Energy density for a massive scalar field in (1+1)D

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Introduction

Vacuum energy density for a massive scalar field

This paper is based on Patrick Hays's paper on a confined massive field in two dimensions. In the paper "Vacuum fluctuations of a confined massive field in two dimensions," the zero-point energy of a massive scalar field confined to a two-dimensional M.I.T. bag model, is computed.

Motivation

We follow the mathematical style of Fulling's paper "Vacuum energy as spectral geometry." The vacuum energy is treated as a purely mathematical problem, an underdeveloped aspect of the spectral theory of self-adjoint second-order differential operators. What I am basically doing is to note the common generalization between the P. Hay's paper and S. Fulling's paper.

Vacuum energy density

Boundary vacuum energy from closed and periodic orbits

We consider a finite interval with either a Dirichlet or a Neumann boundary condition at each end. Thus $H = -\frac{d^2}{dx^2} + m^2$ acts in $L^2(0, L)$ on the domain defined by

$$u^{(1-l)}(0) = 0, \qquad u^{(1-r)}(L) = 0, \qquad l, r \in \{0, 1\} \tag{1}$$

The Green function can be constructed from G_∞ by the method of images. The Green function can be expressed as

$$\begin{aligned} G(\omega^2, x, y) = {} & G_\infty(y) + (-1)^l G_\infty(-y) + (-1)^r G_\infty(2L - y) + (-1)^{l+r} G_\infty(2L + y) && (2) \\ & + (-1)^{l+r} G_\infty(-2L + y) + (-1)^{2l+r} G_\infty(-2L - y) && (3) \\ & + (-1)^{l+2r} G_\infty(4L - y) + (-1)^{2l+2r} G_\infty(4L + y) + \cdots && (4) \end{aligned}$$

and the above Green function has to satisfy the equation:

$$\delta(x - y) = -\frac{\partial^2 G}{\partial x^2} + (m^2 - \lambda) G \tag{5}$$

This is the same as the equation satified by G_∞ and G in [2] except that $-\lambda$ has been replaced by $m^2 - \lambda$. So, we should be abe to use the same formulas as in [2] but we need to replace $\omega(\equiv\sqrt{\lambda})$ by

$$\kappa \equiv \sqrt{\omega^2 - m^2}. \tag{6}$$

Green Function

Let's check from first principles that the new G_∞ satisfies the right Green equations. We want

$$-\frac{\partial^2 G}{\partial x^2} - \kappa^2 G = \delta(x-y), \tag{7}$$

so for $x-y$ we want

$$\frac{\partial^2 G}{\partial x^2} = -\kappa^2 G. \tag{8}$$

Thus

$$G(x,y) = \begin{cases} Ae^{-i\kappa(x-y)}, & x<y, \\ Be^{i\kappa(x-y)}, & x>y. \end{cases} \tag{9}$$

Therefore, our G_∞ is given by

$$G_\infty(\omega^2, x, y) = \frac{i}{2\kappa} e^{-\kappa|x-y|}. \tag{10}$$

When we go to the variable κ the situation is slighly more complicated; κ is not just ω minus a constant

Remark:The Weyl and periodic terms will not be the same as in the massless case, The Hamiltonian, now contains a potential term which comes from the massive scalar field. I will adopt the convention that

$$(H_x - \kappa)G(\omega^2, x, y) = \delta(x - y). \tag{11}$$

All we know is that $G(\omega^2, x, y)$ must satisfy the above equation. Let us not loose sight of our objective, to find the local spectral density from closed and periodic orbits. So, we have

$$\begin{aligned}
\pi\frac{\kappa}{\omega}\sigma(\omega) &\equiv 2\kappa \text{ Im } G(\omega^2, x, x) && (12)\\
&= \sum_{n=0}^{n=\infty}(-1)^{n(l+r)}\cos(2\kappa nL) + \sum_{n=0}^{\infty}(-1)^{l+n(l+r)}\cos(2\kappa(nL+x)) && (13)\\
&+ \sum_{n=1}^{\infty}(-1)^{-l+n(l+r)}\cos(2\kappa(nL-x)) + \sum_{n=1}^{\infty}(-1)^{n(l+r)}\cos(2\kappa nL) && (14)\\
&= 1 + 2\sum_{n=1}^{\infty}(-1)^{n(l+r)}\cos(2\kappa nL) + \sum_{n=-\infty}^{\infty}(-1)^{l+n(l+r)}\cos(2\kappa(x+nL)) && (15)\\
&\equiv \pi\frac{\kappa}{\omega}(\sigma_{av} + \sigma_{per} + \sigma_{bdry}) \equiv \pi\frac{\kappa}{\omega}(\sigma_{av} + \sigma_{osc}) && (16)
\end{aligned}$$

where $\kappa \equiv \sqrt{\omega^2 - m^2}$.

In the case $\xi = \frac{1}{4}$, the contribution of the space derivatives is identical to that of the time derivatives, so we can write

$$T_{00}(t,x) \equiv E(t,x) = -\frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty \sigma(\omega)e^{-\omega t}\,d\omega \equiv E_{Weyl}(t,x) + E_{per}(t,x) + E_{bdry}(t,x). \tag{17}$$

Using Equation 3.914.1 from [4], we obtain the following expression:

$$\int_0^\infty e^{-t\sqrt{m^2+\kappa^2}}\cos(2nL\kappa)\,d\kappa = \frac{mt}{\sqrt{t^2+(2nL)^2}}K_1(m\sqrt{t^2+(2nL)^2}) \tag{18}$$

Let's compute the E_{Weyl} term for the massive case:

$$E_{Weyl}(t) = -\frac{1}{2}\frac{d}{dt}\int_0^\infty \sigma_{Weyl}(\omega)e^{-\omega t}\,d\omega \tag{19}$$

and doing the change of variables, $\omega^2 = \kappa^2 + m^2$, gives

$$E_{Weyl}(t) = -\frac{1}{2\pi}\frac{d}{dt}\int_0^\infty \frac{\sqrt{\kappa^2+m^2}}{\kappa}\cdot\frac{\kappa}{\sqrt{\kappa^2+m^2}}e^{-t\sqrt{\kappa^2+m^2}}\,d\kappa \tag{20}$$

$$= -\frac{1}{2\pi}\frac{d}{dt}\int_0^\infty e^{-t\sqrt{\kappa^2+m^2}}\,d\kappa = -\frac{1}{2\pi}\frac{d}{dt}mK_1(mt). \tag{21}$$

When ν is fixed and $z \to 0$,

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu} \quad (\text{Re } \nu > 0) \tag{22}$$

and in our case, $\nu = 1$ and hence

$$K_1(z = mt) \sim \frac{1}{2}\Gamma(1)\left(\frac{1}{2}mt\right)^{-1} = \frac{1}{mt} \tag{23}$$

Therefore for small mt, our expression becomes

$$E_{Weyl}(t) \sim -\frac{1}{2\pi}\frac{d}{dt}mK_1(mt) \ = -\frac{1}{2\pi}\frac{d}{dt}\left(m\frac{1}{mt}\right) = \frac{1}{2\pi t^2} \tag{24}$$

To put equation 24 into the usual form for renormalization calcualtions, we need to expand it in power (Laurent) series in t. The leading term will be $\mathcal{O}(t^{-2})$ and should match the massless case. The Laurent series in t for equation can be expressed as

$$E_{Weyl}(t) \sim -\frac{1}{2\pi}\frac{d}{dt}\left[\frac{1}{t} + \frac{1}{4}m(mt)\Big(2\log(mt) + 2\gamma - 1 - 2\log(2)\Big) + O\left((mt)^2\right)\right] \tag{25}$$

$$\sim \frac{1}{2\pi t^2} - \frac{m^2}{4\pi} - \frac{1}{8\pi}m^2(2\log(mt) + 2\gamma - 1 - 2\log(2)) + O\left(t\right) \tag{26}$$

$$\sim \frac{1}{2\pi}\left[\frac{1}{t^2} - \frac{m^2}{2}\log\left(\frac{mt}{2}\right) - \frac{m^2}{4}(1+2\gamma)\right] + O\left(t\right) \tag{27}$$

The periodic term for the massive case is given by

$$E_{per}(t) = -\frac{1}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}(-1)^{n(l+r)}\int_0^{\infty}\frac{\omega}{\kappa}\sigma_{per}(\omega)e^{-\omega t}\,d\omega \tag{28}$$

$$= -\frac{1}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}(-1)^{n(l+r)}\int_0^{\infty}e^{-t\sqrt{m^2+\kappa^2}}\cos(2nL\kappa)\,d\kappa \tag{29}$$

$$= -\frac{1}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}(-1)^{n(l+r)}\frac{mt}{\sqrt{(2nL)^2+t^2}}K_1(m\sqrt{(2nL)^2+t^2})$$

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Therefore,

$$\lim_{m\to 0}E_{per}(t) \sim -\frac{1}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}\frac{t}{(2nL)^2+t^2} - \lim_{m\to 0}\frac{1}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}\frac{m^2 t}{4}\left[\left(2\log\left(m\sqrt{(2nL)^2+t^2}\right)\right.\right. \tag{31}$$

$$\left.\left. + 2\gamma - 1 - 2\log(2)\right) + O\left(m^2(4L^2n^2+t^2)\right)\right] \tag{32}$$

$$\sim -\frac{1}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}\frac{t}{(2nL)^2+t^2} = -\frac{1}{\pi}\frac{d}{dt}\frac{1}{4}\left(\frac{\pi\coth\left(\frac{\pi t}{2L}\right)}{L} - \frac{2}{t}\right) \tag{33}$$

$$\sim \frac{\pi}{8L^2}\text{csch}^2\left(\frac{\pi t}{2L}\right) - \frac{1}{2\pi t^2} \tag{34}$$

and we can clearly see that the above result agrees with [2, p. 15].

The periodic term, $E_{per}(t)$, will approach a contant value as $t \to 0$:

$$\lim_{t\to 0} E_{per}(t) \sim -\lim_{t\to 0} \frac{1}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty} \frac{t}{(2nL)^2+t^2} \tag{35}$$

$$-\lim_{t\to 0} \frac{1}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty} \frac{m^2 t}{4}\left[\left(2\log\left(m\sqrt{(2nL)^2+t^2}\right)+2\gamma-1-2\log(2)\right)\right. \tag{36}$$

$$\left.+O(m^2(4L^2n^2+t^2))\right] \tag{37}$$

At this point, let's split the periodic terms into the massless contribution and the massive contribution. The massless contribution to the periodic energy will be denoted by $E_{per}^{m=0}(t)$ and the massive contribution will be denoted by $E_{per}^{m}(t)$.
So,

$$\lim_{t\to 0} E_{per}^{m=0}(t) = \lim_{t\to 0}\left[-\frac{1}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}\frac{t}{(2nL)^2+t^2}\right] = \lim_{t\to 0}\left[-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{d}{dt}\frac{t}{(2nL)^2+t^2}\right] \tag{38}$$

$$= \lim_{t\to 0}\left[-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{4L^2n^2-t^2}{(4L^2n^2+t^2)^2}\right] = \lim_{t\to 0}\left[\frac{\pi\operatorname{csch}^2\left(\frac{\pi t}{2L}\right)}{16L^2}-\frac{1}{4\pi t^2}\right] \tag{39}$$

$$= \lim_{t\to 0}\left(\frac{\pi\operatorname{csch}^2\left(\frac{\pi t}{2L}\right)}{8L^2}-\frac{1}{2\pi t^2}\right) = -\frac{\pi}{24L^2} \tag{40}$$

and this agrees with [2, pg. 16].

Calculation of the term $E_{bdry}(t,x)$

The interesting term is the boundary term, $E_{bdry}(t,x)$, which is given by

$$E_{bdry}(t,x) = -\frac{(-1)^l}{2\pi}\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}(-1)^{n(l+r)}\int_0^\infty \frac{\omega}{\kappa}\cos(2\kappa(x+nL))e^{-\omega t}\,d\omega \tag{41}$$

$$= -\frac{(-1)^l}{2\pi}\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}(-1)^{n(l+r)}\int_0^\infty \cos(2\kappa(x+nL))e^{-t\sqrt{\kappa^2+m^2}}\,d\kappa \tag{42}$$

Now, we do a change of variables so that we can integrate with respect to κ instead of ω. After doing the change of variables we obtain

$$E_{bdry}(t,x) = -\frac{(-1)^l}{2\pi}\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}(-1)^{n(l+r)}\int_0^\infty \cos(2\kappa(x+nL))e^{-t\sqrt{\kappa^2+m^2}}\,d\kappa \tag{43}$$

$$= -\frac{(-1)^l}{2\pi}\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}(-1)^{n(l+r)}\frac{mt}{\sqrt{(2(x+nL))^2+t^2}}K_1(m\sqrt{(2(x+nL))^2+t^2}) \tag{44}$$

since $\omega\,d\omega = \kappa\,d\kappa$.

In the boundary case, things get more complicated because we now have to deal with position, x. Then the boundary term, $E_{bdry}(x,t)$ can be expressed as

$$E_{bdry}(x,t) = \frac{(-1)^l}{2\pi}\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}(-1)^{n(l+r)}\frac{mt}{\sqrt{4(x+nL)^2+t^2}}K_1\left(m\sqrt{4(x+nL)^2+t^2}\right) \tag{45}$$

Let's go back to computing the boundary term. The boundary term can be expressed as

$$E_{bdry}(x,t) = -\frac{(-1)^l}{2\pi}\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}(-1)^{n(l+r)}\frac{mt}{\sqrt{4(x+nL)^2+t^2}}K_1\left(m\sqrt{4(x+nL)^2+t^2}\right) \tag{46}$$

$$\tag{47}$$

Using the asymptotic expansion of $K_1(z)$ for small argument yields

$$E_{bdry}(x,t) \sim \frac{(-1)^l}{2\pi}\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}(-1)^{n(l+r)}\frac{mt}{\sqrt{4(x+nL)^2+t^2}}\frac{1}{m\sqrt{4(x+nL)^2+t^2}} \tag{48}$$

$$= -\frac{(-1)^l}{2\pi}\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}(-1)^{n(l+r)}\frac{t}{4(x+nL)^2+t^2} \tag{49}$$

and this agrees with the result obtained in [2].

Let's go back to the massive case. In the massive case, we quickly discover that we can't obtain an explicit formula for the infinite sum. Then

$$E_{bdry}(x,t) = -\frac{(-1)^l}{2\pi}\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}(-1)^{n(l+r)}\Bigg[\frac{t}{4(Ln+x)^2+t^2} \tag{50}$$

$$+\frac{m^2 t}{4}\left(2\log\left(m\sqrt{4(Ln+x)^2+t^2}\right)+2\gamma-1-2\log(2)\right) \tag{51}$$

$$+O\left(\left(m\sqrt{4(Ln+x)^2+t^2}\right)^2\right)\Bigg] \tag{52}$$

Let's assume that $l+r$ is an odd integer. For the odd case, we have

$$E_{bdry}(x,t) = -\frac{(-1)^l\pi}{16L^2}\Bigg[\coth\left(\frac{\pi(t-2ix)}{2L}\right)\operatorname{csch}\left(\frac{\pi(t-2ix)}{2L}\right) \tag{53}$$

$$+\coth\left(\frac{\pi(t+2ix)}{2L}\right)\operatorname{csch}\left(\frac{\pi(t+2ix)}{2L}\right)\Bigg] \tag{54}$$

$$+\frac{(-1)^l m^2 t}{16L}\left(\operatorname{csch}\left(\frac{\pi(t-2ix)}{2L}\right)+\operatorname{csch}\left(\frac{\pi(t+2ix)}{2L}\right)\right) \tag{55}$$

$$+\frac{(-1)^l}{2\pi}\sum_{n=-\infty}^{\infty}\Bigg[\frac{(-1)^n}{4}m^2\left(2\log\left(m\sqrt{4(Ln+x)^2+t^2}\right)+2\gamma-1-2\log(2)\right) \tag{56}$$

$$+O(m^4(4(Ln+x)^2+t^2))\Bigg] \tag{57}$$

or,

$$E_{bdry}(x,t)\sim\left[\frac{\cot\left(\frac{\pi x}{L}\right)\csc\left(\frac{\pi x}{L}\right)}{8L^2}-\frac{t^2\left(\pi^2\left(2\cot^3\left(\frac{\pi x}{L}\right)+\cot\left(\frac{\pi x}{L}\right)\right)\csc\left(\frac{\pi x}{L}\right)\right)}{64L^4}+\cdots\right] \tag{58}$$

$$+\left[-\frac{t^2\pi m^2\cot\left(\frac{x}{L}\right)\csc\left(\frac{x}{L}\right)}{16L^2}+\cdots\right] \tag{59}$$

$$+\frac{(-1)^l}{2\pi}\sum_{n=-\infty}^{\infty}\Bigg[\frac{(-1)^n}{4}m^2\Bigg(2\log\left(m\sqrt{4(Ln+x)^2+t^2}\right) \tag{60}$$

$$+2\gamma-1-2\log(2)\Bigg)+O(m^4(4(Ln+x)^2+t^2))\Bigg]+O(t^4) \tag{61}$$

and when $t = 0$, we obtain

$$E_{bdry}(x,0) = \frac{\cot\left(\frac{\pi x}{L}\right)\csc\left(\frac{\pi x}{L}\right)}{8L^2} - \frac{(-1)^l}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{4}m^2\Big(2\log(mLn) + 2\gamma - 1\Big) \tag{62}$$

For the even case we obtain,

$$E_{bdry}(x,t) = -\frac{(-1)^l\pi\left(\cosh\left(\frac{\pi t}{L}\right)\cos\left(\frac{2x}{L}\right) - 1\right)}{4L^2\left(\cos\left(\frac{2x}{L}\right) - \cosh\left(\frac{\pi t}{L}\right)\right)^2} - \frac{(-1)^l\pi m^2 t\sinh\left(\frac{\pi t}{L}\right)}{8L(\cos\left(\frac{2x}{L}\right) - \cosh\left(\frac{\pi t}{L}\right))} \tag{63}$$

$$+\frac{(-1)^l}{2\pi}\sum_{n=-\infty}^{\infty}\left[\frac{1}{4}m^2\left(2\log\left(m\sqrt{4(Ln+x)^2+t^2}\right) + 2\gamma - 1 - 2\log(2)\right)\right. \tag{64}$$

$$\left. + O\left(m^4(4(Ln+x)^2+t^2)\right)\right] \tag{65}$$

Can we match this result against some results of [1] or Appendix B of the predecessor paper by Bender and Hays [3]?

Ignoring the mass terms, we obtain the following expression:

$$E_{bdry}(x,t) \sim -\frac{(-1)^l\pi\left(\cosh\left(\frac{\pi t}{L}\right)\cos\left(\frac{2x}{L}\right) - 1\right)}{4L^2\left(\cos\left(\frac{2x}{L}\right) - \cosh\left(\frac{\pi t}{L}\right)\right)^2} \tag{66}$$

and now let's assume that t is very small and that x is fixed. Assuming that $l = 1$, and using a power series expansion we obtain the following expression:

$$E_{bdry}(x,t) \sim \frac{\pi\csc^2\left(\frac{x}{L}\right)}{8L^2} - \frac{t^2\left(\pi^3\left(\cos\left(\frac{2x}{L}\right) + 2\right)\csc^4\left(\frac{x}{L}\right)\right)}{32L^4} + O(t^4). \tag{67}$$

When $t = 0$, we obtain

$$E_{bdry}(x,0) \sim \frac{\pi\csc^2\left(\frac{x}{L}\right)}{8L^2} \tag{68}$$

and the above result agrees with [2].

Then,

$$E_{bdry}(x,t) \sim \frac{\pi \csc^2\left(\frac{x}{L}\right)}{8L^2} - \frac{t^2\left(\pi^3\left(\cos\left(\frac{2x}{L}\right)+2\right)\csc^4\left(\frac{x}{L}\right)\right)}{32L^4} + \frac{\pi^2 m^2 t^2 \csc^2\left(\frac{x}{L}\right)}{16L^2} \tag{69}$$

$$+ \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} \left[\frac{1}{4}\left(2m^2 \log\left(2m\sqrt{(Ln+x)^2}\right) + 2\gamma m^2 - m^2 - 2m^2\log(2)\right)\right. \tag{70}$$

$$\left. + \frac{m^2 t^2}{16(Ln+x)^2}\right] + O\left(t^4\right) \tag{71}$$

and when $t = 0$, we have

$$E_{bdry}(x,0) \sim \frac{\pi \csc^2\left(\frac{x}{L}\right)}{8L^2} + \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{4}\left(2m^2 \log\left(2m\sqrt{(Ln+x)^2}\right) + 2\gamma m^2 - m^2 - 2m^2\log(2)\right) \tag{72}$$

When the mass is sufficiently small or equal to 0, the above analysis yields the correct answers time after time. So far, Dr. Fulling and I haven't spotted any serious errors with the above asymptotic analysis. It seems to me that the above is valid when $m \to 0$ because the above answers also seem to agree with Hay's paper [1].

The boundary term can be expressed as

$$E_{bdry}(x,t) = -\frac{(-1)^l}{2\pi}\frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty}(-1)^{n(l+r)}\frac{mt}{\sqrt{4(x+nL)^2+t^2}}K_1(m\sqrt{4(x+nL)^2+t^2}) \tag{73}$$

$$\tag{74}$$

Let's assume that $l+r$ is an even integer. Then we integrate the local energy denisty and we obtain

$$E_{bdry}(t) = -\frac{(-1)^l}{\pi}\frac{\partial}{\partial t}\sum_{n=0}^{\infty}\int_0^L \frac{mt}{\sqrt{4(x+nL)^2+t^2}}K_1(m\sqrt{4(x+nL)^2+t^2})\,dx \tag{75}$$

and hence,

$$E_{bdry}(t) = -\frac{(-1)^l}{\pi}\sum_{n=0}^{\infty}\int_0^L m\Bigg(\frac{(2Ln-t+2x)(2Ln+t+2x)K_1\left(m\sqrt{t^2+4(Ln+x)^2}\right)}{\left(4(Ln+x)^2+t^2\right)^{3/2}} \tag{76}$$

$$-\frac{mt^2K_0\left(m\sqrt{t^2+4(Ln+x)^2}\right)}{4(Ln+x)^2+t^2}\Bigg)\,dx \tag{77}$$

and doing a change of variables $x' = Ln + x$, we obtain

$$E_{bdry}(t) = -\frac{(-1)^l}{\pi}\sum_{n=0}^{\infty}\int_{Ln}^{L(n+1)} m\left(\frac{(2x'-t)\,(t+2x')\,K_1(m\sqrt{t^2+4x'^2})}{(t^2+4x'^2)^{3/2}} - \frac{mt^2K_0(m\sqrt{t^2+4x'^2})}{t^2+4x'^2}\right)dx' \quad (78)$$

$$= -\frac{(-1)^l}{\pi}\int_0^{\infty} m\left(\frac{(2x'-t)\,(t+2x')\,K_1(m\sqrt{t^2+4x'^2})}{(t^2+4x'^2)^{3/2}} - \frac{mt^2K_0(m\sqrt{t^2+4x'^2})}{t^2+4x'^2}\right)dx' \quad (79)$$

and doing another change of variables $u = 4x'^2 + t^2$, we have

$$E_{bdry}(t) = -\frac{(-1)^l}{\pi}\int_t^{\infty} m\left(\frac{\left(u^2-2t^2\right)K_1\left(mu\right)}{u^3} - \frac{mt^2K_0\left(mu\right)}{u^2}\right)\frac{u}{2\sqrt{u^2-t^2}}\,du \quad (80)$$

$$= -\frac{(-1)^l}{\pi}\int_t^{\infty} m\left(\frac{\left(u^2-2t^2\right)K_1\left(mu\right)}{2u^2\sqrt{u^2-t^2}} - \frac{mt^2K_0\left(mu\right)}{u\sqrt{u^2-t^2}}\right)du \quad (81)$$

$$E_{bdry}(t) = -\left[\frac{1}{4}\pi m^2 t\mathrm{Ei}(-mt) + \frac{1}{4}\pi m\left(mt\mathrm{Ei}(-mt) + e^{-mt}\right)\right] \quad (82)$$

and,

$$\lim_{t\to 0} E_{bdry}(t) = -\left[\frac{(-1)^l}{\pi}\lim_{t\to 0}\left(\frac{1}{4}\pi m^2 t\mathrm{Ei}(-mt) + \frac{1}{4}\pi m\left(mt\mathrm{Ei}(-mt) + e^{-mt}\right)\right)\right] \quad (83)$$

$$= -\frac{(-1)^l}{\pi}\left(\frac{m\pi}{4}\right) = -\frac{(-1)^l m}{4} \quad (84)$$

And for the Dirichlet case ($l = 0$) we obtain

$$E_{bdry}(0) = -\frac{m}{4} \quad (85)$$

Future Work

There are 4 calculations in Section 4 of [2]:

- local spectral density (σ, p. 12),
- "global" eigenvalue density (ρ or N, p. 13 and p. 14),
- total energy (E, pp. 15-16), and
- local energy density (T_{00} or $E(t,x)$, pp. 18-20).

Question: Where do we stand on these four calculations?

Answer: And, my answer is simple, I have been focusing all of my attention on the local energy density calculations. I ignored the other three calcualtions because I thought obtaining the local energy density was the top priority. Hopefully, I will get around to improving the structure of the paper itself, but before I do that I would like to receive more feedback.

The massive analogs of the formulas for ρ_{Weyl}, ρ_{per}, ρ_{bdry} will be computed along the following lines:

- The massive analogs of the formulas for $\sigma(\omega)$ are related to the massless case by simply substituting κ for ω and also multiplying it by the factor $\frac{\pi\omega}{\kappa}$.

Future Work & The Current State of Affairs

At this point, the research notes lack structure, but the notes don't lack direction. The direction that I am taking is now to compute the massive analog of the counting function $N(\omega)$.

Once again, there should be agreement between the massive counting function and the massless counting function when $m \to 0$.

Let's examine the global situation first. The eigenvalue density is

$$\rho(\omega) = \int_0^L \sigma(\omega, x)\, dx = \rho_{Weyl}(\omega) + \rho_{per}(\omega) + \rho_{bdry}(\omega), \tag{86}$$

where

$$\rho_{Weyl}(\omega) = \int_0^L \sigma_{av}\, dx = \int_0^L \frac{\omega}{\pi\kappa}\, dx = \frac{L\omega}{\pi\kappa}, \qquad \rho_{per} = \frac{2L\omega}{\pi\kappa} \sum_{n=1}^{\infty} (-1)^{n(l+r)} \cos(2\kappa n L), \tag{87}$$

The eigenvalue density is

$$\rho(\omega) = \int_0^L \sigma(\omega, x)\, dx = \rho_{Weyl}(\omega) + \rho_{per}(\omega) + \rho_{bdry}(\omega), \tag{88}$$

where

$$\rho_{Weyl}(\omega) = \int_0^L \sigma_{av}\, dx = \int_0^L \frac{\omega}{\pi\kappa}\, dx = \frac{L\omega}{\pi\kappa}, \qquad \rho_{per} = \frac{2L\omega}{\pi\kappa} \sum_{n=1}^{\infty} (-1)^{n(l+r)} \cos(2\kappa n L), \tag{89}$$

$$\rho_{bdry} = \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n(l+r)}\omega}{\kappa^2} [\sin(2\kappa L(n+1)) - \sin(2\kappa L n)], \tag{90}$$

where $\kappa = \sqrt{\omega^2 - m^2}$.

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The eigenvalue counting function $N(\omega)$ is zero for $\omega < m$ and $\int_0^\omega \rho$ for $\omega > m$. Therefore (for $\omega > m$),

$$N_{Weyl}(\omega) = \frac{L}{\pi}\int_0^\kappa \frac{\omega}{\kappa}\cdot\frac{\kappa}{\omega}\,d\kappa = \frac{L\kappa}{\pi} = \frac{L\sqrt{\omega^2 - m^2}}{\pi}. \tag{91}$$

$$N_{per}(\omega) = \frac{2L}{\pi}\sum_{n=1}^{\infty}(-1)^{n(l+r)}\int_0^\kappa \frac{\sqrt{\kappa^2+m^2}}{\kappa}\frac{\kappa}{\sqrt{\kappa^2+m^2}}\cos(2nL\,\kappa)\,d\kappa \tag{92}$$

$$= \frac{2L}{\pi}\sum_{n=1}^{\infty}(-1)^{n(l+r)}\frac{\sin(2nL\,\kappa)}{2Ln} \tag{93}$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n(l+r)}}{n}\sin(2nL\,\kappa) \tag{94}$$

The Fourier series in N_{per} can be evaluated to a sawtooth function. [See GR 1.441.1, GR 1.441.3, and [2] pp. 14 and pg. 9-10].

$$\rho_{bdry}(\omega) = \int_0^L \sigma_{bdry}(\omega)\,dx = \frac{(-1)^l}{\pi}\sum_{n=-\infty}^{\infty}(-1)^{n(l+r)}\int_0^L \frac{\omega}{\kappa}\cos(2\kappa(x+nL))\,dx \tag{95}$$

$$= \frac{(-1)^l}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n(l+r)}\omega}{\kappa^2}[\sin(2\kappa L(n+1)) - \sin(2\kappa nL)] \tag{96}$$

and,

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$$N_{bdry}(\omega) = \int_m^\omega \rho_{bdry}(\omega)\, d\omega = \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \int_0^\kappa \frac{\sqrt{\kappa^2+m^2}}{\kappa^2} \Big[\sin(2\kappa L(n+1)) - \sin(2\kappa n L)\Big] \frac{\kappa}{\sqrt{\kappa^2+m^2}}\, d\kappa \tag{97}$$

$$= \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \int_0^\kappa \frac{1}{\kappa} [\sin(2\kappa L(n+1)) - \sin(2\kappa n L)]\, d\kappa \tag{98}$$

In other words, we end up with the following expression:

$$N_{bdry}(\omega) = \begin{cases} \frac{(-1)^l}{\pi} \sum_{n=0}^{\infty} (-1)^{n(l+r)} \int_0^\kappa \frac{\sin(2\kappa L(n+1)) - \sin(2\kappa L n)}{\kappa}\, d\kappa & \text{if } l+r \text{ is even,} \\ 0 & \text{if } l+r \text{ is odd.} \end{cases} \tag{99}$$

Consider the regularized vacuum energy

$$E(t) = -\frac{d}{dt}\frac{1}{2}\int_0^\infty \rho(\omega) e^{-\omega t}\, d\omega \equiv E_{Weyl}(t) + E_{per}(t) + E_{bdry}(t) \tag{100}$$

where

$$E_{per}(t) = -\frac{L}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}(-1)^{n(l+r)} \int_0^\infty \cos(2\kappa n L) e^{-t\sqrt{\kappa^2+m^2}}\, d\kappa \tag{101}$$

$$= -\frac{L}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}(-1)^{n(l+r)} \frac{mt}{\sqrt{t^2+(2nL)^2}} K_1\left(m\sqrt{t^2+(2nL)^2}\right) \tag{102}$$

Taking the limit of $m \to 0$ of $E_{per}(t)$ yields

$$\lim_{m\to 0} E_{per}(t) = \lim_{m\to 0} -\frac{L}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}(-1)^{n(l+r)}\frac{mt}{\sqrt{t^2+(2nL)^2}}K_1\left(m\sqrt{t+(2nL)^2)}\right) \quad (103)$$

$$\sim \lim_{m\to 0} -\frac{L}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}(-1)^{n(l+r)}\frac{mt}{\sqrt{t^2+(2nL)^2}} \quad (104)$$

$$\times \left[\frac{1}{z}+\frac{1}{4}z(2\log(z)+2\gamma-1-2\log(2))+O\left(z^2\right)\right] \quad (105)$$

$$= -\frac{L}{\pi}\frac{d}{dt}\sum_{n=1}^{\infty}(-1)^{n(l+r)}\frac{t}{t^2+(2nL)^2}, \quad (106)$$

where $z = m\sqrt{t^2+(2nL)^2}$. So, the massive analog of the periodic energy agrees with the massless case.

□

The End

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