Lecture 13

13.2 Much of this section is simply reproving more general statements from linear algebra. For example, if $U \subseteq V \subseteq W$ are vectors spaces, then $dim_U W = (dim_V W)(dim_U V) = [W : V][V : U]$, to use the new notation. This is restated (and reproved) in Theorem 14. If you are having trouble reading this section, I recommend you go back to your linear algebra text and read about bases, extending bases, etc.

13.2.3: Let $x = 1+i$. Then $i = x-1$, so $-1 = (i)^2 = (x-1)^2 = x^2-2x+1$. So x^2-2x+2 is the minimal polynomial of $1+i$ since $1+i$ is a root and $1+i \notin \mathbb{Q}$.

13.2.7: By definition of a field $\mathbb{Q}(\sqrt{2}+\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2},\sqrt{3})$. Since $\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2}) = 3-2 = 1$, $\sqrt{3}-\sqrt{2}) = (\sqrt{3}+\sqrt{2}))^{-1} \in \mathbb{Q}(\sqrt{3}+\sqrt{2})$. Thus $\sqrt{3} = ((\sqrt{3}-\sqrt{2})+(\sqrt{3}+\sqrt{2}))/2 \in \mathbb{Q}(\sqrt{3}+\sqrt{2})$, whence $\sqrt{2} \in \mathbb{Q}(\sqrt{3}-\sqrt{2})$. Therefore $\mathbb{Q}(\sqrt{3}+\sqrt{2}) = \mathbb{Q}(\sqrt{3},\sqrt{2})$.

Let $x = \sqrt{3}+\sqrt{2}$. Then $x-\sqrt{2} = \sqrt{3}$, and $x^2-2x\sqrt{2}+2 = 3$, whence $x^2-1 = 2x\sqrt{2}$. Thus, $x^4-2x^2+1 = 8x^2$ or $0 = x^4-10x^2+1$. Since neither $1, -1$, are roots, we need only worry about quadratic factors whose constant terms are both ± 1 and are monic. Then $x^4-10x^2+1 = (x^2+ax\pm 1)(x^2+bx\pm 1) = x^4+(a+b)x^3+(ab\pm 2)x^2 \pm (a+b)x+1$. Thus $b = -a$ and $-10 = -a^2 \pm 2$ or $a^2 = 8,\ 12$ neither of which are squares in the integers. Therefore $x^4-10x+1$ is irreducible over $\mathbb{Q}$ and $[\mathbb{Q}(\sqrt{3}+\sqrt{2}) : \mathbb{Q}] = 4$.

13.2.8: Let F be a field of characteristic $\neq 2$. Let D_1 and D_2 be elements of F, neither of which is a square in R. Let $x = \sqrt{D_1}+\sqrt{D_2}$. Then $x-\sqrt{D_1} = \sqrt{D_2}$, whence $x^2-2x\sqrt{D_1}+D_1 = D_2$. This yields $x^2+D_1-D_2 = 2x\sqrt{D_1}$. From this we get $x^4+2(D_1-D_2)x^2+(D_1-D_2)^2 = 4D_1x^2$. Therefore, $0 = x^4-2(D_1+D_2)x^2+(D_1-D_2)^2$. By the quadratic formula $x^2 = (2(D_1+D_2) \pm \sqrt{4(D_1+D_2)^2-4(D_1-D_2)^2}/2 = (D_1+D_2 \pm \sqrt{D_1^2+2D_1D_2+D_2^2-(D_1^2-2D_1D_2+D_2^2)} = D_1+D_2 \pm \sqrt{4D_1D_2}$. Thus $x^2 \in \mathbb{Z}$ if and only if D_1D_2 is a square. Therefore, $F(\sqrt{D_1},\sqrt{D_2}) : F] = 4$ if D_1D_2 is not a square and 2 if it is.

13.3 I think the book does a good presentation of why, if α is constructable over $F \subseteq \mathbb{R}$, then $[F(\alpha) : F] = 2^s$. What they should have told you is that the converse is also true, but not provable with what we know now.

I also think they did a poor job of showing you how to use the theorems. I was particularly annoyed at them for using the triple angle formula as it gives you no idea how to deal with other angles. So here goes.

Recall DeMoivre's Formula: $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$. This is simply a restatement of $(e^{i\theta})^n = e^{in\theta}$ since $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Now recall that one can equate the real parts with the real parts and imaginary parts with the imaginary parts. Since the powers of i are i, -1, $-i$, 1 and repeat, by the binomial theorem the odd numbered terms in $(\cos(\theta) + i\sin(\theta))^n$ are all real. So let's derive the triple angle formula. $\cos(\theta) = Re((\cos(\theta/3) + i\sin(\theta/3))^3) = (\cos(\theta/3))^3 - 3\cos(\theta/3)\sin^2(\theta/3) = (\cos^3(\theta/3) - 3\cos(\theta/3)(1 - cos^2(\theta/3) = 4\cos^3(\theta/3) - 3\cos(\theta/3)$ as they claimed.

Let's show that a regular pentagon is constructable using the unproved converse. The central angle of a pentagon is $360/5 = 72$. So we can construct a regular pentagon if and only if we can construct a 72 degree angle. $1 = \cos(360) = Re((\cos(72) + i\sin(72))^5) = \cos^5(72) - 10\cos^3(72)\sin^2(72) + 5\cos(72)\sin^4(72) = \cos^5(72) - 10\cos^3(72)(1 - \cos^2(72)) + 5\cos(72)(1 - \cos^2(72))^2 = 16\cos^5(72) - 20\cos^3(72) + 5\cos(72)$. Let $x = \cos(72)$. Then $0 = 16x^5 - 20x^3 + 5x - 1 = (x-1)(16x^4 + 16x^3 - 4x^2 - 4x + 1$. Let $y = 2x$. Then $16x^4 + 15x^3 - 4x^2 - 4x + 1 = y^4 + 2y^3 - y^2 - 2y + 1 = f(y)$. Since f has no integer roots, we need only check to see if it factors into quadratics to see if it is irreducible, but in this case that does not matter as either way $[\mathbb{Q}(\cos(72)) : \mathbb{Q}]$ is even (2 or 4). So a regular pentagon is constructable.

Since $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\cos(\theta/2) = \sqrt{(1 + \cos(\theta))/2}$, if we know an angle is constructable, then we know that twice it or half it is also constructable. That means that a regular 10-gon, a regular 20-gon, a regular 40-gon, etc. are constructable because the central angles all halve from 72.

What about regular 15-gons and 30-gons? We need only check one of them since the central angle of the second is 1/2 of the first. Since $360/30 = 12$ and $5(12) = 60$, we need only do another 5th power. So $1/2 = \cos(60) = Re((\cos(12)+i\sin(12))^5) = \cos^5(12)-10\cos^3(12)\sin^2(12)+5\cos(12)\sin^4(12) = \cos^5(12)-10\cos^3(12)(1-\cos^2(12))+5\cos(12)(1-\cos^2(12))^2 = 16\cos^5(12)-20\cos^3(12)+5\cos(12)$. Let $x = \cos(12)$. Then $0 = 32x^5 - 40x^3 + 10x - 1 = y^5 - 5y^3 + 5y - 1 = (y-1)(y^4 + y^3 - 4y^2 - 4y + 1)$ where $y = 2x$. Since $y^4 + y^3 - 4y^2 - 4y + 1$ has no integer roots, it is either irreducible or the product of two irreducible quadratics. Either way, $\cos(12)$ is constructable and so are regular 15- and 30-gons.