MÜNTZ SPACES AND REMEZ INEQUALITIES

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Abstract. Two relatively long standing conjectures concerning Müntz polynomials are resolved. The central tool is a bounded Remez type inequality for non-dense Müntz spaces.

1. Introduction

Müntz's beautiful, classical theorem characterizes sequences $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{1.1}$$

for which the Müntz space $M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C[0,1]$. Here, and in what follows, $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ denotes the collection of finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \dots$ with real coefficients and $C[A]$ is the space of all real-valued continuous functions on $A \subset [0,\infty)$ equipped with the uniform norm. Throughout this paper $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ denotes a sequence satisfying (1.1). Müntz's Theorem [11, 17, 24, 27] states the following.

Theorem. *$M(\Lambda)$ is dense in $C[0,1]$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.*

The original Müntz Theorem proved by Müntz [17] in 1914, by Szász [24] in 1916, and anticipated by Bernstein [3] was only for sequences of exponents tending to infinity. The point 0 is special in the study of Müntz spaces. Even replacing $[0,1]$ by an interval $[a,b] \subset [0,\infty)$ in Müntz's Theorem is a non-trivial issue. This is, in large measure, due to Clarkson and Erdős [12] and Schwartz [22] whose works include the result that if $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ then every function belonging to the uniform closure of $M(\Lambda)$ on $[a,b]$ can be extended analytically throughout the region $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < b\}$.

There are many variations and generalizations of Müntz's Theorem [1, 4, 5, 6, 7, 8, 9, 16, 18, 22, 23, 25, 26]. There are also still many open problems.

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In Section 3 of this paper we show that the interval $[0,1]$ in Müntz's Theorem can be replaced by an arbitrary compact set $A \subset [0,\infty)$ of positive Lebesgue measure. That is, if $A \subset [0,\infty)$ is a compact set of positive Lebesgue measure, then $M(\Lambda)$ is dense in $C[A]$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.

If A contains an interval then this follows from the already mentioned results of Clarkson, Erdős, and Schwartz. However, their results and methods cannot handle the case when, for example, $A \subset [0,1]$ is a Cantor type set of positive measure.

In the case that $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, analyticity properties of the functions belonging to the uniform closure of $M(\Lambda)$ on A are also established.

Speculations about the above extension of Müntz's Theorem are probably as old as Müntz's Theorem itself.

Somorjai [23] and Bak and Newman [2, 19] proved that

$$R(\Lambda) := \{p/q : p, q \in M(\Lambda)\}$$

is always dense in $C[0,1]$. This surprising result says that while the set $M(\Lambda)$ of Müntz polynomials may be far from dense, the set $R(\Lambda)$ of Müntz rationals is always dense in $C[0,1]$ no matter what the underlying sequence Λ. In light of this result, Newman, in 1978 [19, p. 50] raises "the very sane, if very prosaic question". Are the functions

$$\prod_{j=1}^{k} \left(\sum_{i=0}^{n_j} a_{i,j} x^{i^2} \right), \qquad a_{i,j} \in \mathbb{R}, \quad n_j \in \mathbb{N}$$

dense in $C[0,1]$ for some fixed $k \geq 2$? In other words does the "extra multiplication" have the same power that the "extra division" has in the Bak-Newman-Somorjai result? Newman speculated that it did not.

Denote the set of the above products by H_k. Since every natural number is the sum of four squares, H_4 contains all the monomials x^n, $n = 0, 1, 2, \dots$. However, H_k is not a linear space, so Müntz's Theorem itself cannot be applied. Section 4 of this paper deals with products of Müntz spaces and answers the above question of Newman in the negative. For

$$\text{(1.2)} \qquad \Lambda_j := \{\lambda_{i,j}\}_{i=0}^{\infty}, \qquad 0 = \lambda_{0,j} < \lambda_{1,j} < \lambda_{2,j} < \cdots, \qquad j = 1, 2, \dots$$

we define the sets

$$M(\Lambda_1, \Lambda_2, \dots, \Lambda_k) := \left\{ p = \prod_{j=1}^{k} p_j : p_j \in M(\Lambda_j) \right\}.$$

In Section 4 a bounded Remez type inequality is established for $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ whenever

$$\text{(1.3)} \qquad \sum_{i=1}^{\infty} \frac{1}{\lambda_{i,j}} < \infty, \qquad j = 1, 2, \dots, k.$$

This obviously implies that if (1.2) and (1.3) hold and $A \subset [0,\infty)$ is a compact set of positive Lebesgue measure then $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ is not dense in $C[A]$. In particular, H_4 is not dense in $C[0,1]$ which answers Newman's problem negatively. In addition, under the assumptions (1.2) and (1.3), our methods give an "almost characterization" of the uniform closure of $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$ on A in terms of analyticity properties.

2. Bounded Remez Type Inequality for $M(\Lambda)$

Let $\mathcal{P}_n$ denote the set of all algebraic polynomials of degree at most n with real coefficients. For a fixed $s \in (0,1)$ let

$$\mathcal{P}_n(s) := \{p \in \mathcal{P}_n : m(\{x \in [0,1] : |p(x)| \le 1\}) \ge s\}$$

where $m(\cdot)$ denotes linear Lebesgue measure. The classical Remez inequality concerns the problem of bounding the uniform norm of a polynomial $p \in \mathcal{P}_n$ on $[0,1]$ given that its modulus is bounded by 1 on a subset of $[0,1]$ of Lebesgue measure at least s. That is, how large can $\|p\|_{[0,1]}$ (the uniform norm of p on $[0,1]$) be if $p \in \mathcal{P}_n(s)$? The answer is given in terms of the Chebyshev polynomials. The extremal polynomials for the above problem are the Chebyshev polynomials $\pm T_n(x) := \pm\cos(n \arccos h(x))$, where h is a linear function which scales $[0,s]$ or $[1-s,1]$ onto $[-1,1]$. For various proofs, extensions, and applications see [13, 14, 15, 20, 21].

We announce the following bounded Remez type inequality for $M(\Lambda)$ whose proof, which is quite difficult, will appear elsewhere.

Theorem 2.1. *Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Let $s > 0$. Then there exists a constant c depending only on $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ and s (and not on ϱ, A, or the "length" of p) so that*

$$\|p\|_{[0,\varrho]} \le c\|p\|_A$$

for every $p \in M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$, and for every set $A \subset [\varrho, 1]$ of Lebesgue measure at least s.

In the above theorem, and throughout the paper, $\|p\|_A := \sup_{x \in A} |p(x)|$.

One might note that the existence of such a bounded Remez type inequality for a Müntz space $M(\Lambda)$ is equivalent to the non-denseness of $M(\Lambda)$ in $C[0,1]$. We believe that this result should be a basic tool for dealing with problems about Müntz spaces. In this paper we demonstrate the power of Theorem 2.1 by settling two long standing conjectures as fairly straightforward corrolaries.

3. Müntz's Theorem on Compact Sets of Positive Measure

Theorem 3.1. *Suppose $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ and $A \subset [0,\infty)$ is a set of positive Lebesgue measure. Then $M(\Lambda)$ is not dense in $C[A]$. Moreover, if the gap condition*

$$\inf\{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0 \tag{3.1}$$

holds, then every function $f \in C[A]$ from the uniform closure of $M(\Lambda)$ on A is of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \qquad x \in A \cap [0, r_A)$$

where $r_A := \sup\{x \in [0,\infty) : m(A \cap (x,\infty)) > 0\}$ *is the essential supremum of* A. *If the gap condition (3.1) does not hold, then every function* $f \in C[A]$ *from the uniform closure of* $M(\Lambda)$ *on* A *can still be extended analytically throughout the region* $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}$.

Proof. Suppose $f \in C[A]$ and suppose there is a sequence $\{p_i\}_{i=1}^\infty \subset M(\Lambda)$ which converges to f uniformly on A. Then the sequence $\{p_i\}_{i=1}^\infty$ is uniformly Cauchy on A. Therefore, Theorem 2.1 and the definition of r_A yield that $\{p_i\}_{i=1}^\infty$ is uniformly Cauchy on every closed subinterval of $[0, r_A)$. If the gap condition (3.1) holds then the characterization of the uniform closure of $M(\Lambda)$ on A follows from the results of Clarkson and Erdős [12]. If the gap condition (3.1) does not hold, then results of Schwartz [22] yield the theorem. □

Theorem 3.2. *Suppose* $A \subset [0,\infty)$ *is a compact set of positive Lebesgue measure. Then* $M(\Lambda)$ *is dense in* $C[A]$ *if and only if* $\sum_{i=1}^\infty 1/\lambda_i = \infty$.

Proof. Suppose $\sum_{i=1}^\infty 1/\lambda_i = \infty$. Let $f \in C[A]$. By Tietze's Extension Theorem there exists an $\tilde{f} \in C[0,1]$ so that $\tilde{f}(x) = f(x)$ for every $x \in A$. By Müntz's Theorem there is a sequence $\{p_i\}_{i=1}^\infty \subset M(\Lambda)$ which converges to $\tilde{f}$ uniformly on $[0,1]$, hence on A. This finishes the trivial part of the theorem.

Suppose now that $\sum_{i=1}^\infty 1/\lambda_i < \infty$. Then Theorem 3.1 yields that $M(\Lambda)$ is not dense in $C[A]$. □

4. Products of Müntz Spaces

We prove the following Remez type inequality for $M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$.

Theorem 4.1. *Suppose (1.2) and (1.3) hold. Let* $s > 0$. *Then there exists a constant* c *depending only on* $\Lambda_1, \Lambda_2, \dots, \Lambda_k, s$, *and* k *(and not on* ϱ *or* A*) so that*

$$\|p\|_{[0,\varrho]} \le c\|p\|_A$$

for every $p \in M(\Lambda_2, \Lambda_2, \dots, \Lambda_k)$ *and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* s.

Proof. Theorem 2.1 implies that there exist constants $\alpha_j > 0$ depending only on $\Lambda_1, \Lambda_2, \dots, \Lambda_k, s$, and k so that

$$m(\{x \in [y,1] : |p(x)| > \alpha_j^{-1}|p(y)|\}) \ge 1 - y - \frac{s}{2k}$$

for every $p \in M(\Lambda_j)$ and $y \in [0, 1-s]$. Now let $p \in M(\Lambda_1, \Lambda_2, \dots, \Lambda_k)$, that is, $p = \prod_{j=1}^k p_j$ with $p_j \in M(\Lambda_j)$. Then, for every $y \in [0, 1-s]$,

$$\begin{aligned}
&m(\{x \in [y,1] : |p(x)| > (\alpha_1\alpha_2\cdots\alpha_k)^{-1}|p(y)|\}) \\
\ge &m\left(\cap_{j=1}^k \left\{x \in [y,1] : |p_j(x)| > \alpha_j^{-1}|p_j(y)|\right\}\right) \\
\ge &1 - y - k\frac{s}{2k} = 1 - y - \frac{s}{2}.
\end{aligned}$$

Hence $y \in [0, \inf A]$ and $m(A) \ge s$ imply

$$m(\{x \in A : |p(x)| > (\alpha_1\alpha_2\cdots\alpha_k)^{-1}|p(y)|\}) \ge \frac{s}{2} > 0$$

and the theorem follows with $c = \alpha_1\alpha_2\cdots\alpha_k$. □

Theorem 4.1 immediately solves Newman's problem [19].

Corollary 4.2. *Suppose (1.2) and (1.3) hold and* $A \subset [0,1]$ *is a set of positive Lebesgue measure. Then* $M(\Lambda_1, \Lambda_2, \ldots, \Lambda_k)$ *is not dense in* $C[A]$.

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