BERNSTEIN-TYPE INEQUALITIES FOR LINEAR COMBINATIONS OF SHIFTED GAUSSIANS

Tamás Erdélyi

Abstract. Let $\mathcal{P}_n$ be the collection of all polynomials of degree at most n with real coefficients. A subtle Bernstein-type extremal problem is solved by establishing the inequality

$$\|U_n^{(m)}\|_{L_q(\mathbb{R})} \leq (c^{1+1/q}m)^{m/2} n^{m/2} \|U_n\|_{L_q(\mathbb{R})}$$

for all $U_n \in \widetilde{G}_n$, $q \in (0, \infty]$, and $m = 1, 2, \dots$, where c is an absolute constant and

$$\widetilde{G}_n := \left\{ f : f(t) = \sum_{j=1}^{N} P_{m_j}(t) e^{-(t-\lambda_j)^2} \,, \quad \lambda_j \in \mathbb{R}\,, \;\; P_{m_j} \in \mathcal{P}_{m_j}\,, \;\; \sum_{j=1}^{N} (m_j + 1) \leq n \right\}.$$

Some related inequalities and direct and inverse theorems about the approximation by elements of $\widetilde{G}_n$ in $L_q(\mathbb{R})$ are also discussed.

1. Introduction and Notation

In his book [1] Braess writes "The rational functions and exponential sums belong to those concrete families of functions which are the most frequently used in nonlinear approximation theory. The starting point of consideration of exponential sums is an approximation problem often encountered for the analysis of decay processes in natural sciences. A given empirical function on a real interval is to be approximated by sums of the form

$$\sum_{j=1}^{n} a_j e^{\lambda_j t} \,,$$

where the parameters a_j and λ_j are to be determined, while n is fixed."

In [3] the authors prove the right Bernstein-type inequality for exponential sums.

Let

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t} \,, \quad a_j, \lambda_j \in \mathbb{R} \right\}.$$

Key words and phrases. Bernstein inequality, exponential sums, shifted Gaussians.

2000 Mathematics Subject Classifications: Primary: 41A17

Research is supported, in part, by NSF under Grant No. DMS-0070826

So E_n is the collection of all $n+1$ term exponential sums with constant first term. Schmidt [10] proved that there is a constant $c(n)$ depending only on n so that

$$\|f'\|_{[a+\delta,b-\delta]} \leq c(n)\delta^{-1}\|f\|_{[a,b]}$$

for every $f \in E_n$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$. Here, and in what follows, $\|\cdot\|_{[a,b]}$ denotes the uniform norm on $[a,b]$. The main result, Theorem 3.2, of [3] shows that Schmidt's inequality holds with $c(n) = 2n-1$. That is,

$$\sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a, b-y\}}, \qquad y \in (a,b). \tag{1.1}$$

In this Bernstein-type inequality even the point-wise factor is sharp up to a multiplicative absolute constant; the inequality

$$\frac{1}{e-1} \frac{n-1}{\min\{y-a, b-y\}} \leq \sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}}, \qquad y \in (a,b),$$

is established by Theorem 3.3 in [3].

Bernstein-type inequalities play a central role in approximation theory via a machinery developed by Bernstein, which turns Bernstein-type inequalities into inverse theorems of approximation. See, for example, the books by Lorentz [7] and by DeVore and Lorentz [5]. From (1.1) one can deduce in a standard fashion that if there is a sequence $(f_n)_{n=1}^{\infty}$ of exponential sums with $f_n \in E_n$ that approximates f on an interval $[a,b]$ uniformly with errors

$$\|f - f_n\|_{[a,b]} = O\left(n^{-m}(\log n)^{-2}\right), \qquad n = 2, 3, \ldots,$$

where $m \in \mathbb{N}$ is a fixed integer, then f is m times continuously differentiable on (a,b). Let $\mathcal{P}_n$ be the collection of all polynomials of degree at most n with real coefficients. Inequality (1.1) can be extended to E_n replaced by

$$\widetilde{E}_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{N} P_{m_j}(t)e^{\lambda_j t}, \quad a_0, \lambda_j \in \mathbb{R}, \ P_{m_j} \in \mathcal{P}_{m_j}, \ \sum_{j=1}^{N}(m_j+1) \leq n \right\}.$$

In fact, it is well-known that $\widetilde{E}_n$ is the uniform closure of E_n on any finite subinterval of the real number line.

For a function f defined on a set A let

$$\|f\|_A := \|f\|_{L_\infty A} := \|f\|_{L_\infty(A)} := \sup\{|f(x)| : x \in A\},$$

and let

$$\|f\|_{L_p A} := \|f\|_{L_p(A)} := \left(\int_A |f(x)|^p \, dx\right)^{1/p}, \qquad p > 0,$$

whenever the Lebesgue integral exists. In this paper we focus on the classes

$$G_n := \left\{ f : f(t) = \sum_{j=1}^{n} a_j e^{-(t-\lambda_j)^2}\,, \quad a_j, \lambda_j \in \mathbb{R} \right\},$$

$$\widetilde{G}_n := \left\{ f : f(t) = \sum_{j=1}^{N} P_{m_j}(t) e^{-(t-\lambda_j)^2}\,, \quad \lambda_j \in \mathbb{R}\,,\ P_{m_j} \in \mathcal{P}_{m_j}\,,\ \sum_{j=1}^{N} (m_j+1) \le n \right\},$$

and

$$\widetilde{G}_n^* :=$$

$$\left\{ f : f(t) = \sum_{j=1}^{N} P_{m_j}(t) e^{-(t-\lambda_j)^2}\,, \quad \lambda_j \in [-n^{1/2}, n^{1/2}]\,,\ P_{m_j} \in \mathcal{P}_{m_j}\,,\ \sum_{j=1}^{N} (m_j+1) \le n \right\}.$$

Note that $\widetilde{G}_n$ is the uniform closure of G_n on any finite subinterval of the real number line. Let $W(t) := \exp(-t^2)$. Combining Corollaries 1.5 and 1.8 in [6] and recalling that for the weight W the Mhaskar-Rachmanov-Saff number a_n defined by (1.4) in [6] satisfies $a_n \le c_1 n^{1/2}$ with a constant c_1 independent of n, we obtain that

$$\inf_{P \in \mathcal{P}_n} \|(P-g)W\|_{L_q(\mathbb{R})} \le c_2 n^{-m/2} \|g^{(m)} W\|_{L_q(\mathbb{R})}$$

with a constant c_2 independent of n, whenever the norm on the right-hand side is finite for some $m \in \mathbb{N}$ and $q \in [1, \infty]$. As a consequence

$$\inf_{f \in \widetilde{G}_n^*} \|f - gW\|_{L_q(\mathbb{R})} \le c_3 n^{-m/2} \sum_{k=0}^{m} \|(1+|t|)^{m-k} (gW)^{(k)}(t)\|_{L_q(\mathbb{R})}$$

with a constant c_3 independent of n whenever the norms on the right-hand side are finite for each $k = 0, 1, \dots, m$ with some $q \in [1, \infty]$. Replacing gW by g, we conclude that

$$\inf_{f \in \widetilde{G}_n^*} \|f - g\|_{L_q(\mathbb{R})} \le c_3 n^{-m/2} \sum_{k=0}^{m} \|(1+|t|)^{m-k} g^{(k)}(t)\|_{L_q(\mathbb{R})} \tag{1.2}$$

with a constant c_3 independent of n whenever the norms on the right-hand side are finite for each $k = 0, 1, \dots, m$ with some $q \in [1, \infty]$.

2. New Results

Theorem 2.1. *There is an absolute constant c_4 such that*

$$|U_n'(0)| \le c_4 n^{1/2}\, \|U_n\|_{\mathbb{R}}$$

for all U_n of the form $U_n = P_n Q_n$ with $P_n \in \widetilde{G}_n$ and an even $Q_n \in \mathcal{P}_n$. As a consequence

$$\|P_n'\|_{\mathbb{R}} \le c_4 n^{1/2}\, \|P_n\|_{\mathbb{R}}$$

for all $P_n \in \widetilde{G}_n$.

We remark that a closer look at the proof shows that $c_4 = 5$ in the above theorem is an appropriate choice.

Theorem 2.2. *There is an absolute constant* c_5 *such that*

$$\|U_n'\|_{L_q(\mathbb{R})} \le c_5^{1+1/q} n^{1/2} \, \|U_n\|_{L_q(\mathbb{R})}$$

for all $U_n \in \widetilde{G}_n$ *and* $q \in (0,\infty)$*.*

Theorem 2.3. *There is an absolute constant* c_6 *such that*

$$\|U_n^{(m)}\|_{L_q(\mathbb{R})} \le (c_6^{1+1/q} nm)^{m/2} \|U_n\|_{L_q(\mathbb{R})}$$

for all $U_n \in \widetilde{G}_n$*,* $q \in (0,\infty]$*, and* $m = 1,2,\ldots$*.*

We remark that a closer look at the proofs shows that $c_5 = 180\pi$ in Theorem 2.2 and $c_6 = 180\pi$ in Theorem 2.3 are appropriate choices.

Our next theorem may be viewed as a slightly weak version of the right inverse theorem of approximation that can be coupled with the direct theorem of approximation formulated in (1.2).

Theorem 2.4. *Suppose* $q \in [1,\infty]$*,* m *is a positive integer,* $\varepsilon > 0$*, and* f *is a function defined on* $\mathbb{R}$*. Suppose also that*

$$\inf_{f_n \in \widetilde{G}_n} \|f_n - f\|_{L_q(\mathbb{R})} \le c_7 n^{-m/2} (\log n)^{-1-\varepsilon}\,, \qquad n = 2,3,\ldots\,,$$

with a constant c_7 *independent of* n*. Then* f *is* m *times differentiable almost everywhere on* $\mathbb{R}$*. Also, if*

$$\inf_{f_n \in \widetilde{G}_n^*} \|f_n - f\|_{L_q(\mathbb{R})} = c_7 n^{-m/2} (\log n)^{-1-\varepsilon}\,, \qquad n = 2,3,\ldots\,,$$

with a constant c_7 *independent of* n*, then, in addition to the fact that* f *is* m *times differentiable almost everywhere on* $\mathbb{R}$*, we also have*

$$\|(1+|t|)^{m-j} f^{(j)}(t)\|_{L_q(\mathbb{R})} < \infty\,, \qquad j = 0,1,\ldots,m\,.$$

Theorem 2.5. *There is an absolute constant* c_8 *such that*

$$\|U_n'\|_{L_q[y-\delta/2,y+\delta/2]} \le c_8^{1+1/q} \left(\frac{n}{\delta}\right) \|U_n\|_{L_q[y-\delta,y+\delta]}$$

for all $U_n \in \widetilde{G}_n$*,* $q \in (0,\infty]$*,* $y \in \mathbb{R}$*, and* $\delta \in (0,n^{1/2}]$*.*

In [9] H. Mhaskar writes “Professor Ward at Texas A&M University has pointed out that our results implicitly contain an inequality, known as Bernstein inequality, in terms of the number of neurons, under some conditions on the minimal separation. Professor Erdélyi at Texas A&M University has kindly sent us a manuscript in preparation, where he proves this inequality purely in terms of the number of neurons, with no further conditions. This inequality leads to the converse theorems in terms of the number of neurons, matching our direct theorem in this theory. Our direct theorem in [8] is sharp in the sense of n-widths. However, the converse theorem applies to individual functions rather than a class of functions. In particular, it appears that even if the cost of approximation is measured in terms of the number of neurons, if the degrees of approximation of a particular function by Gaussian networks decay polynomially, then a linear operator will yield the same order of magnitude in the error in approximating this function. We find this astonishing, since many people have told us based on numerical experiments that one can achieve a better degree of approximation by non-linear procedures by stacking the centers near the bad points of the target functions".

3. Proofs

To prove Theorem 2.1 we need two lemmas. Our first lemma can be proved by a (not completely straightforward) modification of the proof of Theorem 3.2 in [3]. This is carefully done in Section 4.

Lemma 3.1. *We have*

$$|U_n'(0)| \leq \frac{2n+m}{\delta} \|U_n\|_{[-\delta,\delta]}$$

for all $U_n = \widetilde{P}_n R_m$ *with* $\widetilde{P}_n \in \widetilde{E}_n$ *and an even* $R_m \in \mathcal{P}_m$*, and for all* $\delta \in (0,\infty)$.

Our next lemma is a simple observation.

Lemma 3.2. *For the even polynomials* $S_{2n} \in \mathcal{P}_{2n}$ *defined by* $S_{2n}(x) = (1 - x^2/n)^n$ *we have* $S_{2n}(0) = 1$ *and* $0 \leq S_{2n}(x) \leq \exp(-x^2)$ *for every* $x \in [n^{-1/2}, n^{-1/2}]$.

Proof of Theorem 2.1. Observe that every $P_n \in \widetilde{G}_n$ is of the form $P_n(t) = \widetilde{P}_n(t)\exp(-t^2)$ with some $\widetilde{P}_n \in \widetilde{E}_n$. It is sufficient to prove the existence of an absolute constant c_9 such that

$$|U_n'(0)| \leq c_9 n^{1/2} \|U_n\|_{[-\delta,\delta]}\,, \qquad \delta := n^{1/2}\,, \tag{3.1}$$

for all U_n of the form $U_n = P_n Q_n$ with $P_n \in \widetilde{G}_n$ and an even $Q_n \in \mathcal{P}_n$. Note that every such U_n is of the form $U_n(t) = \widetilde{P}_n(t) Q_n(t)\exp(-t^2)$ with $\widetilde{P}_n \in \widetilde{E}_n$ and an even $Q_n \in \mathcal{P}_n$. Combining this observation with Lemma 3.2, it is sufficient to prove (3.1) for all U_n of the form $U_n = \widetilde{P}_n Q_n S_{2n} := \widetilde{P}_n R_{3n}$ with $\widetilde{P}_n \in \widetilde{E}_n$ and an even $R_{3n} := Q_n S_{2n} \in \mathcal{P}_{3n}$. However (3.1) in this situation follows from Lemma 3.1 □

In the proof of Theorem 2.2 we need the following well known result which, in fact, may be viewed as a simple exercise in approximation theory (it follows from part c of E.19 on page 413 of [2], for instance). A more direct proof of the lemma below is presented in Section 4 with $c_{10} = (2\pi)^{-1}$.

Lemma 3.3. *For every $n \in \mathbb{N}$, $\delta \in (0, \infty)$, and $q \in (0, \infty)$, there are even polynomials $V_{n,\delta,q} \in \mathcal{P}_n$ and an absolute constant $c_{10} > 0$ such that*

$$1 = |V_{n,\delta,q}(0)| \geq c_{10}^{1+1/q} \left(\frac{1+qn}{\delta} \right)^{1/q} \|V_{n,\delta,q}\|_{L_q[-\delta,\delta]} .$$

Our next lemma can be proved by a (not completely straightforward) modification of the proof of Theorem 3.2 in [3] as well. This is carefully done in Section 4 with $c_{11} = 2$.

Lemma 3.4. *There is an absolute constant c_{11} such that*

$$|U_n'(0)| \leq \left(\frac{c_{11}}{\delta} \right)^{1+1/q} (2n+m)(1+q(2n+m))^{1/q} \|U_n\|_{L_q[-\delta,\delta]}$$

holds for all U_n of the form $U_n = \widetilde{P}_n R_m$ with $\widetilde{P}_n \in \widetilde{E}_n$ and an even $R_m \in \mathcal{P}_m$, and for all $q \in (0, \infty)$.

Combining Lemmas 3.4 and 3.2 we obtain the lemma below with $c_{12} = 5c_{11} = 10$.

Lemma 3.5. *There is an absolute constant c_{12} such that*

$$|U_n'(0)|^q \leq c_{12}^{1+q} n^{q/2-1/2} (1+qn) \, \|U_n\|^q_{L_q[-n^{1/2}, n^{1/2}]}$$

holds for all U_n of the form $U_n = \widetilde{P}_n Q_n$ with $\widetilde{P}_n \in \widetilde{G}_n$ and an even $Q_n \in \mathcal{P}_n$, and for all $q \in (0, \infty)$.

Proof of Theorem 2.2. Let

$$I_n := [-n^{1/2}, n^{1/2}] .$$

Recalling the notation of Lemma 3.3, we define

$$Q_n := V_{n,\delta,q} \qquad \text{with} \qquad \delta := 3n^{1/2} .$$

Using $Q_n'(0) = 0$ and Lemma 3.5, we obtain

$$(3.2) \qquad |\widetilde{P}_n'(0)|^q = |(\widetilde{P}_n' Q_n)(0)|^q = |(\widetilde{P}_n Q_n)'(0)|^q \leq c_{12}^{1+q} n^{q/2-1/2} (1+qn) \, \|\widetilde{P}_n Q_n\|^q_{L_q(I_n)}$$

for every $\widetilde{P}_n \in \widetilde{G}_n$. Now let $P_n \in \widetilde{G}_n$. Applying (3.2) with $\widetilde{P}_n \in \widetilde{G}_n$ defined by $\widetilde{P}_n(t) := P_n(t+y)$, we obtain

$$\begin{aligned} |P_n'(y)|^q &\leq c_{12}^{1+q} n^{q/2-1/2} (1+qn) \int_{I_n} |P_n(t+y) Q_n(t)|^q \, dt \\ &\leq c_{12}^{1+q} n^{q/2-1/2} (1+qn) \int_{I_{4n}} |P_n(u) Q_n(u-y)|^q \, du \end{aligned}$$

for all $P_n \in \widetilde{G}_n$ and $y \in I_n$. Integrating on I_n with respect to y, then using Fubini's Theorem and Lemma 3.3, we conclude

(3.3)
$$\begin{aligned}
&\|P_n'\|^q_{L_q(I_n)} \\
&= \int_{I_n} |P_n'(y)|^q \, dy \le c_{12}^{1+q} n^{q/2-1/2}(1+qn) \int_{I_n} \int_{I_{4n}} |P_n(u)Q_n(u-y)|^q \, du \, dy \\
&\le c_{12}^{1+q} n^{q/2-1/2}(1+qn) \int_{I_{4n}} |P_n(u)|^q \int_{I_n} |Q_n(u-y)|^q \, dy \, du \\
&\le c_{12}^{1+q} n^{q/2-1/2}(1+qn) c_{10}^{-1-q} \frac{3n^{1/2}}{1+qn} \int_{I_{4n}} |P_n(u)|^q \, du \\
&\le c_{13}^{1+q} n^{q/2} \|P_n\|^q_{L_q(I_{4n})}
\end{aligned}$$

for all $P \in \widetilde{G}_n$, where c_{13} is an absolute constant. Now we divide the real number line into subintervals of length $2n^{1/2}$ and apply the shifted versions of (3.3) on each subinterval to finish the proof of the theorem. □

Proof of Theorem 2.3. This follows from Theorem 2.2 by induction on m. Note that if $U_n \in \widetilde{G}_n$, then $U_n^{(m-1)} \in \widetilde{G}_{nm}$. □

To prove Theorem 2.4 we need the following inequality that follows from part g of E.4 on pages 120–121 in [2].

Lemma 3.6. *We have*

$$|f(t)| \le \exp(\gamma(|t|+\delta)) \left(\frac{2|t|}{\delta}\right)^n \|f\|_{[-\delta,\delta]}\,, \qquad t \in \mathbb{R} \setminus [-\delta,\delta]\,,$$

for all $f \in \widetilde{E}_n$ of the form

$$f(t) = a_0 + \sum_{j=1}^N P_{m_j}(t) e^{\lambda_j t}\,, \quad a_0 \in \mathbb{R}\,, \ \ \lambda_j \in [-\gamma,\gamma]\,, \ \ P_{m_j} \in \mathcal{P}_{m_j}\,, \ \ \sum_{j=1}^N (m_j+1) \le n\,,$$

and for all $\gamma > 0$.

Corollary 3.7. *We have*

$$|f(t)\exp(-t^2)| \le \exp(-t^2/4)\|f(x)\exp(-x^2)\|_{[-2n^{1/2},2n^{1/2}]}\,,$$

$$t \in \mathbb{R} \setminus [-6n^{1/2}, 6n^{1/2}]\,,$$

for every $f \in \widetilde{E}_n$ of the form

$$f(t) = a_0 + \sum_{j=1}^N P_{m_j}(t) e^{\lambda_j t}\,,$$

$$a_0 \in \mathbb{R}\,, \ \ \lambda_j \in [-n^{1/2}, n^{1/2}]\,, \ \ P_{m_j} \in \mathcal{P}_{m_j}\,, \ \ \sum_{j=1}^N (m_j+1) \le n\,.$$

Proof of Corollary 3.7. Assume that f in the corollary satisfies

$$\|f(t)\exp(-t^2)\|_{[-2n^{1/2},2n^{1/2}]} \le 1\,. \tag{3.4}$$

Elementary calculus shows that

$$\|t^n \exp\left(-t^2/2\right)\|_{\mathbb{R}} \le \left(\frac{n}{e}\right)^{n/2}\,. \tag{3.5}$$

Applying Lemma 3.6 with $\delta := 2n^{1/2}$ and $\gamma := n^{1/2}$, then using (3.4) and (3.5), we obtain

$$\begin{aligned}
|f(t)\exp\left(-t^2\right)| &= |f(t)|\exp\left(-t^2\right) \\
&\le \exp(n^{1/2}(|t| + 2n^{1/2}))\left(\frac{2|t|}{2n^{1/2}}\right)^n \exp\left(-t^2/2\right)\exp\left(-t^2/2\right) \\
&\le \exp\left(n^{1/2}(|t| + 2n^{1/2}) - n/2 - t^2/4\right)\exp\left(-t^2/4\right) \\
&\le \exp\left(-t^2/4\right), \qquad t \in \mathbb{R} \setminus [-6n^{1/2}, 6n^{1/2}]\,,
\end{aligned}$$

and the corollary is proved. □

The following result is stated as Theorem 2.2 in [4], and plays a role in the proof of Lemma 3.9.

Lemma 3.8. *There is an absolute constant c_{14} such that*

$$\|f\|_{[a+\delta,b-\delta]} \le \left(\frac{c_{14}(1+qn)}{\delta}\right)^{1/q} \|f\|_{L_q[a,b]}$$

holds for all $f \in \widetilde{E}_n$, $q \in (0,\infty)$, and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$.

Lemma 3.9. *Let $q \in (0,\infty]$. There is a constant c_{15} independent of n such that*

$$\|f(t)\exp\left(-t^2\right)\|_{L_q(\mathbb{R})} \le c_{15}\|f(t)\exp\left(-t^2\right)\|_{L_q[-6n^{1/2},6n^{1/2}]}$$

for every $f \in \widetilde{E}_n$ of the form considered in Corollary 3.7. In conclusion

$$\|f\|_{L_q(\mathbb{R})} \le c_{15}\|f\|_{L_q[-6n^{1/2},6n^{1/2}]}$$

for every $f \in \widetilde{G}_n^*$.

Proof. This follows from Corollary 3.7 and Lemma 3.8. □

The Nikolskii-type inequality below is also needed in the proof of Theorem 2.4. A proof of it can be given by a routine combination of the second Bernstein-type inequality of Theorem 2.1 and the Mean Value Theorem.

Lemma 3.10. *There is an absolute constant c_{16} such that*

$$\|P_n\|_{\mathbb{R}} \le c_{16}^{1+1/q} n^{1/(2q)} \|P_n\|_{L_q(\mathbb{R})}$$

for all $P_n \in \widetilde{G}_n$ and for all $q > 0$.

Proof of Theorem 2.4. Suppose

$$\inf_{f_n \in \widetilde{G}_n} \|f_n - f\|_{L_q(\mathbb{R})} \le c_7 n^{-m/2} (\log n)^{-1-\varepsilon}\,, \qquad n = 2, 3, \dots\,, \tag{3.6}$$

with a constant c_7 independent of n. Choose a sequence (f_n) with $f_n \in \widetilde{G}_n$ such that

$$\|f_n - f\|_{L_q(\mathbb{R})} \le c_7 n^{-m/2} (\log n)^{-1-\varepsilon}\,, \qquad n = 2, 3, \dots\,. \tag{3.7}$$

Let

$$h_n := f_{2^n} - f_{2^{n-1}}\,, \qquad n = 2, 3, \dots\,.$$

Then $h_n \in \widetilde{G}_{2^{n+1}}$. Since $q \in [1, \infty]$, (3.6) implies

$$\|h_n\|_{L_q(\mathbb{R})} \le 2c_7 2^{-(n+1)m/2} (n+1)^{-1-\varepsilon}\,. \tag{3.8}$$

Combining this with Theorem 2.3, we obtain

$$\|h_n^{(m)}\|_{L_q(\mathbb{R})} \le c_{17} 2^{(n+1)m/2} 2^{-(n+1)m/2} (n+1)^{-1-\varepsilon} = c_{17} (n+1)^{-1-\varepsilon} \tag{3.9}$$

with a constant c_{17} independent of n. Also, combining Theorem 2.3 and Lemma 3.10 (recall that $q \in [1, \infty]$), and then using (3.7), we deduce that

$$\begin{aligned} \|h_n^{(j)}\|_{\mathbb{R}} &\le c_{18} 2^{(n+1)j/2} \|h_n\|_{\mathbb{R}} \\ &\le c_{19} 2^{(n+1)j/2} 2^{(n+1)/2} \|h_n\|_{L_q(\mathbb{R})} \\ &\le c_{20} 2^{(n+1)j/2} 2^{(n+1)/2} 2^{-(n+1)m/2} (n+1)^{-1-\varepsilon} \\ &\le c_{21} (n+1)^{-1-\varepsilon}\,, \qquad j = 0, 1, \dots, m-1\,, \end{aligned} \tag{3.10}$$

with some constants c_{18}, c_{19}, c_{20}, and c_{21} independent of n. Since

$$\sum_{n=1}^{\infty} (n+1)^{-1-\varepsilon} < \infty\,,$$

it follows easily from (3.10) that the sequence $(f_{2^n}^{(j)})$ is uniformly Cauchy on $\mathbb{R}$ for each $j = 0, 1, \dots, m-1$, while (3.9) implies that the sequence $(f_{2^n}^{(m)})$ is Cauchy in $L_q(\mathbb{R})$. Hence there are functions $F_j \in C(\mathbb{R})$, $j = 1, 2, \dots, m-1$, and $F_m \in L_q(\mathbb{R})$ such that

$$\lim_{n \to \infty} \|f_{2^n}^{(j)} - F_j\|_{\mathbb{R}} = 0\,, \qquad j = 0, 1, \dots, m-1\,,$$

and

$$\lim_{n\to\infty} \|f_{2^n}^{(m)} - F_m\|_{L_q(\mathbb{R})} = 0\,.$$

Therefore, if $q \in [1,\infty]$, the sequence $(u_{j,n})_{n=1}^\infty$ with

$$u_{j,n}(t) := \int_0^t f_{2^n}^{(j)}(\tau)\,d\tau = f_{2^n}^{(j-1)}(t) - f_{2^n}^{(j-1)}(0)$$

converges uniformly to

$$U_j(t) := \int_0^t F_j(\tau)\,d\tau = F_{j-1}(t) - F_{j-1}(0)$$

on every finite closed subinterval of $\mathbb{R}$ for every $j = 1,2,\dots,m$. Therefore $F'_{j-1} = F_j$ everywhere on $\mathbb{R}$ for all $j = 1,2,\dots,m-1$, while $F'_{m-1} = F_m$ almost everywhere on $\mathbb{R}$. Since $F_0 = f$, we have $F_j = f^{(j)} \in C(\mathbb{R})$ for every $j = 0,1,\dots,m-1$, and $F_m = f^{(m)} \in L_q(\mathbb{R})$. This finishes the proof of the first statement of the theorem.

The proof of the second statement of the theorem is quite similar. We use the notation introduced in the proof of the first statement of the theorem, but $\widetilde{G}_n$ is replaced by $\widetilde{G}_n^*$. Theorem 2.3 and (3.8) imply that

$$\begin{aligned}(3.11)\qquad \|h_n^{(j)}\|_{L_q(\mathbb{R})} &\le c_{22} 2^{(n+1)j/2} \|h_n\|_{L_q(\mathbb{R})}\\ &\le c_{23} 2^{(n+1)j/2} 2^{-(n+1)m/2} (n+1)^{-1-\varepsilon}\\ &\le c_{23} 2^{-(n+1)(m-j)/2} (n+1)^{-1-\varepsilon}\,, \qquad j = 0,1,\dots,m\,,\end{aligned}$$

with a constants c_{22} and c_{23} independent of n. For $j = 0,1,\dots,m$ and $k = 0,1,\dots,m-j$, we define

$$u_{j,k,n}(t) := t^k f_{2^n}^{(j)}(t) \qquad \text{and} \qquad v_{j,k,n}(t) := u_{j,k,n}(t) - u_{j,k,n-1}(t) = t^k h_n^{(j)}(t)\,.$$

Note that $v_{j,k,n} \in \widetilde{G}^*_{(m+1)2^{n+1}}$. Applying Lemma 3.9 to $v_{j,k,n}$, then recalling (3.11), we obtain

$$\begin{aligned}\|v_{j,k,n}\|_{L_q(\mathbb{R})} &\le c_{15} \|v_{j,k,n}\|_{L_q[-6(m+1)^{1/2}2^{(n+1)/2},6(m+1)^{1/2}2^{(n+1)/2}]}\\ &\le c_{24} 2^{(n+1)(m-j)/2} \|h_n^{(j)}\|_{L_q[-6(m+1)^{1/2}2^{(n+1)/2},6(m+1)^{1/2}2^{(n+1)/2}]}\\ &\le c_{25} 2^{(n+1)(m-j)/2} 2^{-(n+1)(m-j)/2} (n+1)^{-1-\varepsilon}\\ &\le c_{25} (n+1)^{-1-\varepsilon}\end{aligned}$$

with constants c_{24} and c_{25} independent of n. Since

$$\sum_{n=1}^\infty (n+1)^{-1-\varepsilon} < \infty\,,$$

we can deduce that the sequence $(u_{j,k,n})_{n=1}^\infty$ is Cauchy in $L_q(\mathbb{R})$ for each $j = 0,1,\dots,m$ and $k = 0,1,\dots,m-j$. Hence there are functions $F^*_{j,k} \in L_q(\mathbb{R})$, $j = 0,1,\dots,m$, $k = 0,1,\dots,m-j$, such that

$$\lim_{n\to\infty} \|t^k f_{2^n}^{(j)}(t) - F^*_{j,k}(t)\|_{L_q(\mathbb{R})} = 0\,.$$

As a consequence, there is a subsequence $(f_{2^{n_l}})_{l=1}^{\infty}$ such that

$$\lim_{l\to\infty} t^k f_{2^{n_l}}(t) = F_{j,k}^*(t)$$

almost everywhere on $\mathbb{R}$ for every $j = 0, 1, \dots, m$ and $k = 0, 1, \dots, m - j$. Hence

$$F_{j,k}^*(t) = t^k F_{j,0}^*(t) = t^k f^{(j)}(t)$$

for every $j = 0, 1, \dots, m$ and $k = 0, 1, \dots, m - j$. So

$$\|t^k f^{(j)}(t)\|_{L_q(\mathbb{R})} < \infty$$

for every $j = 0, 1, \dots, m$ and $k = 0, 1, \dots, m - j$. This proves the second part of the theorem. $\square$

Proof of Theorem 2.5. The proof of the theorem is a straightforward modification of that of Theorem 2.3 and is left to the reader. $\square$

4. Additional Details for the Proofs of Lemmas 3.1, 3.3, and 3.4

In this section we present the details of the proofs of Lemmas 3.1, 3.3, and 3.4.

Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of real numbers. The collection of all linear combinations of of $e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ over $\mathbb{R}$ will be denoted by

$$E(\Lambda_n) := \text{span}\{e^{\lambda_0 t}, e^{\lambda_1 t}, \dots, e^{\lambda_n t}\}\,.$$

Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of positive real numbers. The collection of all linear combinations of of

$$\{\sinh(\lambda_0 t), \sinh(\lambda_1 t), \dots, \sinh(\lambda_n t)\}$$

over $\mathbb{R}$ will be denoted by

$$H(\Lambda_n) := \text{span}\{\sinh(\lambda_0 t), \sinh(\lambda_1 t), \dots, \sinh(\lambda_n t)\}\,.$$

The lemma below can be proved by a simple compactness argument.

Lemma 4.1. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of positive real numbers. Let $a, b \in \mathbb{R}$, $0 < a < b$. Let w be a not identically 0 continuous function defined on $[a, b]$. Let $q \in (0, \infty]$. Then there exists a $0 \neq S \in H(\Lambda_n)$ such that*

$$\frac{|S'(0)|}{\|Sw\|_{L_q[a,b]}} = \sup\left\{\frac{|P'(0)|}{\|Pw\|_{L_q[a,b]}} : \;\; P \in H(\Lambda_n)\right\}\,.$$

Our next lemma is an essential tool in proving our key lemma, Lemmas 4.3.

Lemma 4.2. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ be a set of positive real numbers. Let $a, b \in \mathbb{R}$, $0 < a < b$. Let $q \in (0, \infty]$. Let S be the same as in Lemma 4.1. Then S has exactly n zeros in (a, b) by counting multiplicities.*

To prove Lemma 4.2 below we need the following two facts.

(a) Every $f \in H(\Lambda_n)$ has at most n positive real zeros by counting multiplicities.

(b) If $t_1 < t_2 < \cdots < t_m$ are positive real numbers and $k_1, k_2, \ldots, k_m$ are positive integers such that $\sum_{j=1}^m k_j = n$, then there is a $0 \neq f \in H(\Lambda_n)$ having a zero at t_j with multiplicity k_j for each $j = 1, 2, \ldots, m$.

Proof of Lemma 4.2. To avoid some extra technical details we prove only that S has exactly n zeros in $[a, b]$ (rather than $((a, b))$ by counting multiplicities. Suppose to the contrary that $t_1 < t_2 < \cdots < t_m$ are real numbers lying in $[a, b]$ such that t_j is a zero of S with multiplicity k_j for each $j = 1, 2, \ldots, m$, $k := \sum_{j=1}^m k_j < n$, and S has no other zeros in $[a, b]$ different from $t_1, t_2, \ldots, t_m$. Let $t_0 := 0$ and $k_0 := 2(n-k)+1$. Choose an $0 \neq R \in H(\Lambda_n)$ such that R has a zero at t_j with multiplicity k_j for each $j = 0, 1, \ldots, m$ (it is easy to see that such an $0 \neq R \in H(\Lambda_n)$ exists). Then $R'(0) = 0$ and R has no positive zeros different from $t_1, t_2, \ldots, t_m$. We normalize R so that $R(t)$ and $S(t)$ have the same sign for every $t \in [a, b]$. Let $S_\varepsilon := S - \varepsilon R$. Note that $R \in H(\Lambda_n)$ still has a zero at each t_j with multiplicity k_j for each $j = 1, 2, \ldots, m$, hence S and R are of the form

$$S(t) = \widetilde{S}(t) \prod_{j=1}^m (t - t_j)^{k_j} \qquad \text{and} \qquad R(t) = \widetilde{R}(t) \prod_{j=1}^m (t - t_j)^{k_j}\,,$$

where both $\widetilde{S}$ and $\widetilde{R}$ are continuous functions on $[a, b]$ having no zeros on $[a, b]$. Hence, if $\varepsilon > 0$ is sufficiently small, then $|S_\varepsilon(t)| < |T(t)|$ at every $t \in [a, b] \setminus \{t_1, t_2, \ldots, t_m\}$, so

$$\|S_\varepsilon w\|_{L_q[a,b]} < \|Sw\|_{L_q[a,b]}\,.$$

This, together with $S'_\varepsilon(0) = S'(0)$, contradicts the maximality of S. $\square$

The fact that for any $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n$,

$$(\sinh(\lambda_0 t), \sinh(\lambda_1 t), \ldots, \sinh(\lambda_n t))$$

is a Descartes system on $(0, \infty)$ is stated as Lemma 4.5 and proved in [3]. The heart of the proof of our theorems is the following comparison lemma. The proof of the next couple of lemmas is based on basic properties of Descartes systems, in particular on Descartes' Rule of Sign, and on a technique used earlier by P.W. Smith and A. Pinkus. G.G. Lorentz ascribes this result to Pinkus, although it was P.W. Smith [19] who published it. I have learned about the the method of proofs of these lemmas from P. Borwein (oral communication), who also ascribes it to Pinkus. This is the proof we present here.

Lemma 4.3. *Let $\Lambda_n := \{\lambda_0 < \lambda_1 < \cdots < \lambda_n\}$ and $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ be sets of positive real numbers satisfying $\lambda_j \leq \gamma_j$ for each $j = 0, 1, \ldots, n$. Let $a, b \in \mathbb{R}$, $0 \leq a < b$. Let w be a not identically 0 continuous function defined on $[a, b]$. Let $q \in (0, \infty]$. Then*

$$\sup\left\{\frac{|P'(0)|}{\|Pw\|_{L_q[a,b]}} : \ P \in H(\Gamma_n)\right\} \leq \sup\left\{\frac{|P'(0)|}{\|Pw\|_{L_q[a,b]}} : \ P \in H(\Lambda_n)\right\} .$$

Proof of Lemma 4.3. We may assume that $0 < a < b$. The general case when $0 \leq a < b$ follows by a standard continuity argument. Let $k \in \{0, 1, \ldots, n\}$ be fixed and let

$$\gamma_0 < \gamma_1 < \cdots < \gamma_n , \qquad \gamma_j = \lambda_j , \quad j \neq k , \qquad \text{and} \qquad \lambda_k < \gamma_k < \lambda_{k+1}$$

(let $\lambda_{n+1} := \infty$). To prove the lemma it is sufficient to study the above cases since the general case follows from this by a finite number of pairwise comparisons. By Lemmas 4.1 and 4.2, there is an $S \in H(\Gamma_n)$ such that

$$\frac{|S'(0)|}{\|Sw\|_{L_q[a,b]}} = \sup\left\{\frac{|P'(0)|}{\|Pw\|_{L_q[a,b]}} : \ P \in H(\Gamma_n)\right\} ,$$

where S has exactly n zeros in (a, b) by counting multiplicities. Denote the distinct zeros of S in (a, b) by $t_1 < t_2 < \cdots < t_m$, where t_j is a zero of S with multiplicity k_j for each $j = 1, 2, \ldots, m$, and $\sum_{j=1}^{m} k_j = n$. Then S has no other zeros in $(0, \infty)$ different from $t_1, t_2, \ldots, t_m$. Let

$$S(t) := \sum_{j=0}^{n} a_j \sinh(\gamma_j t) , \qquad a_j \in \mathbb{R} .$$

Without loss of generality we may assume that $S(b) > 0$. Note that $S(b) \neq 0$ since $S \in H(\Gamma_n)$, and S has exactly n zeros in (a, b) (by counting multiplicities). Because of the extremal property of S, $S'(0) \neq 0$. Since $H(\Gamma_n)$ is the span of a Descartes system on $[a, b]$, it follows from Descartes' Rule of Signs that

$$(-1)^{n-j} a_j > 0 , \qquad j = 0, 1, \ldots, n .$$

Choose $R \in H(\Lambda_n)$ of the form

$$R(t) = \sum_{j=0}^{n} b_j \sinh(\lambda_j t), \qquad b_j \in \mathbb{R} ,$$

so that R has a zero at each t_j with multiplicity k_j for each $j = 1, 2, \ldots, m$, and we normalize so that $R(a) = S(a)$ (this $R \in H(\Lambda_n)$ is uniquely determined). Since $H(\Lambda_n)$ is the span of a Descartes system on $[a, b]$, Descartes' Rule of Signs implies

$$(-1)^{n-j} b_j > 0 , \qquad j = 0, 1, \ldots, n .$$

We have

$$(S - R)(t) = a_k \sinh(\gamma_k t) - b_k \sinh(\lambda_k t) + \sum_{\substack{j=0 \\ j \neq k}}^{n} (a_j - b_j) \sinh(\lambda_j t) .$$

Since $S-R$ has altogether at least $n+1$ zeros at $t_1, t_2, \dots, t_m$, and a (by counting multiplicities), it does not have any zero on $\mathbb{R}$ different from $t_1, t_2, \dots, t_m$, and a. Since

$$(\sinh(\lambda_0 t), \sinh(\lambda_1 t), \dots, \sinh(\lambda_k t), \sinh(\gamma_k t), \sinh(\lambda_{k+1} t), \dots, \sinh(\lambda_n t))$$

is a Descartes system on $(-\infty, \infty)$, Descartes' Rule of Signs implies

$$(a_0 - b_0,\ \ a_1 - b_1,\ \ \dots,\ \ a_{k-1} - b_{k-1},\ \ -b_k,\ \ a_k,\ \ a_{k+1} - b_{k+1},\ \ \dots,\ \ a_n - b_n)$$

strictly alternates in sign. Since $(-1)^{n-k} a_k > 0$, this implies that $a_n - b_n > 0$, so

$$(S-R)(t) > 0\,, \qquad t > t_m\,.$$

Since each of S, R, and $S-R$ has a zero at t_j with multiplicity k_j for each $j = 1, 2, \dots, m$; $\sum_{j=1}^m k_j = n$, and $S-R$ has a sign change (a zero with multiplicity 1) at a, we can deduce that each of S, R, and $S-R$ has the same sign on each of the intervals (t_j, t_{j+1}) for every $j = 0, 1, \dots, m$ with $t_0 := a$ and $t_{m+1} := \infty$. Hence $|R(t)| \leq |S(t)|$ holds for all $t \in [a,b]$ with strict inequality at every t different from $t_1, t_2, \dots, t_m$, while $|R(t)| \geq |S(t)|$ at every $t \in [0,a]$. These, together with $R(0) = S(0) = 0$ imply that $\|Rw\|_{L_q[a,b]} \leq \|Sw\|_{L_q[a,b]}$ and $|R'(0)| \geq |S'(0)|$. Therefore

$$\frac{|R'(0)|}{\|Rw\|_{L_q[a,b]}} \geq \frac{|S'(0)|}{\|Sw\|_{L_q[a,b]}} = \sup\left\{ \frac{|P'(0)|}{\|Pw\|_{L_q[a,b]}} :\ \ P \in H(\Gamma_n) \right\}.$$

Since $R \in H(\Lambda_n)$, the lemma follows from this. $\square$

Let, as before, $\mathcal{P}_n$ denote the collection of all algebraic polynomials of degree at most n with real coefficients. Our next lemma may be viewed as an exercise and we do not present its complete proof here. It follows from Lemmas 3.8, 3.6, and Theorem 2.3 on page 173 of [1].

Lemma 4.4. *Let $a, b \in \mathbb{R}$, $a < b$. Let w be a not identically 0 continuous function defined on $[a,b]$. Let $q \in (0, \infty]$. For an $\varepsilon > 0$ let*

$$H_n(\varepsilon) := \operatorname{span}\{\sinh(\varepsilon t), \sinh(2\varepsilon t), \dots, \sinh(n\varepsilon t)\}\,.$$

Suppose (P_k) is a sequence with $P_k \in H_n(\varepsilon_k)$ and $\|P_k w\|_{L_q[a,b]} \leq 1$, where (ε_k) is a sequence of positive real numbers converging to 0. Then (P_k) has a subsequence (P_{k_j}) that converges to an odd polynomial $\widetilde{P} \in \mathcal{P}_{2n}$ uniformly on $[a,b]$, while $(P'_{k_j}(0))$ converges to $\widetilde{P}'(0)$.

Combining Lemmas 4.3 and 4.4, we easily obtain

Lemma 4.5. *Let $\Gamma_n := \{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ be a set of positive real numbers. Let w be a not identically 0 continuous even function defined on $[-1,1]$. Let $q \in (0,\infty]$. Then*

$$\sup\left\{\frac{|P'(0)|}{\|Pw\|_{L_q[-1,1]}} : \quad P \in H(\Gamma_n)\right\} \le \sup\left\{\frac{|S'(0)|}{\|Sw\|_{L_q[-1,1]}} : \quad S \in \mathcal{P}_{2n}\right\}.$$

The following result follows easily from Bernstein's inequality and the Nikolskii-type inequality of Theorem A.4.3 on page 394 of [2].

Lemma 4.6. *We have*

$$|P'(0)| \le n\|P\|_{[-1,1]}$$

and

$$|P'(0)| \le 2n\,\|P\|_{[-1/2,1/2]} \le 2n\left(\frac{2e}{\sqrt{3}\pi}(1+qn)\right)^{1/q}\|P\|_{L_q[-1,1]}$$

for every $P \in \mathcal{P}_n$.

Lemma 4.7. *We have*

$$|U'(0)| \le \frac{2n+m}{\delta}\|U\|_{[-\delta,\delta]}$$

and

$$|U'(0)| \le \max\{2^{1/q-1},1\}\,\frac{2(2n+m)}{\delta}\,\left(\frac{2e}{\sqrt{3}\pi\delta}\,(1+q(2n+m))\right)^{1/q}\|U\|_{L_q[-\delta,\delta]}$$

holds for all U of the form $U = PR$ with $P \in E(\Lambda_n)$ and an even $R \in \mathcal{P}_m$, and for all $q \in (0,\infty]$.

Proof of Lemma 4.7. Without loss of generality we may assume that $\delta := 1$, the general case follows simply by a linear scaling. We may also assume that $\lambda_j + \lambda_k \ne 0$ for every $0 \le j \le k \le n$, the general case follows by a simple continuity argument. Choose the set of positive numbers $\{\gamma_0 < \gamma_1 < \cdots < \gamma_n\}$ so that

$$\{\gamma_0, \gamma_1, \ldots, \gamma_n\} = \{|\lambda_0|, |\lambda_1|, \ldots, |\lambda_n|\}.$$

Let U be of the form $U = PR$ with $P \in E(\Lambda_n)$ and an even $R \in \mathcal{P}_m$. Let $f(t) := U(t) + U(-t)$, $Q(t) := P(t) + P(-t)$, and $w := R$. Then $f(t) = Q(t)w(t)$ with $Q \in H(\Gamma_n)$ and an even $w \in \mathcal{P}_m$. We have

$$\begin{aligned}
\frac{|U'(0)|}{\|U\|_{L_q[-1,1]}} &\le \max\{2^{1/q-1},1\}\,\frac{|f'(0)|}{\|f\|_{L_q[-1,1]}} = \max\{2^{1/q-1},1\}\,\frac{|Q'(0)w(0)|}{\|Qw\|_{L_q[-1,1]}} \\
&\le \max\{2^{1/q-1},1\}\sup\left\{\frac{|S'(0)w(0)|}{\|Sw\|_{L_q[-1,1]}} : \quad S \in \mathcal{P}_{2n}\right\} \\
&\le \max\{2^{1/q-1},1\}\sup\left\{\frac{|V'(0)|}{\|V\|_{L_q[-1,1]}} : \quad V \in \mathcal{P}_{2n+m}\right\} \\
&\le \max\{2^{1/q-1},1\}2(2n+m)\,\left(\frac{2e}{\sqrt{3}\pi}\,(1+q(2n+m))\right)^{1/q}
\end{aligned}$$

and

$$\frac{|U'(0)|}{\|U\|_{[-1,1]}} \leq \sup\left\{\frac{|V'(0)|}{\|V\|_{[-1,1]}} : \; V \in \mathcal{P}_{2n+m}\right\} \leq 2n+m\,,$$

and the lemma is proved. □

Proof of Lemmas 3.1 and 3.4. Lemma 4.7 yields Lemma 3.1 and Lemma 3.4 with $c_{11} := 2$. □

In the proof of Theorem 2.2 we have used Lemma 3.3. The Lemma below shows that Lemma 3.3 holds with $c_{10} = (2\pi)^{-1}$. absolute constant as well.

Lemma 4.8. *For every $n \in \mathbb{N}$, $\delta \in (0,\infty)$, and $q \in (0,\infty)$, there are even polynomials $V_{n,\delta,q} \in \mathcal{P}_n$ such that*

$$1 = |V_{n,\delta,q}(0)| \geq \left(\frac{1+qn}{2\pi(2+q)\delta}\right)^{1/q} \|V_{n,\delta,q}\|_{L_q[-\delta,\delta]}\,.$$

Proof of Lemma 4.8. Without loss of generality we may assume that $\delta := 1$, the lemma in the general case follows from this by a linear scaling. Let $T_n \in \mathcal{P}_n$ be the nth Chebyshev polynomial defined by

$$T_n(\cos t) := \cos(nt)\,, \qquad t \in \mathbb{R}\,.$$

The polynomials $2^{-1/2}T_0, T_1, T_2, \ldots$ form an orthonormal system on $[-1,1]$ with respect ‘to the weight $(2/\pi)(1-x^2)^{-1/2}$. Let $k := \lceil 2/q \rceil$ be the smallest integer not less than $2/q$. Let $m := \lfloor n/(2k) \rfloor$ be the greatest integer not greater than $n/(2k)$. For $qn \geq 1$ we define

$$V_{n,1} := \left(\sum_{j=0}^{m} (-1)^j T_{2j}(0)\right)^{-k} \left(\sum_{j=0}^{m} (-1)^j T_{2j}\right)^{k}.$$

For $qn < 1$ we define $V_{n,1} := 1$. Then $V_{n,1} \in \mathcal{P}_n$ and $V_{n,1}(0) = 1$. When $qn \geq 1$ we have

$$\begin{aligned}
\int_{-1}^{1} |V_{n,1}(t)|^q \, dt &= (m+1)^{-kq} \int_{-1}^{1} \left|\sum_{j=0}^{m} (-1)^j T_{2j}(t)\right|^{kq} dt \\
&\leq (m+1)^{-kq} \max_{t\in[-1,1]} \left|\sum_{j=0}^{m} (-1)^j T_{2j}(t)\right|^{kq-2} \int_{-1}^{1} \left(\sum_{j=0}^{m} (-1)^j T_{2j}(t)\right)^2 dt \\
&\leq (m+1)^{-kq}(m+1)^{kq-2}\frac{\pi}{2}(m+1/2) \leq \frac{\pi}{2}(m+1)^{-1} \leq \frac{\pi(2+q)}{qn} \\
&\leq \frac{2\pi(2+q)}{qn+1}\,,
\end{aligned}$$

while when $qn < 1$ we have

$$\int_{-1}^{1} |V_{n,1}(t)|^q \, dt \leq 2 \leq \frac{2\pi(2+q)}{qn+1}\,,$$

and the proof is finished. □

References

1. D. Braess, *Nonlinear Approximation Theory*, Springer-Verlag, Berlin, 1986.
2. P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer-Verlag, Graduate Texts in Mathematics, Volume 161, 486 p., New York, NY, 1995.
3. P. Borwein and T. Erdélyi, *A sharp Bernstein-type inequality for exponential sums*, J. Reine Angew. Math. **476** (1996), 127–141.
4. P. Borwein and T. Erdélyi, *Pointwise Remez- and Nikolskii-type inequalities for exponential sums*, Math. Annalen **316** (2000), 39–60.
5. R.A. DeVore & G.G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, 1993.
6. Z. Ditzian and D. Lubinsky, *Jackson and smoothness theorems for Freud weights*, Constr. Approx. **13** (1997), 99–152.
7. G.G. Lorentz, *Approximation of Functions, 2nd ed.*, Chelsea, New York, N.Y., 1986.
8. H.N. Mhaskar, *Introduction to the Theory of Weighted Polynomial Approximation*, World Scientific, Singapore, 1996.
9. H.N. Mhaskar, *When is approximation by Gaussian networks necessarily a linear process?*, manuscript (2003).
10. E. Schmidt, *Zur Kompaktheit der Exponentialsummen*, J. Approx. Theory **3** (1970), 445–459.

Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA

E-mail address: `terdelyi@math.tamu.edu`