as an equation in the space $\mathcal{D}'_*$, but in that case the second H on the left side and the H on the right side are elements of $\mathcal{D}'_*$ while the first H is an element of $\mathcal{E}_*$. Thus it is a good idea to rewrite it as

$$H(x)\widetilde{H}(x) = \widetilde{H}(x), \tag{6.11}$$

where $\widetilde{H}$ is H as an element of $\mathcal{D}'_*$. Then denoting the derivative in $\mathcal{D}'_*$ with a star, we obtain

$$\frac{dH(x)}{dx}\,\widetilde{H}(x) + H(x)\,\frac{d^*\widetilde{H}(x)}{dx} = \frac{d^*\widetilde{H}(x)}{dx}\,. \tag{6.12}$$

Here the first derivative is the ordinary derivative, which is 0, while the distributional derivative in the space $\mathcal{D}'_*$ is

$$\frac{d^*\widetilde{H}(x)}{dx} = \delta_+(x), \tag{6.13}$$

and therefore we obtain (6.2), which of course is true — but contains no factor $\frac{1}{2}$. Finally, (6.9) cannot be considered in the space $\mathcal{D}'$ (because H is not in the right Moyal algebra, $\mathcal{E}$), and thus the usual distributional differentiation in $\mathcal{D}'$ is not valid. Thus it is not possible to prove in this way that λ in (6.7) must equal $\frac{1}{2}$. (In particular, it is not legal to multiply by H again and conclude that $\frac{1}{2} = \frac{1}{4}$, etc., as we were tempted to do in Sec. 2.)

Now we return to the integral (2.1). Of course it is a Fourier transform, but since it is classically divergent we need to say in which space we are working, or, what is the same, which regularization of the function 1 we are using. If we work in $\mathcal{W}'$ and, consequently, look for a result in

$\mathcal{S}'_*$, it is natural because of symmetry arguments to consider the regularization $\widetilde{1}$. Hence,

$$\begin{aligned}\int_0^\infty \cos(2kx)\,dx &= \frac{1}{2}\int_{-\infty}^\infty \cos(2kx)\,dx \\ &= \frac{1}{2}\int_{-\infty}^\infty e^{2ikx}\,dx \\ &= \frac{1}{2}\mathcal{F}\{\widetilde{1};2k\} \\ &= \pi\widetilde{\delta}(2k) \\ &= \frac{\pi}{2}\,\widetilde{\delta}(k).\end{aligned}$$

The result $(\pi/2)\widetilde{\delta}(k)$ holds for k positive or negative. If we want the result for $k > 0$ in the space $\mathcal{S}'$, we need to apply the projection multiplication $M_H\colon \mathcal{S}'_* \to \mathcal{S}'$; that is we need to multiply by the Heaviside function. Use of (6.6) then yields

$$H(k)\int_0^\infty \cos(2kx)\,dx = \frac{\pi}{2}\,M_H\big(\widetilde{\delta}(k)\big) = \frac{\pi}{4}\,\delta(k). \tag{6.14}$$

That is, *both* (2.2) and (2.4) are correct, depending upon context! A pragmatic statement, avoiding abstract spaces, is that the definition of a delta function located at an endpoint of the interval of integration is a matter of convention ((2.5) versus the standard equation without the $\frac{1}{2}$; or, $M_H(\widetilde{\delta})$ versus δ in the notation of this section). Having chosen a convention, one must stay with it throughout a calculation. In particular, when an integral like (2.1) arises, one must be careful to evaluate it in terms of δ *using the convention chosen*. In the original application [3], it was found that the most convenient conventions were to interpret $\delta(k)$ as $2H(k)\widetilde{\delta}(k)$ (so that (2.4) is correct), but, in the conjugate variable, to interpret $\delta(x)$ as $H(x)\widetilde{\delta}(x)$. The consistency of all results could then be checked.

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