What Is Vacuum Energy, That Mathematicians Should be Mindful of It?

I shall discuss vacuum energy as a purely mathematical problem, suppressing or postponing physics issues.

The setting

Let H be a second-order, elliptic, self-adjoint PDO, on scalar functions, in a d-dimensional region Ω.

Prototype: A **billiard**. $H = -\nabla^2$, $\Omega \subset \mathbf{R}^d$, boundary conditions (say Dirichlet, $u = 0$ on $\partial\Omega$).

Generalizations:

- electromagnetic field (vector functions) *(other talks today)*
- other boundary conditions
- Riemannian manifold (Laplace–Beltrami operator)
- potential: $-\nabla^2 + V(x)$

Technical assumptions:

- smoothness as needed
- self-adjointness (spectral decomposition of $L^2(\Omega)$)
- positivity ($H \geq 0$; 0 is not an eigenvalue) for simplicity

Total energy

A finite total energy is expected when

- Spectrum is discrete.
- Ω is compact (or V is confining).

Example 1: The (Dirichlet) interval

$$\Omega = (0, L), \quad H = -\frac{d^2}{dx^2}, \quad u(0) = 0 = u(L).$$

Spectral decomposition
(eigenvalues and normalized eigenvectors)

$$H\varphi_n = E_n \varphi_n\,, \qquad \|\varphi_n\|^2 = \int_\Omega |\varphi_n(x)|^2\,dx = 1.$$

$$u(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x), \quad c_n = \langle \varphi_n, u \rangle = \int_\Omega \overline{\varphi_n(x)} u(x)\,dx.$$

Define $\omega_n = \sqrt{E_n}$.

Ex. 1: Fourier sine series.

Functional calculus and integral kernels

$$f(H)u \equiv \sum_{n=1}^{\infty} f(E_n)\langle\varphi_n, u\rangle\varphi_n\,.$$

At least formally, $f(H)u(x) = \displaystyle\int_\Omega G(x,\tilde{x})u(\tilde{x})\,d\tilde{x}$,

$$G(x,y) = \sum_{n=1}^{\infty} f(E_n)\varphi_n(x)\overline{\varphi_n(y)}.$$

If f is sufficiently rapidly decreasing, this converges to a smooth function.

Trace: $$\operatorname{Tr} G \equiv \int_\Omega G(x,x)\,dx = \sum_{n=1}^{\infty} f(E_n).$$

Cylinder (Poisson) kernel

Let $f_t(E) = e^{-t\sqrt{E}}$. $\quad f_t(H)u_0$ is the solution of

$$\frac{\partial^2 u}{\partial t^2} = Hu, \qquad u(0,x) = u_0(x),$$

that is well-behaved as $t \to +\infty$.

Kernel $$T(t,x,y) = \sum_{n=1}^{\infty} e^{-t\omega_n} \varphi_n(x) \overline{\varphi_n(y)}.$$

Trace $$\operatorname{Tr} T = \int_\Omega T(t,x,x)\,dx = \sum_{n=1}^{\infty} e^{-t\omega_n}.$$

Asymptotics ($t \downarrow 0$)

$$\operatorname{Tr} T \sim \sum_{s=0}^{\infty} e_s t^{-d+s} + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}}^{\infty} f_s t^{-d+s} \ln t.$$

- Gilkey & Grubb, *Commun. PDEs* **23** (1998), 777.
- Fulling & Gustafson, *Electr. J. DEs* **1999**, # 6.
- Bär & Moroianu, *Internat. J. Math.* **14** (2003), 397.

Define the vacuum energy as $E = -\frac{1}{2} e_{1+d}$ (modulo "local" terms to be determined by physical considerations).

Formally, E is the "finite part" of

$$\frac{1}{2} \sum_{n=1}^{\infty} \omega_n = -\frac{1}{2} \frac{d}{dt} \sum_n e^{-\omega_n t} \bigg|_{t=0} .$$

Ex. 1: (case $L = \pi$)

$$T(t,x,y) = \frac{2}{\pi} \sum_{k=1}^{\infty} \sin(kx)\sin(ky) e^{-kt}$$

$$= \frac{t}{\pi} \sum_{N=-\infty}^{\infty} \left[\frac{1}{(x-y-2N\pi)^2 + t^2} - \frac{1}{(x+y-2N\pi)^2 + t^2} \right]$$

(image sum = sum over classical paths)

$$= \frac{1}{2\pi} \left[\frac{\sinh t}{\cosh t - \cos(x-y)} - \frac{\sinh t}{\cosh t - \cos(x+y)} \right].$$

So (reverting to general L)

$$\mathrm{Tr}\, T = \frac{1}{2} \frac{\sinh(\pi t/L)}{\cosh(\pi t/L) - 1} - \frac{1}{2}$$

$$\sim \frac{L}{\pi t} - \frac{1}{2} + \frac{\pi t}{12L} + O(t^3).$$

Thus $\quad E = -\dfrac{\pi}{24L} \quad$ ($O(t)$ term times $-\frac{1}{2}$).

(There are no logarithms in this problem.)

Energy density

(remains meaningful when Ω is noncompact and H has some continuous spectrum)

Leave out the integration in the trace:

$$T(t,x,x) = \int_0^\infty e^{-t\sqrt{E}}\, dP(E,x,x)$$
$$\sim \sum_{s=0}^{\infty} e_s(x) t^{-d+s} + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}}^{\infty} f_s(x) t^{-d+s} \ln t.$$

Define $\quad E(x) = -\frac{1}{2} e_{1+d}(x).$

In quantum field theory (with $\xi = \frac{1}{4}$)

$$E(x) = \text{finite part of } \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + u\,Hu\right].$$

Example 2: The (Dirichlet) half-line

$$\Omega = (0,\infty), \quad H = -\frac{d^2}{dx^2}, \quad u(0) = 0.$$

$$P(E,x,y) = \int_0^{\sqrt{E}} \frac{2}{\pi}\, \sin(kx) \sin(ky)\, dk$$

(Fourier sine transform).

$$T(t,x,y) = \frac{t}{\pi}\left[\frac{1}{(x-y)^2+t^2} - \frac{1}{(x+y)^2+t^2}\right],$$

$$T(t,x,x) \sim \frac{1}{\pi t} - \frac{t}{\pi(2x)^2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{t}{2x}\right)^{2k} \quad \text{as } t \downarrow 0,$$

so $\quad E(x) = \dfrac{1}{8\pi x^2}\,.$

Contrast heat kernel: $\quad K(t,x,x) \sim (4\pi t)^{-d/2} + O(t^\infty)$ (for fixed $x \notin \partial\Omega$) regardless of boundary conditions!

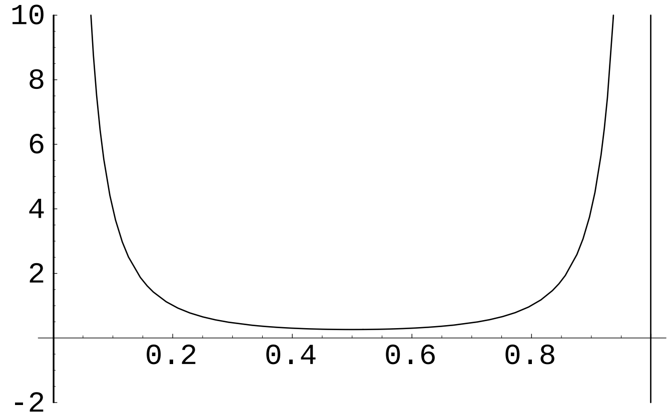
Ex. 1: $\quad E(x) = -\dfrac{\pi}{24L^2} + \dfrac{\pi}{8L^2}\,\csc^2\left(\dfrac{\pi x}{L}\right).$

$$\frac{\pi}{8L^2}\,\csc^2\left(\frac{\pi x}{L}\right) \sim \frac{1}{8\pi x^2} \text{ as } x \to 0, \quad \text{similar as } x \to L.$$

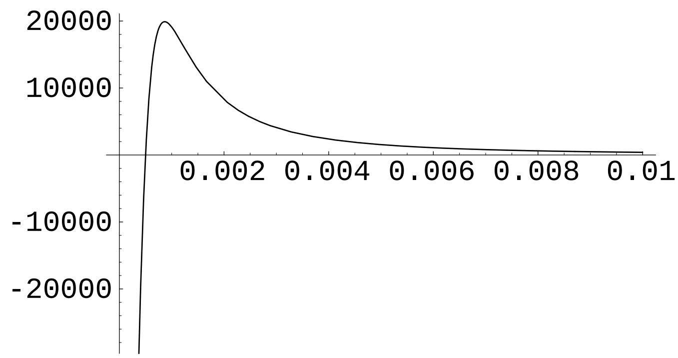
$E(x) =$ bulk (true Casimir) energy $+$ boundary energy.

$$\int_0^L E(x)\,dx = E + \infty\,!$$

The physicist says: Two kinds of renormalization.
The mathematician says: Nonuniform convergence.



Boundary energy density for $\Omega = (0, 1)$



Regularized energy density $E(t, x)$ for $\Omega = (0, \infty)$

$$E(t, x) = -\frac{1}{2}\frac{\partial}{\partial t}T(t, x, x) = -\frac{1}{2\pi}\frac{t^2 - 4x^2}{(t^2 + 4x^2)^2}\,.$$

This regularization method has no special *physical* significance. But similar results are found by physical modeling of “softer” boundaries.

- Ford & Svaiter, *Phys. Rev. D* **58** (1998) 065007.
- Graham & Olum, *Phys. Rev. D* **67** (2003) 085014.

Spectral density, counting function, etc.

$$\operatorname{Tr} T = \int_0^\infty e^{-t\omega}\, dN, \quad T(t,x,x) = \int_0^\infty e^{-t\omega}\, dP(x,x),$$
$$\operatorname{Tr} K = \int_0^\infty e^{-tE}\, dN, \; K(t,x,x) = \int_0^\infty e^{-tE}\, dP(x,x).$$

$N(E) \;=\; N(\omega^2) \;=$ number of eigenvalues $\;\le\; E$,
$P(E,x,y) =$ projection kernel onto spectrum $\le E$.

$$\operatorname{Tr} T \sim \sum_{s=0}^\infty e_s t^{-d+s} + \sum_{\substack{s=d+1 \\ s-d \text{ odd}}}^\infty f_s t^{-d+s} \ln t,$$

$$\operatorname{Tr} K \sim \sum_{s=0}^\infty b_s t^{(-d+s)/2},$$

and similarly for the local quantities.

Recall: *Semiclassical approximation* reveals oscillatory structures in N and P correlated with *periodic* and *closed* classical orbits.

- Schaden & Spruch, *Phys. Rev. A* **58** (1998) 935.
- Mazzitelli et al., *Phys. Rev. A* **67** (2003) 013807.
- Jaffe & Scardicchio, *Nucl. Phys. B* **704** (2005) 552.

Theorem. *The b_s are proportional to coefficients in the high-frequency asymptotics of Riesz means of N (or P) with respect to E. The e_s and f_s are proportional to coefficients in the asymptotics of Riesz means with respect to ω. If $d-s$ is even or positive,*

$$e_s = \pi^{-1/2} 2^{d-s} \Gamma((d-s+1)/2) b_s \,.$$

If $d-s$ is odd and negative,

$$f_s = \frac{(-1)^{(s-d+1)/2} 2^{d-s+1}}{\sqrt{\pi}\,\Gamma((s-d+1)/2)}\, b_s \,,$$

but e_s is undetermined by the b_r.

These new e_s (of which the first is the vacuum energy) are a new set of moments of the spectral distribution. *What are they good for, mathematically?* Unlike the old ones, they are *nonlocal* in their dependence on the geometry of Ω (and the coefficients of H). Thus they embody (at least partially) the global dynamical structure of the system; they are a half-way house between the heat-kernel coefficients and a full semiclassical closed-orbit analysis.

But what about the zeta function?

Let $f_s(H) = H^{-s}$, $\quad \zeta(s,H) \equiv \operatorname{Tr} f_s(H)$. Then

$$\zeta(s,H) = \zeta(2s,\sqrt{H}).$$

Zeta functions are related to integral kernels by

$$\int_0^\infty t^{s-1} T(t,H)\,dt = \Gamma(s)\zeta(s,\sqrt{H}), \quad \text{etc.}$$

Thus b_n and e_n are residues at poles of $\Gamma(s)\zeta(s,H)$ (at $s = \frac{1}{2}(d-n)$) and $\Gamma(s)\zeta(s,\sqrt{H})$ (at $s = d-n$), respectively. So (when there's no logarithm)

$$\Gamma\left(\frac{d-n}{2}\right)^{-1} b_n = \frac{1}{2}\,\Gamma(d-n)^{-1} e_n\,.$$

$\Gamma(d-n)$ may have a pole where $\Gamma\bigl(\frac{1}{2}(d-n)\bigr)$ does not; the information in the corresponding e_n is thereby expunged from the heat-kernel expansion. That quantity is not a *residue* of the zeta function but a *value* of zeta at a regular point — a more subtle object to calculate. (Logarithmic terms give rise to coinciding poles of ζ and Γ.)

- Gilkey, *Duke Math. J.* **47** (1980), 511.

Questions for investigation

1. How (if at all) is *chaos* reflected in vacuum energy?
2. What determines the *sign* of vacuum energy in each situation? (seems to be related to the phase of the periodic-orbit oscillations)©
3. Do *other spectral functions* give new geometrical information? $\left(e^{-tE^{1/3}}\,? \quad (e^{tE}-1)^{-1}\,?\right)$
4. What is the *boundary behavior* of regularized vacuum energy density in generic, multidimensional situations?©
5. What is the behavior of vacuum energy density near *edges and corners*; how does it contribute to renormalized total energy? (exterior of a cube?)©
6. Is the prediction of *low-lying spectrum* (and long-time dynamics) more accurate than stationary-phase proofs suggest? (quantum graphs?)
7. How does vacuum energy depend on *mass* (in Klein–Gordon sense)?

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Mass dependence of vacuum energy

Let $H = H_0 + \mu \quad (\mu = m^2$ in usual notation).
Let $T(\mu, t)$ stand for either $\operatorname{Tr} T$ or $T(t, x, x)$;
$K(\mu, t)$ similarly for the heat kernel.
Mass dependence of K is trivial:

$$K(\mu, t) = K(0, t) e^{-\mu t} \quad \left(\frac{\partial K}{\partial \mu} = -tK \right).$$

$$T = \sum_n e^{-t\sqrt{E_n + \mu}} \quad \text{or} \quad \int e^{-t\sqrt{E+\mu}} \, dP(E).$$

Proposition: $$\frac{\partial^2}{\partial \mu \, \partial t} \left(\frac{T}{t} \right) = \frac{T}{2} \, .$$

Let $F(s, t)$ be the Laplace transform of $T(\mu, t)/t$ with respect to μ.

$$s \, \frac{dF}{dt} - \frac{\partial}{\partial t} \, \frac{T(0, t)}{t} = \frac{t}{2} \, F.$$

$$\frac{dF}{dt} - \frac{t}{2s} \, F = \frac{\partial}{\partial t} \, \frac{T(0, t)}{st} \, .$$

$$F(s,t) = C(s)e^{t^2/4s} + e^{t^2/4s} \int_{t_0}^{t} e^{-v^2/4s} \frac{\partial}{\partial v} \frac{T(0,v)}{sv} \, dv.$$

Since T and hence $F \to 0$ as $t \to \infty$, we may choose $t_0 = \infty$ and conclude $C(s) = 0$.

Theorem:

$$F(s,t) = -e^{t^2/4s} \int_{t}^{\infty} e^{-v^2/4s} \frac{\partial}{\partial v} \frac{T(0,v)}{sv} \, dv.$$

Thus, in principle, $T(\mu, t)$ can be calculated from $T(0, v)$.