The Zero Set of an Electrical Field from a Finite Number of Point Charges: One, Two, and Three Dimensions

Tamás Erdélyi, Joseph Rosenblatt, Rebecca Rosenblatt

Abstract

We study the structure of the zero set of a finite point charge electrical field $F = (X, Y, Z)$ in $\mathbb{R}^3$. Indeed, mostly we focus on a finite point charge electrical field $F = (X, Y)$ in $\mathbb{R}^2$. The well-known conjecture is that the zero set of $F = (X, Y)$ is finite. We show that this is true in a Special Case: when the point charges for $F = (X, Y)$ lie on a line. In addition, we give fairly complete structural information about the zero sets of X and Y for $F = (X, Y)$ in the Special Case. A highlight of the paper states that in the Special Case the zero set of $F = (X, Y)$ contains at most $9M^2 4^M$ points, where M is the number of point charges.

1 Introduction

This article gives the results that we know to hold for the zero sets of finite point charge electrical fields $F = (X, Y, Z)$ in $\mathbb{R}^3$. While some general facts in this setting are clear, in order to get complete proofs we have to focus for now on the case where the charges are in a plane. That is, $F = (X, Y)$ is a finite point charge field in $\mathbb{R}^2$. The well-known, still unresolved, conjecture is that in this case the zero set of F is finite. We give full details proving this conjecture in the Special Case where the charges themselves lie on a line in $\mathbb{R}^2$.

While the Special Case may seem to be an easy case to analyze, it turns out that the details are complicated, especially if one wants to know the structure of not only the zero set of $F = (X, Y)$, but also the zero sets of X and Y. We give different ways of seeing that the zero set of $F = (X, Y)$ is finite in the Special Case when the point charges lie on a line. In addition, we obtain structural information about the zero sets of X and Y at infinity by analyzing power series expansions, using binomial series, geometric series, and also critical moments that are given by the coefficients of the field and the positions of the charges. Indeed, in order to complete the analysis in the case where the point charges lie on a line, we need to consider the structure of the zero set of $F = (X, Y)$ with the point charges in general position in $\mathbb{R}^2$. Details will be discussed for what we know and what yet remains to be shown.

The outline of this article is as follows. In Section 2, we set up the problem and explain how it relates to a larger class of problems of distinguishing and/or reconstructing electrical fields and also gravitational fields. In Section 3, we show that the zero set of $F = (X, Y)$ cannot contain a non-trivial curve when the point charges lie on a line, the Special Case. We show that it follows that the zero set of $F = (X, Y)$ must be finite when the point charges lie on a line. In Section 4, in our Special Case, we derive facts about the asymptotic directions at infinity of the zero sets of X and Y and explain why these directions do not overlap.

2 Finite Gravitational and Electrical Fields

The original problem we considered that gave rise to this work is how to determine a gravitational field from a limited set of measurements. In general, we would want to be able to have a calculus for this when the mass generating this field is in motion. But for now we only consider the case where we have a finite number of masses in fixed positions, and idealize these to being point masses. So this means we have a finite set of points (x_j, y_j, z_j), $j = 1, 2, \ldots, M$, and corresponding masses $m_j, j = 1, 2, \ldots, M$. We take a point (x, y, z) and assume that there is a unit mass at that point. Then the force at (x, y, z) from the mass at (x_j, y_j, z_j) is proportional to the mass m_j and inversely proportionally to the square of the Euclidean distance

$$r_j = \|(x_j, y_j, z_j) - (x, y, z)\|_2 = ((x_j - x)^2 + (y_j - y)^2 + (z_j - z)^2)^{1/2} .$$

There is a direction of this force too of course: from (x, y, z) toward the mass at (x_j, y_j, z_j). We take units so that the gravitational constant G is 1. Hence the force field

$$F(x, y, z) = (X(z, y, z), Y(x, y, z), Z(x, y, z)) = \sum_{j=1}^{M} \frac{m_j}{r_j^2} \frac{(x_j, y_j, z_j) - (x, y, z)}{r_j} .$$

So we have these equations for the components of the force field:

$$X(x, y, z) = \sum_{j=1}^{M} \frac{m_j(x_j - x)}{r_j^3} , \quad Y(x, y, z) = \sum_{j=1}^{M} \frac{m_j(y_j - y)}{r_j^3} , \quad Z(x, y, z) = \sum_{j=1}^{M} \frac{m_j(z_j - z)}{r_j^3} .$$

There are actually two problems to deal with here. The first is to distinguish two gravitational fields by a limited set of measurements, and the second is to determine the full structure of an unknown gravitational field, by taking a limited set of measurements. The first problem is the one we focus on in these notes. The second one is in a class of problems of this type, many of which seem to be very difficult to solve.

The problem of distinguishing gravitational fields can be phrased in terms of *finite (point) electrical fields* (i.e. electrical fields given by a finite number of point charges). Indeed, suppose we have two finite point mass gravitational fields F_1 and F_2 and have a given set $\mathcal{V}$ such that $F_1(x, y, z) = F_2(x, y, z)$ for all $(x, y, z) \in \mathcal{V}$. That is, $F_1(x, y, z) - F_2(x, y, z) = 0$ for all $(x, y, z) \in \mathcal{V}$. But $F_1 - F_2$ has the form of the vector field given by a finite set of point electrical charges in $\mathbb{R}^3$. Here the units have been chosen so that the electrical constant K is 1. What we want to know is this:

Question: Taking an unknown finite electrical field, what method of taking a limited set of measurements will guarantee that if the measurements are all zero, then the field itself is zero everywhere?

To answer this question completely, we would want to known the geometry of the zero set of a field given just as F is above except that the coefficients m_j are allowed to be real numbers a_j.

We say that the electrical field is *empty* when there are actually no point charges. Since a finite electrical field can always be seen as the difference of two gravitational fields, this is the same as saying that the gravitational fields are identical.

The structure of the zero set of such a field in $\mathbb{R}^3$ is certainly different than in $\mathbb{R}^2$. We make some remarks about the three dimensional case in the following, but this article actually focuses on the two dimensional case, indeed for complete results on a Special Case of the two dimensional case.

2.1 Finite Electrical Fields in Three Dimensions

We take our field $F = (X, Y, Z)$ to be a finite point charge electrical field. So now, consistent with positive charges repulsing each other, rather than in the case of gravitational fields where positive masses attract each other, we have

$$X(x,y,z)=\sum_{j=1}^{M}\frac{a_j(x-x_j)}{r_j^3}\,,\quad Y(x,y,z)=\sum_{j=1}^{M}\frac{a_j(y-y_j)}{r_j^3}\,,\quad Z(x,y,z)=\sum_{j=1}^{M}\frac{a_j(z-z_j)}{r_j^3}\,,$$

with

$$r_j=((x-x_j)^{1/2}+(y-y_j)^2+(z-z_j)^2)^{1/2}\,,\qquad j=1,2,\dots,M\,.$$

We know that in $\mathbb{R}^3$, X, Y, and Z can have zero sets consisting of points, lines, curves, and even two dimensional surfaces. How these intersect is not clear. The main question for the three dimensional case was whether or not the common zeros must be only points or curves (perhaps even just lines) of vectors? Also, how many points, and how many curves and lines, will there be at most, say given only M? The basic structure is limited in a more general setting by A. I. Yanushauskas [10]. Yanushauskas used the fact that the force field in this case is the gradient of a harmonic function to show that the zero set consists of a locally finite set of points and analytic curves.

Proposition 2.1. *If a finite electrical field in $\mathbb{R}^3$ vanishes on a set containing a two dimensional surface (or even as much as an open disc), then the field is identically zero everywhere and the point charge set is empty.*

But we still have structural issues to consider in order to understand the geometry of such a zero set.

Conjecture: The zero set of X, Y, or Z is asymptotically planar or linear at infinity, with only a finite number of planes or lines being possible. Moreover, these planes or lines are distinct among the three components and the overlap of these three zero sets can only consist of a finite set of points and asymptotically linear curves.

Of course, we could have the component zero sets being asymptotically a plane or line in one direction and another plane or line in another direction. If this conjecture is true then the common zero set would either be bounded or consist of a finite number of curves each of which is asymptotically linear at infinity.

Remark 2.2. Examples show that the zero set in this case can contain curves (e.g. circles and straight lines). But counting these and understanding what types of curves one can get is a future project. The main question otherwise is whether or not the unbounded portion is asymptotically linear. □

Without any additional work, we can address the Conjecture and the structural issues by using the basic theory of real analytic varieties. See Whitney [6], Whitney [7], and Bruhat and Whitney [8]. But this will only give us local information about the zero set. Instead, we will use a product argument, and the structure of real algebraic varieties, to get more information. See Whitney [6] for an early advanced view, and Gibson [5] for details at a very elementary level.

An algebraic curve in the Euclidean plane is the set of the points whose coordinates are the solutions of a bivariate polynomial equation $P(x, y) = 0$. This equation is often called the implicit equation of the curve, in contrast to the curves that are the graph of a function defining explicitly y as a function of x, or vice versa. With a curve given by such an implicit equation, the first problem would be to determine the shape of the curve and to "draw it". These problems are not as easy to solve as in the case of the graph of a function, for which y may easily be computed for various values of x. The fact that the defining equation is a polynomial implies that the curve has some structural properties that may help in solving these problems. Every algebraic curve may be uniquely decomposed into a finite number of smooth monotone arcs (called *branches*) sometimes connected by some points sometimes called *remarkable points*, and possibly a finite number of isolated points called *acnodes*. A smooth monotone arc is the graph of a smooth function which is defined and monotone on an open interval of the x-axis or the y-axis. In each direction, an arc is either unbounded (usually called an *infinite arc*) or has an endpoint which is either a *singular point* (this will be defined below) or a point with a tangent parallel to one of the coordinate axes. The singular points of a curve of degree d defined by a polynomial $P(x,y)$ of degree d are the solutions of the system of equations

$$\frac{\partial P}{\partial x}(x,y) = \frac{\partial P}{\partial y}(x,y) = P(x,y) = 0\,.$$

First, we need a basic algebraic principle. We take here $n = 2, 3$ for concreteness. We are also thinking of applying this to the components of the vector field. So when $n = 2$, our functions below are $A_j(x,y) = a_j(x - x_j)$ or $A_j(x,y) = a_j(y - y_j)$, and $B_j(x,y) = ((x-x_j)^2 + (y-y_j)^2)^3$. When $n = 3$, our functions A_j include also $A_j(x,y,z) = a_j(z - z_j)$ and $B_j(x,y,z) = ((x - x_j)^2 + (y-y_j)^2 + (z-z_j)^2)^3$. Let $\mathbb{R}[x_1, x_2, \dots, x_n]$ denote the ring of polynomials in the variables $x_1, x_2, \dots, x_n$ with real coefficients.

Proposition 2.3. *Let*

$$R = \sum_{j=1}^{M} \frac{A_j}{B_j^{1/2}}\,, \qquad A_j, B_j \in \mathbb{R}[x_1, x_2, \dots, x_n]\,,$$

with $n = 2$ or $n = 3$, where $B_j \geq 0$ on $\mathbb{R}^n$ for all $j = 1, 2, \dots, M$. Then the zero set

$$\{(x_1, x_2, \dots, x_n) : R(x_1, x_2, \dots, x_n) = 0\}$$

is a subset of a real algebraic variety in $\mathbb{R}^n$. As a consequence, we have the following.

a) If $n = 2$, the zero set $\{(x,y) : R(x,y) = 0\}$ is a subset of a finite number of points and curves.

b) If $n = 3$, the zero set $\{(x,y,z) : R(x,y,z) = 0\}$ is a subset of a finite number of points, curves, and two dimensional surfaces.

Proof. Take R and put the terms over the common denominator $D^{1/2}$ where $D = B_1 B_2 \cdots B_M$. This expresses $R = D^{-1/2} \sum_{j=1}^{M} A_j D_j^{1/2}$ where each D_j is D with B_j divided out. We see then that $\{R = 0\}$ is a subset of where $S = \sum_{j=1}^{M} A_j D_j^{1/2} = 0$. Let Σ be the collection of the 2^M functions $\sigma : \{1, 2, \dots, M\} \to \{-1, 1\}$. Now we take the product

$$P = \prod_{\sigma \in \Sigma} \left(\sum_{j=1}^{M} \sigma(j) A_j D_j^{1/2} \right).$$

Here the product has 2^M factors. We show that P is a polynomial (of degree at most $3M \cdot 2^M$). Let

$$\mathcal{P}(\chi_1, \chi_2, \dots, \chi_M) := \prod_{\sigma \in \Sigma} \left(\sum_{j=1}^{M} \sigma(j) \chi_j \right).$$

Observe that for every fixed $k \in \{1, 2, \dots, M\}$ the product $\mathcal{P}$ is a polynomial in χ_k and its value remains the same if we replace χ_k by $-\chi_k$, as the factors of the product remain the same, they are only permuted. Hence $\mathcal{P}$ is a polynomial in χ_k^2 for every fixed $k \in \{1, 2, \dots, M\}$. Now apply this with $\chi_j = A_j D_j^{1/2}$ to conclude that P is a polynomial (of degree at most $3M \cdot 2^M$), indeed. To finish the proof observe that $\sum_{j=1}^{M} A_j D_j^{1/2}$ is a factor of this product defining P, namely the sum when $\sigma(j) = 1$ for each j. So $\{R = 0\}$ is a subset of the real algebraic variety given by $P = 0$. □

Remark 2.4. a) It is an interesting issue that deriving P is not so straightforward. Most elementary cases of equations with square roots are handled just by isolating the terms on one side of the equality, squaring, rearranging the terms, and continuing this process. But with multiple terms (five or more) that are square roots, this does not work.

b) We believe that in fact in a) and b) above, not only is the zero set contained in the associated structure, but is itself in fact of the same structure. At this time, we do not have a complete proof of this conjecture. □

Remark 2.5. The asymptotic directions of the zero sets of the component functions of the electric field when $n = 2$ is one structural aspect that is examined in detail below. But the case of the surfaces being asymptotically linear in the case that $n = 3$ is not resolved here. Nonetheless, if this is true, then two gravitational fields in $\mathbb{R}^3$ can be distinguished by taking a set of three non-colinear vectors v_1, v_2, and v_3, and then translating them by a vector w which is far from the origin. Now, the intersection of the component zero sets can only consist of a finite set of curve sections of an associated real algebraic variety containing where $F = (0, 0, 0)$. Also, these curves are probably asymptotically linear. If so, this shows that if two finite point gravitational fields F_1 and F_2 have $F_1(v_i + w) = F_2(v_i + w)$ for $i = 1, 2$, and 3, and w of large norm, then the gravitational fields are the same everywhere. Note: there is the additional ambiguity of how far from the origin one must choose w, but taking these values in a sequence going to infinity would resolve that. □

2.2 Finite Electric Fields in the Plane

In $\mathbb{R}^2$, we consider a finite set of points (x_j, y_j) and a finite set of coefficients a_j that are real numbers. We let the Euclidean distance $r_j = \|(x, y) - (x_j, y_j)\|_2$, We take a vector field F given by

$$F(x, y) = (X(z, y), Y(x, y)) = \sum_{j=1}^{M} \frac{a_j}{r_j^2} \frac{(x, y) - (x_j, y_j)}{r_j} .$$

So for the real-valued components of this vector field, we have:

$$X(x, y) = \sum_{j=1}^{M} \frac{a_j(x - x_j)}{r_j^3} \qquad \text{and} \qquad Y(x, y) = \sum_{j=1}^{M} \frac{a_j(y - y_j)}{r_j^3} .$$

Examples suggest that now the zero sets of X and Y consists of points, lines, and curves, both bounded and unbounded. However, the common zero set is more limited.

Conjecture: The zero set of a two-dimensional finite electrical field consists of a finite set of points.

Of course, we would want to know also what the corresponding Maxwell Conjecture would be: how many points can there be given only M? See Gabrielov, Novikov, and Shapiro [2]. See also Proposition 4.1 in [1] and Killian [3].

This conjecture should not be so difficult to solve. However, despite a number of attempts to find simple methods to resolve it, and to estimate the number of points, the conjecture has not been proven to be correct. In any case, what this result would say about distinguishing two finite gravitational fields in an ecliptic is that a finite, possibly large, number of measurements suffice to distinguish these fields. Certainly, measurement in an entire disc would suffice because F could not be zero on a disc without being zero throughout the plane. Even showing that the same thing is true if a disc is replaced by a line segment seems not so easy to prove.

Indeed, the conjecture for finite electrical fields in the plane says that if you take a measurement far from the point charges, and it is zero in both components, then the field is identically zero everywhere and there are actually no point charges at all. In practice, since one does not know how far out one must be, the right thing to do is to take a fixed sequence of points tending to infinity and take the measurements there. The conjecture says that these cannot all be zero without the field being identically zero everywhere.

Remark 2.6. Another version of the measurement problem is not easy to answer. How can we take a limited number of measurements of two gravitational fields and based on these measurements conclude the fields must be a rotation and/or a translation of one another? This question can be rephrased for finite electrical fields in a suitable fashion. □

One can consider there being two separate issues with describing the zero set of a finite electrical field in the plane . One issue is how to show that all the zeros must be in a large disc i.e. there cannot be zeros arbitrarily far out in the plane. This is not the case for $X = 0$ or for $Y = 0$ separately; they typically do have portions arbitrarily far from the origin.

The second issue is how to show that there can only be a finite set of zeros within a fixed distance of the origin. The second issue is not completely resolved at this time unless we had a result like Proposition 4.1 in [1].

Actually, what we seek to prove, with some technical work that is much more difficult to complete than it ought to be, is that at infinity the zero set of X or of Y consists of a finite number of curves that are to some degree at least asymptotically linear. In the process, or as a result, we would want to find equations that give the slopes of these lines and show that they are distinct from one another when we consider X or when we consider Y. This is how we could show that the common zeros of X and Y must lie in a large disc.

The obvious approach to these questions would seem to be just a smart use of implicit differentiation. For example, where $X = 0$, if there is a solution $y = y(x)$ or $x = x(y)$, we should be able to compute dy/dx or dx/dy and work with this. In particular, if we believe that a given curve $y = y(x)$ in the zero set of X must be asymptotically linear at infinity, we would just need to show that y is asymptotically αx, for some α. At this time, we cannot see how to carry this out in general.

So instead, what we do is consider the implicit equation $Y = 0$ and $X = 0$ and expand this as an infinite series. This series has to be adapted to the region in question; one series does not seem to suffice for the whole plane. Then we consider these series asymptotically and get equations that determine the directions of solutions to $X = 0$ at large distances from the origin. These equations only have a finite number of solutions. Then we carry out the same thing for $Y = 0$. We then show that the asymptotic directions are different for the two zero sets. This approach gives us at least part of the results that we want.

Remark 2.7. One aspect of this analysis that is worth noting is that certain *moments* are closely tied to the structure of the zero sets. In the Special Case we call $\sum_{j=1}^{M} a_j x_j^k$ the kth *moment*. If $k = 0$, then we get the ground state moment $\sum_{j=1}^{M} a_j$. If M is large, then many of these could be zero without forcing all the charges a_j to be zero. The critical moment is the one given by L such that $\sum_{j=1}^{M} a_j x_j^k = 0$ for all $0 \leq k < L$ and $\sum_{j=1}^{M} a_j x_j^L \neq 0$. As the number of point charges increases, the structure of the zero sets of X and Y become more complicated, and as L increases, the structure of the zero sets of X and Y at infinity become more complicated. We have found no way to see that these moments play an important role in the structure of the zero sets without doing asymptotic analysis using power series expansions. $\square$

2.3 A Special Case in the Plane

The *Special Case* will mean that we take points in $\mathbb{R}^2$ that are all on a line and consider the zeros of $F = (X, Y)$. The *General Case* will mean that we take any points in $\mathbb{R}^2$ and consider the zeros of $F = (X, Y)$. Oddly enough, the Special Case is not so easy to analyze, even though it seems quite restrictive.

By rotating and translating, or by a appropriate choice of the coordinate system, the Special Case becomes the following. Take distinct points $P_j = (x_j, 0), j = 1, 2, \dots, M$, on the positive x-axis. Assume that $0 < x_1 < x_2 < \cdots < x_M$. Put non-zero electric charges a_j at these points. Then the field in the plane is given by

$$F(x, y) = \sum_{j=1}^{M} \frac{a_j((x, y) - (x_j, 0))}{((x - x_j)^2 + y^2)^{3/2}} .$$

Where is $F(x, y) = (0, 0)$? Here the function $F(x, y) = (X(x, y), Y(x, y))$ where

$$X(x, y) = \sum_{j=1}^{M} \frac{a_j(x - x_j)}{((x - x_j)^2 + y^2)^{3/2}} \quad \text{and} \quad Y(x, y) = \sum_{j=1}^{M} \frac{a_j y}{((x - x_j)^2 + y^2)^{3/2}} .$$

Can we prove that X and Y only vanish simultaneously at only a finite number of values (x, y)?

2.4 Zeros on the Line of the Point Charges

Proposition 2.8. *The function $F(x, 0)$ vanishes only at finitely many $x \in \mathbb{R}$.*

Proof. This is the subcase that $y = 0$. This is actually the case of a finite electrical field on a line. But now $Y(x, y) = Y(x, 0) = 0$ and

$$X(x, y) = X(x, 0) = \sum_{j=1}^{M} \frac{a_j(x - x_j)}{((x - x_j)^2)^{3/2}} = \sum_{j=1}^{M} \frac{a_j(x - x_j)}{|x - x_j|} \frac{1}{(x - x_j)^2} .$$

Let $I_0 := (-\infty, x_1)$, $I_M := (x_M, \infty)$, and $I_j := (x_j, x_{j+1})$, $j = 1, 2, \dots, M - 1$. The function $h(x) := \dfrac{x - x_j}{|x - x_j|}$ is identically -1 or identically 1 on any of these intervals I_j for $j = 0, 1, \dots, M$. Hence, $X(x, y) = X(x, 0)$ is a rational function on any of these I_j and can only have a finite number of zeros on I_j. This means that $F(x, 0)$ can be zero only finitely many times too. $\square$

2.5 Zeros Off the Line of the Point Charges

Now we work to argue that the zeros must be in some disc. Suppose y is not zero but $F(x, y) = (0, 0)$. Then $Y(x, y) = 0$ implies that

$$\sum_{j=1}^{M} \frac{a_j}{((x - x_j)^2 + y^2)^{3/2}} = 0$$

since $y \neq 0$. So

$$0 = X(x,y) = \sum_{j=1}^{M} \frac{a_j(x-x_j)}{((x-x_j)^2+y^2)^{3/2}} = \sum_{j=1}^{M} \frac{a_j(-x_j)}{((x-x_j)^2+y^2)^{3/2}}\,.$$

Hence, if we have zeros at an arbitrarily large distance away from the origin (necessarily not on the x-axis and so $y \neq 0$), then multiply

$$0 = \sum_{j=1}^{M} \frac{a_j}{((x-x_j)^2+y^2)^{3/2}} \qquad \text{and} \qquad 0 = \sum_{j=1}^{M} \frac{a_j x_j}{((x-x_j)^2+y^2)^{3/2}}$$

by $\left(x^2+y^2\right)^{3/2}$. Then take a limit along a sequence of values (x,y) whose distance from the origin is going to ∞, for which the forces are zero, and conclude that $\sum_{j=1}^{M} a_j = 0$ and $\sum_{j=1}^{M} a_j x_j = 0$.

We examine the cases $M=2$ and $M=3$ briefly in the rest of this section. Suppose $M=2$ so that there are only two point charges. Then the existence of zeros arbitrarily far from the origin, would give $a_1+a_2=0$ and $a_1x_1+a_2x_2=0$. Since $x_1 \neq x_2$, this mean that both $a_1=a_2=0$. So if there are zeros, then they must lie in some large disc.

How do we argue that there are only finitely many of these even in this very simple case? The zero set can be non-empty for sure. Actually, with two point charges, it is not hard to see we cannot have zeros off the axis, and there is only a finite number on the axis. The number is very limited too, maybe at most two (or three?) zeros.

We can extend the argument above to handle three points, at least in some Special Cases. Assume there are zeros arbitrarily far out. We then have $\sum_{j=1}^{M} a_j = 0$ and $\sum_{j=1}^{M} a_j x_j = 0$. Using points (x,y) that are zeros, we have

$$\sum_{j=1}^{M} \frac{a_j x_j}{((x-x_j)^2+y^2)^{3/2}} = 0\,.$$

So using $-\sum_{j=2}^{M} a_j x_j = a_1 x_1$, we have

$$\sum_{j=2}^{M} a_j x_j \left(\frac{1}{((x-x_j)^2+y^2)^{3/2}} - \frac{1}{((x-x_1)^2+y^2)^{3/2}} \right) = 0\,.$$

Now we can use the Mean Value Theorem to rewrite

$$\frac{1}{((x-x_j)^2+y^2)^{3/2}} - \frac{1}{((x-p_1)^2+y^2)^{3/2}} = (x_j - x_1)\left(\frac{-3t_j}{(t_j^2+y^2)^{5/2}} \right)$$

for some t_j between $x-x_1$ and $x-x_j$. Again, taking zeros of large radius, that also have the x value large, and multiplying by $\dfrac{(x^2+y^2)^{5/2}}{-3x}$ gives

$$\sum_{j=2}^{M} a_j x_j (x_j - x_1) = \sum_{j=2}^{M} a_j x_j^2 - \sum_{j=2}^{M} a_j x_j x_1 = 0\,.$$

Since $\sum_{j=2}^{M} a_j x_j = -a_1 p_1$, we get $\sum_{j=1}^{M} a_j x_j^2 = 0$. So in this Special Case we have $\sum_{j=1}^{M} a_j = 0$, $\sum_{j=1}^{M} a_j x_j = 0$, and $\sum_{j=1}^{M} a_j x_j^2 = 0$. The conclusion is that if $a_1 = a_2 = a_3 = 0$. So there cannot be zeros arbitrarily far out, with x values also large unless $a_1 = a_2 = a_3 = 0$. We need a separate argument to treat the case where the x values do not get large. $\square$

3 Special Case: No Curves and Finiteness of the Zero Set

It turns out that there are a couple of ways, closely related to each other, to show that in the Special Case there cannot be any non-trivial curves in the zero set $F = (X, Y) = (0, 0)$. We choose here one particular approach.

First, in Section 3.1, we prove that this zero set is, in fact, countable. The advantage of this argument is that we do not even need to define what we mean by "non-trivial curve". Then in Section 3.2, we show that the zero set is in fact finite.

3.1 The Zero Set $\{X = Y = 0\}$ Is Countable

With $0 < x_1 < x_2 < \cdots < x_M$, let

$$X_m(x,y) := \sum_{j=1}^{M} \frac{a_j(x-x_j)}{((x-x_j)^2+y^2)^{(2m+1)/2}}\,, \qquad m = 1,2,\dots\,,$$

and

$$Y_m(x,y) := \sum_{j=1}^{M} \frac{a_j y}{((x-x_j)^2+y^2)^{(2m+1)/2}}\,, \qquad m = 1,2,\dots\,.$$

We have

$$\begin{aligned}
\frac{\partial X_m}{\partial x} &= \sum_{j=1}^{M} a_j \frac{((x-x_j)^2+y^2)-(2m+1)(x-x_j)^2}{((x-x_j)^2+y^2)^{(2m+3)/2}}\,, \\
\frac{\partial X_m}{\partial y} &= \sum_{j=1}^{M} a_j \frac{-(2m+1)y(x-x_j)}{((x-x_j)^2+y^2)^{(2m+3)/2}}\,, \\
\frac{\partial Y_m}{\partial x} &= \sum_{j=1}^{M} a_j \frac{-(2m+1)y(x-x_j)}{((x-x_j)^2+y^2)^{(2m+3)/2}}\,, \\
\frac{\partial Y_m}{\partial y} &= \sum_{j=1}^{M} a_j \frac{((x-x_j)^2+y^2)-(2m+1)y^2}{((x-x_j)^2+y^2)^{(2m+3)/2}}\,.
\end{aligned}$$

Let

$$H := \{(x,y) : \ \ X_1 = Y_1 = 0, \ \ y \neq 0\}$$

and

$$H_m := \{(x,y) : X_u = Y_u = 0, \ \ y \neq 0 \ \ u = 1,2,\dots,m, \ \ X_{m+1}^2 + Y_{m+1}^2 > 0\}\,,$$

$m = 1,2,\dots$. Observe that for $(x,y) \in H_m$ we have

$$\frac{\partial X_m}{\partial x} = -\frac{\partial Y_m}{\partial y} = (2m+1)yY_{m+1}$$

and

$$\frac{\partial X_m}{\partial y} = \frac{\partial Y_m}{\partial x} = -(2m+1)yX_{m+1}\,.$$

In this section we use only undergraduate real analysis such as Vandermonde determinants and the Implicit Function Theorem and prove a structural property of the set H. In particular, we prove that H is a countable set. The basic idea is under the assumptions

$$X_1(x_0,y_0) = Y_1(x_0,y_0) = 0\,, \qquad \frac{\partial X_1}{\partial x}(x_0,y_0) \neq 0$$

the algebraic varieties

$$\{(x,y) : X_1(x,y) = 0\} \qquad \text{and} \qquad \{(x,y) : Y_1(x,y) = 0\}\,,$$

which are graphs of functions in a neighborhood of (x_0, y_0) by the Implicit Function Theorem, intersect each other in an orthogonal fashion. However, proving our results in this section requires more technical details.

Proposition 3.1. *We have*

$$H = \bigcup_{m=1}^{M-1} H_m$$

unless $a_1 = a_2 = \cdots = a_M = 0$.

Proof. Suppose $(x_0, y_0) \notin H$. For the sake of brevity let

$$r_j := ((x_0 - x_j)^2 + y_0^2)^{1/2}\,, \qquad j = 1, 2, \ldots, M\,.$$

We have

$$\sum_{j=1}^{M} \frac{a_j}{r_j} \frac{1}{(r_j^2)^k} = 0\,, \qquad k = 1, 2, \ldots, M\,, \tag{3.1}$$

and

$$\sum_{j=1}^{M} \frac{a_j x_j}{r_j} \frac{1}{(r_j^2)^k} = 0\,, \qquad k = 1, 2, \ldots, M\,. \tag{3.2}$$

Let $\{R_1 < R_2 < \ldots < R_m\} = \{r_1, r_2, \ldots, r_M\}$. Since our point charges $(x_j, 0)$ are on a line, for every $u \in \{1, 2, \ldots, m\}$ there are at most 2 values of $j \in \{1, 2, \ldots, M\}$ for which $r_j = R_u$. Hence by (3.1) and (3.2) we have

$$\sum_{u=1}^{m} \frac{b_u}{R_u} \frac{1}{(R_u^2)^k} = 0\,, \qquad k = 1, 2, \ldots, m\,, \tag{3.3}$$

and

$$\sum_{u=1}^{m} \frac{c_u}{R_u} \frac{1}{(R_u^2)^k} = 0\,, \qquad k = 1, 2, \ldots, m\,, \tag{3.4}$$

where

$$b_u = a_j \qquad \text{and} \qquad c_u = x_j a_j \tag{3.5}$$

if there is only one value $j \in \{1, 2, \ldots, M\}$ such that $r_j = R_u$, and

$$b_u = a_{j_1} + a_{j_2} \qquad \text{and} \qquad c_u = x_{j_1} a_{j_1} + x_{j_2} a_{j_2} \tag{3.6}$$

if there are two distinct values $j_1, j_2 \in \{1, 2, \ldots, M\}$ such that $r_{j_1} = r_{j_2} = R_u$. Using (3.3)–(3.6) and the well known non-vanishing property of Vandermonde determinants we conclude that

$$b_u = c_u = 0\,, \qquad u = 1, 2, \ldots, m\,,$$

from which

$$a_j = 0, \qquad j = 1, 2, \ldots, M\,,$$

follows. $\square$

Proposition 3.2. *Each point in H_m is an isolated point in H_m for every $m = 1, 2, \ldots, M-1$, unless $a_1 = a_2 = \cdots = a_M = 0$.*

Proof. Suppose $a_1^2 + a_2^2 + \cdots + a_M^2 > 0$. Let m be a fixed positive integer. Suppose $(x_0, y_0) \in H_m$, that is, either

$$X_u(x_0, y_0) = Y_u(x_0, y_0) = 0\,, \quad y_0 \neq 0\,, \quad u = 1, 2, \ldots, m\,, \quad X_{m+1}(x_0, y_0) \neq 0\,, \tag{3.7}$$

or

$$X_u(x_0, y_0) = Y_u(x_0, y_0) = 0\,, \quad y_0 \neq 0\,, \quad u = 1, 2, \ldots, m\,, \quad Y_{m+1}(x_0, y_0) \neq 0\,. \tag{3.8}$$

If (3.7) holds, then we have

$$\frac{\partial X_m}{\partial y}(x_0, y_0) = -(2m+1) y_0 X_{m+1}(x_0, y_0) \neq 0 \tag{3.9}$$

and

$$\frac{\partial Y_m}{\partial x}(x_0, y_0) = -(2m+1) y_0 X_{m+1}(x_0, y_0) \neq 0 \tag{3.10}$$

Also,

$$\begin{aligned} \frac{\partial Y_m}{\partial y}(x_0, y_0) &= \frac{Y_m(x_0, y_0)}{y_0} - (2m+1) y_0 Y_{m+1}(x_0, y_0) \\ &= -(2m+1) y_0 Y_{m+1}(x_0, y_0) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \frac{\partial X_m}{\partial x}(x_0, y_0) &= (2m+1) y_0 Y_{m+1}(x_0, y_0) - \frac{(2m) Y_m(x_0, y_0)}{y_0} \\ &= (2m+1) y_0 Y_{m+1}(x_0, y_0)\,. \end{aligned} \tag{3.12}$$

By the Implicit Function Theorem and (3.9), there are $\delta > 0$, $\varepsilon > 0$, and a function $y = f(x)$ differentiable on $(x_0 - \delta, x_0 + \delta)$ such that the set

$$\begin{aligned} C_1 :&= \{(x, y) : X_m(x, y) = 0, \;\; x \in (x_0 - \delta, x_0 + \delta), \;\; y \in (y_0 - \varepsilon, y_0 + \varepsilon)\} \\ &= \{(x, f(x)) : \;\; x \in (x_0 - \delta, x_0 + \delta)\} \end{aligned}$$

is the graph of a differentiable function $y = f(x)$ on $(x_0 - \delta, x_0 + \delta)$, and the equation for the tangent line to the curve C_1 at the point (x_0, y_0) is

$$y - y_0 = \alpha(x - x_0)\,,$$

where

$$\alpha = \left(\frac{\partial X_m}{\partial x} \Big/ \frac{\partial X_m}{\partial y} \right)(x_0, y_0 = \frac{(2m+1) y_0 Y_{m+1}(x_0, y_0)}{-(2m+1) y_0 X_{m+1}(x_0, y_0)} = \frac{-Y_{m+1}(x_0, y_0)}{X_{m+1}(x_0, y_0)}\,.$$

Similarly, by the Implicit Function Theorem and (3.10), there are $\delta > 0$, $\varepsilon > 0$, and a function $x = g(y)$ differentiable on $(y_0 - \delta, y_0 + \delta)$ such that the set

$$\begin{aligned} C_2 :&= \{(x, y) : Y_m(x, y) = 0, \;\; y \in (y_0 - \delta, y_0 + \delta), \;\; x \in (x_0 - \varepsilon, x_0 + \varepsilon)\} \\ &= \{(g(y), y) : \;\; y \in (y_0 - \delta, y_0 + \delta)\} \end{aligned}$$

is the graph of a differentiable function $x = g(y)$ on $(y_0 - \delta, y_0 + \delta)$, and the equation for the tangent line to the curve C_2 at the point (x_0, y_0) is

$$x - x_0 = \beta(y - y_0)\,,$$

where by (3.11) and (3.12) we have

$$\beta = \left(\frac{\partial Y_m}{\partial y} \Big/ \frac{\partial Y_m}{\partial x}\right)(x_0, y_0) = \frac{-(2m+1)y_0 Y_{m+1}(x_0, y_0)}{-(2m+1)y_0 X_{m+1}(x_0, y_0)} = \frac{Y_{m+1}(x_0, y_0)}{X_{m+1}(x_0, y_0)}\,.$$

Observe that $\alpha = -\beta$. We conclude that the curves C_1 and C_2 are orthogonal to each other at (x_0, y_0). As $H_m \subset C_1 \bigcap C_2$, the point $(x_0, y_0) \in H_m$ is an isolated point of H_m.

If (3.8) holds, then we have

$$\begin{aligned} \frac{\partial Y_m}{\partial y}(x_0, y_0) &= \frac{Y_m(x_0, y_0)}{y_0} - (2m+1)y_0 Y_{m+1}(x_0, y_0) \\ &= -(2m+1)y_0 Y_{m+1}(x_0, y_0) \neq 0 \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \frac{\partial X_m}{\partial x}(x_0, y_0) &= (2m+1)y_0 Y_{m+1}(x_0, y_0) - \frac{(2m)Y_m(x_0, y_0)}{y_0} \\ &= (2m+1)y_0 Y_{m+1}(x_0, y_0) \neq 0\,. \end{aligned} \tag{3.14}$$

Also,

$$\frac{\partial X_m}{\partial y}(x_0, y_0) = \frac{\partial Y_m}{\partial x}(x_0, y_0) = -(2m+1)y_0 X_{m+1}(x_0, y_0) \tag{3.15}$$

By the Implicit Function Theorem and (3.13), there are $\delta > 0$, $\varepsilon > 0$, and a function $y = f(x)$ differentiable on $(x_0 - \delta, x_0 + \delta)$ such that the set

$$\begin{aligned} C_3 :&= \{(x, y) : Y_m(x, y) = 0,\;\; x \in (x_0 - \delta, x_0 + \delta),\;\; y \in (y_0 - \varepsilon, y_0 + \varepsilon)\} \\ &= \{(x, f(x)) :\;\; x \in (x_0 - \delta, x_0 + \delta)\} \end{aligned}$$

is the graph of a differentiable function $y = f(x)$ on $(x_0 - \delta, x_0 + \delta)$, and the equation for the tangent line to the curve C_3 at the point (x_0, y_0) is

$$y - y_0 = \alpha(x - x_0)\,,$$

where

$$\alpha = \left(\frac{\partial Y_m}{\partial x} \Big/ \frac{\partial Y_m}{\partial y}\right)(x_0, y_0) = \frac{-(2m+1)y_0 X_{m+1}(x_0, y_0)}{-(2m+1)y_0 Y_{m+1}(x_0, y_0)} = \frac{X_{m+1}(x_0, y_0)}{Y_{m+1}(x_0, y_0)}\,.$$

Similarly, by the Implicit Function Theorem and (3.14), there are $\delta > 0$, $\varepsilon > 0$, and a function $x = g(y)$ differentiable on $(y_0 - \delta, y_0 + \delta)$ such that the set

$$\begin{aligned} C_4 :&= \{(x, y) : X_m(x, y) = 0,\;\; y \in (y_0 - \delta, y_0 + \delta),\;\; x \in (x_0 - \varepsilon, x_0 + \varepsilon)\} \\ &= \{(g(y), y) :\;\; y \in (y_0 - \delta, y_0 + \delta)\} \end{aligned}$$

is the graph of a differentiable function $x = g(y)$ on $(y_0 - \delta, y_0 + \delta)$, and the equation for the tangent line to the curve C_4 at the point (x_0, y_0) is

$$x - x_0 = \beta(y - y_0)$$

where by (3.15) we have

$$\beta = \left(\frac{\partial X_m}{\partial y} \Big/ \frac{\partial X_m}{\partial x} \right)(x_0, y_0) = \frac{-(2m+1)y_0 X_{m+1}(x_0, y_0)}{(2m+1)y_0 Y_{m+1}(x_0, y_0)} = \frac{-X_{m+1}(x_0, y_0)}{Y_{m+1}(x_0, y_0)} .$$

Observe that $\alpha = -\beta$. We conclude that the curves C_3 and C_4 are orthogonal to each other at (x_0, y_0). As $H_m \subset C_3 \bigcap C_4$, the point $(x_0, y_0) \in H_m$ is an isolated point of H_m. $\square$

Now by combining Propositions 3.1, 3.2, and Section 2.4, our final conclusion in this section is the following.

Proposition 3.3. *In the Special Case, the set* $\{(x, y) : X(x, y) = Y(x, y) = 0\}$ *is countable, and hence the zero set of* $F = (X, Y)$ *does not contain a non-trivial curve.*

3.2 Bézout and Finiteness of the Zero Set

Now we use the above result on the absence of curves in the joint zero set to show that the zero set is actually finite in the Special Case. The argument is along the lines of the proof of Proposition 2.3.

Proposition 3.4. *In the Special Case, the zero set of* $F = (X, Y)$ *is a finite set.*

Proof. Again, suppose $0 < x_1 < x_2 < \cdots < x_M$ and the real numbers $a_1, a_2, \ldots, a_M$ are not all zero. Let, as before,

$$X(x, y) = \sum_{j=1}^{M} \frac{a_j (x - x_j)}{((x - x_j)^2 + y^2)^{3/2}} , \quad \text{and} \quad Y(x, y) = \sum_{j=1}^{M} \frac{a_j y}{((x - x_j)^2 + y^2)^{3/2}} ,$$

Let Σ be the collection of the 2^M functions $\sigma : \{1, 2, \ldots, M\} \to \{-1, 1\}$. Let

$$X_\sigma(x, y) := \sum_{j=1}^{M} \frac{\sigma(j) a_j (x - x_j)}{((x - x_j)^2 + y^2)^{3/2}} , \quad \text{and} \quad Y_\sigma(x, y) := \sum_{j=1}^{M} \frac{\sigma(j) a_j y}{((x - x_j)^2 + y^2)^{3/2}} .$$

Let

$$D_j(x, y) := \prod_{\substack{k=1 \\ k \neq j}}^{M} ((x - x_k)^2 + y^2))^{3/2} , \qquad j = 1, 2, \ldots, M ,$$

and

$$D(x, y) := \prod_{k=1}^{M} ((x - x_k)^2 + y^2))^{3/2} .$$

We have

$$\begin{aligned}
&X_\sigma(x, y)^2 + Y_\sigma(x, y)^2 \\
&= \left(\sum_{j=1}^{M} \frac{\sigma(j) a_j (x - x_k) D_j(x, y)}{D(x, y)} \right)^2 + \left(\sum_{j=1}^{M} \frac{\sigma(j) a_j y D_j(x, y)}{D(x, y)} \right)^2 \\
&= \frac{1}{D(x, y)^2} \left(\left(\sum_{j=1}^{M} \sigma(j) a_j (x - x_j) D_j(x, y) \right)^2 + \left(\sum_{j=1}^{M} \sigma(j) a_j y D_j(x, y) \right)^2 \right) .
\end{aligned}$$

Observe that the function

$$F(x,y) := \prod_{\sigma\in\Sigma}\left(\left(\sum_{j=1}^{M}\sigma(j)a_j(x-x_j)D_j(x,y)\right)^2 + \left(\sum_{j=1}^{M}\sigma(j)a_k y D_j(x,y)\right)^2\right)$$

is an even polynomial in each of the variables

$$\chi_j := D_j(x,y)\,, \qquad j=1,2,\ldots,M\,,$$

as it remains the same when χ_j is replaced by $-\chi_j$. Hence $F(x,y)$ is a polynomial in each of the variables

$$\chi_j^2 = D_j(x,y)^2, \qquad j=1,2,\ldots,M\,.$$

We conclude that

$$G(x,y) := \prod_{\sigma\in\Sigma}\left(X_\sigma(x,y)^2 + Y_\sigma(x,y)^2\right) = \frac{P(x,y)}{D(x,y)^{2^{M+1}}}\,,$$

where P is a polynomial of degree at most $(3M)2^M$. We claim that the set

$$E := \{(x,y) : P(x,y)=0\}$$

cannot contain a non-trivial curve. Indeed, observe that E is a finite union of the countable sets $X_\sigma \cap Y_\sigma$, hence E is also countable. Thus $E = \{(x,y) : P(x,y)=0\}$ is an algebraic variety not containing a non-trivial curve. Such an algebraic variety contains only finitely many points. As the zero set of $F=(X,Y)$ is a subset of E, it is also finite. $\square$

With the help of Bézout's Theorem we can give an upper bound for the number of points in the zero set of $F=(X,Y)$.

Proposition 3.5. *In the Special Case the zero set of $F=(X,Y)$ contains at most $9M^2 4^M$ points.*

To prove the above proposition we need the following classic version of Bézout's Theorem. Two polynomials $f\in\mathbb{R}[x,y]$ and $g\in\mathbb{R}[x,y]$ are coprime if there is no non-constant polynomial which is a factor of both f and g. Associated with a polynomial $f\in\mathbb{R}[x,y]$ we define its zero set $Z(f) := \{(x,y) : \; f(x,y)=0\}$.

Proposition 3.6. (Bézout's Theorem) *Given any two coprime polynomials $f,g\in\mathbb{R}[x,y]$ of degrees d_1 and d_2, respectively, $Z(f)\cap Z(g)$ contains at most d_1d_2 points.*

A consequence of the above Bézout's Theorem is the following.

Proposition 3.7. *Let $f\in\mathbb{R}[x,y]$ be a polynomial of degree d. If $Z(f)$ contains only finitely many points, then $Z(f)$ contains at most $d(d-1)$ points.*

Proof. Without loss of generality we may assume that f is square-free, that is, f is a product of irreducible polynomial factors none of which is repeated, as keeping only one of each repeated factors in a factorization of f the set $Z(f)$ remains the same. As $Z(f)$ contains only finitely many points, we may also assume that f does not have a factor that is a polynomial of y of degree at least 1. Observe that if $Z(f)$ contains only finitely many points, then each of these points are singular, that is,

$$Z(f) = \left\{(x,y) : f(x,y) = \frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0\right\}\,,$$

otherwise the Implicit Function Theorem would imply that $Z(f)$ contains a curve (graph of a function) with infinitely many points. For the sake of brevity let

$$g(x,y) := \frac{\partial f}{\partial x}(x,y)\,.$$

As f is square-free, f and g are coprime. Hence Bézout's Theorem implies that $Z(f) = Z(f) \cap Z(g)$ contains at most $d(d-1)$ points. $\square$

Now we are ready to prove Proposition 3.5.

Proof. Observe that the degree of P in the proof of Proposition 3.4 is at most $d := (3M)2^M$. Hence, by Proposition 3.7 we can deduce that the zero set $E = \{(x,y) : P(x,y) = 0\}$ contains at most $d^2 = 9M^2 4^M$ points. As the zero set of $F = (X,Y)$ is a subset of E, it contains at most $d^2 = 9M^2 4^M$ points as well. $\square$

Observe that no general fact about analytic varieties has been used. All that is needed is Bézout's Theorem. See Gibson [5] for Bézout's Theorem.

3.3 Impact of the Next Dimension

The zero sets of X and Y do not meet in an orthogonal fashion in general. But under simple assumptions they do, and in fact aspects of the field in one higher dimension can play a role.

Consider an analysis of this by taking charges in the xy-plane, with the force field in $\mathbb{R}^3$. Now we have $F = (X,Y,Z)$ but our charges are not in general position in $\mathbb{R}^3$. Instead they are restricted to being in the xy-plane: that is, the charges are at points $(x_j, y_j, 0)$, $j = 1, 2, \ldots, M$.

Again, the force field in $\mathbb{R}^3$ has a potential

$$\phi(x,y,z) = \frac{-1}{2}\sum_{j=1}^{M} \frac{a_j}{r_j}\,.$$

But now, unlike in two dimensions, this potential is harmonic.

Suppose $X = Y = 0$ at a point $(x_0, y_0, 0)$. Suppose again as before that both $y_X = y_X(x)$ and $y_Y = y_Y(x)$ can be implicitly determined by $X = 0$ and $Y = 0$ in a neighborhood of $(x_0, y, 0)$. For the sake of this argument, we assume that this happens because

$$\frac{\partial X}{\partial y} \neq 0 \qquad \text{and} \qquad \frac{\partial Y}{\partial y} \neq 0$$

at the point $(x_0, y_0, 0)$. This means that

$$\frac{\partial^2 \phi}{\partial y \partial x} \neq 0 \qquad \text{and} \qquad \frac{\partial^2 \phi}{\partial y^2} \neq 0$$

at the point $(x_0, y_0, 0)$. We also have

$$0 = \frac{\partial X}{\partial x} + \frac{\partial X}{\partial y}\frac{dy_X}{dx} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y \partial x}\frac{dy_X}{dx}$$

and

$$0 = \frac{\partial Y}{\partial x} + \frac{\partial Y}{\partial y}\frac{dy_Y}{dx} = \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y^2}\frac{dy_Y}{dx}$$

at the point $(x_0, y_0, 0)$. Hence

$$\frac{dy_X}{dx}\frac{dy_Y}{dx} = \left(-\frac{\partial^2\phi}{\partial x^2}\Big/\frac{\partial^2\phi}{\partial y\partial x}\right)\left(-\frac{\partial^2\phi}{\partial x\partial y}\Big/\frac{\partial^2\phi}{\partial y^2}\right) = \left(\frac{\partial^2\phi}{\partial x^2}\Big/\frac{\partial^2\phi}{\partial y^2}\right)$$

holds at the point $(x_0, y_0, 0)$. We have just proved the following result.

Proposition 3.8. *If $X = Y = 0$,*

$$\frac{\partial X}{\partial y} \neq 0\,, \qquad \frac{\partial Y}{\partial y} \neq 0\,, \qquad \textit{and} \qquad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

at the point $(x_0, y_0, 0)$, then

$$\frac{dy_X}{dx}\frac{dy_Y}{dx} = -1$$

holds at the point $(x_0, y_0, 0)$, and the zero sets $\{X = 0\}$ and $\{Y = 0\}$ in the xy-plane meet at the point $(x_0, y_0, 0)$ in an orthogonal fashion.

However, in general $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$ is not zero. Rather

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0\,.$$

But

$$\frac{\partial^2\phi}{\partial z^2} = \sum_{j=1}^{M}\frac{a_j}{r_j^3} - 3\sum_{j=1}^{M}\frac{a_j z^2}{r_j^5}\,.$$

So when $z = 0$, we have $\frac{\partial^2\phi}{\partial z^2} = \sum_{j=1}^{M}\frac{a_j}{r_j^3}$. But $Z = \sum_{j=1}^{M}\frac{a_j z}{r_j^3}$. So at a point $(x_0, y_0, 0)$ which is a limit of points (x, y, z) with $z \neq 0$ and $Z(x, y, z) = 0$ we have $\frac{\partial^2\phi}{\partial z^2} = 0$, and so

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0\,.$$

Hence, as above, this means that

$$\frac{dy_X}{dx}\frac{dy_Y}{dx} = -1$$

at the point $(x_0, y_0, 0)$, and so again we have the same result as in Proposition 3.8, that is, the zero sets of $\{X = 0\}$ and $\{Y = 0\}$ meet at the point $x_0, y_0, 0)$ in an orthogonal fashion.

However, examples should show that there are are points $(x, y, 0)$ in the xy-plane where both X and Y are zero, but

$$\sum_{j=1}^{M}\frac{a_j}{r_j^3} = \sum_{j=1}^{M}\frac{a_j}{((x - x_j)^2 + (y - y_j)^2)^{3/2}}$$

is not zero. So at such points the zero sets $\{X = 0\}$ and $\{Y = 0\}$ would not necessarily meet in an orthogonal fashion.

4 Special Case: Asymptotic Directions of Component Zero Sets

Let $0 \neq \alpha \in (-\infty, \infty)$. The line $y = \alpha x$ is called an asymptotic direction for a set $A \subset \mathbb{R}^2$ if there are $(p_m, q_m) \in A$ such that

$$\lim_{|p_m| \to \infty} \frac{q_m}{p_m} = \lim_{|q_m| \to \infty} \frac{q_m}{p_m} = \alpha \,.$$

The x-axis is called an asymptotic direction for a set $A \subset \mathbb{R}^2$ if there are $(p_m, q_m) \in A$ such that

$$\lim_{|p_m| \to \infty} \frac{q_m}{p_m} = 0 \,.$$

The y-axis is called an asymptotic direction for a set $A \subset \mathbb{R}^2$ if there are $(p_m, q_m) \in A$ such that

$$\lim_{|q_m| \to \infty} \frac{p_m}{q_m} = 0 \,.$$

For the sake of brevity we will use the notation $\beta := 1/\alpha$. Note that the case $\beta = 0$ corresponds to the asymptotic direction given by the y-axis. The goal in this section is is simple: to show that the set of possible asymptotic directions for $\{X = 0\}$ and the set of the possible asymptotic directions for $\{Y = 0\}$ are distinct. We succeed in showing this with the exception of the asymptotic directions $y = \pm x$. We call the domains $\{(x, y) : |x| < |y|\}$ and $\{(x, y) : |y| < |x|\}$ Type I domain and Type II domain, respectively. To find all possible asymptotic directions for the zero sets $\{Y = 0\}$ and $\{X = 0\}$ we proceed differently in Type I and Type II domains, but the cases of the zero sets $\{Y = 0\}$ and $\{X = 0\}$ are quite similar to each other in both domains. Our results in this section have their own intrinsic interest, even though the boundedness of the zero set $\{X = Y = 0\}$ contained in Propositions 3.4 and 3.5 is not completely recaptured in this section as we cannot show that the lines $|y| = |x|$ cannot be a common asymptotic direction to both of the zero sets $\{Y = 0\}$ and $\{X = 0\}$.

4.1 Notation

Let, as before, $0 < x_1 < x_2 < \cdots < x_M$ and

$$X(x, y) := \sum_{j=1}^{M} \frac{a_j (x - x_j)}{((x - x_j)^2 + y^2)^{3/2}} \qquad \text{and} \qquad Y(x, y) := \sum_{j=1}^{M} \frac{a_j y}{((x - x_j)^2 + y^2)^{3/2}} \,.$$

By the Binomial Theorem the Taylor series expansion

$$(1 + z)^{-3/2} = \sum_{n=0}^{\infty} \binom{-3/2}{n} z^n \,, \qquad z \in (-1, 1) \,,$$

converges uniform on the interval $[-1 + \delta, 1 - \delta]$ for every $\delta > 0$. Let $\mu_u := (-1)^u \sum_{j=1}^{M} a_j x_j^u$. By the non-vanishing property of the Vandermonde determinants there is an integer $0 \leq L \leq M$ such that $\mu_u = 0$ for $u = 0, 1, \ldots, L-1$, and $\mu_L \neq 0$. This is the choice of L throughout Sections 4.1-4.6. Typically, as L gets larger, the complexity to detect possible asymptotic directions for the zero sets $\{X = 0\}$ and $\{Y = 0\}$ increases.

4.2 Type I Domain, Possible Asymptotes for $Y = 0$

Proposition 4.1. *Let $\delta \in (0, 1/2)$ be fixed. If*

$$Y(p_m, q_m) = 0, \qquad \left|\frac{p_m}{q_m}\right| < 1 - \delta, \qquad q_m \neq 0, \qquad \lim_{|q_m|\to\infty} \frac{p_m}{q_m} = \beta\,,$$

then

$$\frac{d^L}{d\beta^L}\left(\frac{1}{(1+\beta^2)^{3/2}}\right) = 0\,.$$

Proof. We have

$$\begin{aligned} 0 = \ & q_m^{L+1}|q_m| Y(p_m, q_m) = q_m^L \sum_{j=1}^{M} a_j \left(1 + \left(\frac{p_m - x_j}{q_m}\right)^2\right)^{-3/2} \\ = \ & q_m^L \sum_{j=1}^{M} a_j \sum_{n=0}^{\infty} \binom{-3/2}{n} \left(\frac{p_m - x_j}{q_m}\right)^{2n} \end{aligned}$$

for all sufficiently large m. Note that we have adjusted with the factor $q_m^{L+1}|q_m|$. We also have

$$\begin{aligned} \sum_{j=1}^{M} a_j (p_m - x_j)^{2n} &= \sum_{j=1}^{M} a_j \sum_{u=0}^{2n} \binom{2n}{u} (-1)^u x_j^u p_m^{2n-u} = \sum_{u=0}^{2n}\sum_{j=1}^{M} \binom{2n}{u} (-1)^u a_j x_j^u p_m^{2n-u} \\ &= \sum_{u=L}^{2n}\sum_{j=1}^{M} \binom{2n}{u} (-1)^u x_j^u p_m^{2n-u} = \sum_{u=L}^{2n} \binom{2n}{u} \mu_u p_m^{2n-u}\,, \end{aligned} \tag{4.1}$$

and hence

$$\begin{aligned} 0 = \ & q_m^L \sum_{j=1}^{M} a_j \sum_{n=0}^{\infty} \binom{-3/2}{n} \left(\frac{p_m - x_j}{q_m}\right)^{2n} = q_m^L \sum_{n=0}^{\infty} \binom{-3/2}{n} \sum_{j=1}^{M} a_j \left(\frac{p_m - x_j}{q_m}\right)^{2n} \\ = \ & \sum_{n=0}^{\infty} \binom{-3/2}{n} \sum_{u=L}^{2n} \binom{2n}{u} \mu_u \left(\frac{p_m}{q_m}\right)^{2n-u} q_m^{L-u} \end{aligned}$$

for all sufficiently large m. Separating the term for which $u = L$, we obtain

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} \binom{-3/2}{n} \binom{2n}{L} \mu_L \left(\frac{p_m}{q_m}\right)^{2n-L} + \sum_{n=0}^{\infty} \binom{-3/2}{n} \sum_{u=L+1}^{2n} \binom{2n}{u} \mu_u \left(\frac{p_m}{q_m}\right)^{2n-u} q_m^{L-u} \\ &= \frac{\mu_L}{L!} \frac{d^L}{dz^L}\left((1+z^2)^{-3/2}\right)\Big|_{z=p_m/q_m} + \sum_{u=L+1}^{\infty} \frac{\mu_u}{u!} \frac{d^u}{dz^u}\left((1+z^2)^{-3/2}\right)\Big|_{z=p_m/q_m} q_m^{L-u} \end{aligned} \tag{4.2}$$

for all sufficiently large m. Here we interchanged the order of summations which, is legal as one can easily check that under our conditions the double sum converges absolutely. See Section 4.6. Observe that

$$|\mu_u| = \left|\sum_{j=1}^{M} a_j x_j^u\right| \leq \left(\sum_{j=1}^{M} |a_j|\right) x_M^u = A x_M^u\,. \tag{4.3}$$

The Cauchy Integral Formula and $|p_m/q_m| < 1-\delta$ imply that

$$\left|\frac{1}{u!}\frac{d^u}{dz^u}\left(\left(1+z^2\right)^{-3/2}\right)\Big|_{z=p_m/q_m}\right| \le \frac{\delta}{2}\max_{|z|=1-\delta/2}\left|\left(1+z^2\right)^{-3/2}\right|\left(\frac{\delta}{2}\right)^{-(u+1)} \le \left(\frac{2}{\delta}\right)^{u+2}. \quad (4.4)$$

Observe also that

$$|q_m|^{L-u} = |q_m|^{L-u+1/2}|q_m|^{-1/2} \le |q_m|^{-u/(2L+2)}|q_m|^{-1/2}, \qquad u \ge L+1,\ \ |q_m| > 1. \quad (4.5)$$

Combining 4.3, 4.4, and 4.5, we obtain

$$\begin{aligned}
&\left|\sum_{u=L+1}^{\infty}\frac{\mu_u}{u!}\frac{d^u}{dz^u}\left(\left(1+z^2\right)^{-3/2}\right)\Big|_{z=p_m/q_m} q_m^{L-u}\right| \\
&\le \sum_{u=L+1}^{\infty}|\mu_u|\left|\frac{1}{u!}\frac{d^u}{dz^u}\left(\left(1+z^2\right)^{-3/2}\right)\Big|_{z=p_m/q_m}\right||q_m|^{L-u} \\
&\le \sum_{u=L+1}^{\infty}Ax_M^u\left(\frac{2}{\delta}\right)^{u+2}|q_m|^{L-u} \le |q_m|^{-1/2}\sum_{u=L+1}^{\infty}Ax_M^u\left(\frac{2}{\delta}\right)^{u+2}|q_m|^{-u/(2L+2)} \\
&\le |q_m|^{-1/2}A\left(\frac{2}{\delta}\right)^2\sum_{u=L+1}^{\infty}\left(\frac{2x_M}{\delta|q_m|^{1/(2L+2)}}\right)^u.
\end{aligned} \quad (4.6)$$

As $|q_m| \to \infty$, 4.6 implies that

$$\lim_{|q_m|\to\infty}\left(\sum_{u=L+1}^{\infty}\frac{\mu_u}{u!}\frac{d^u}{dz^u}\left(\left(1+z^2\right)^{-3/2}\right)\Big|_{z=p_m/q_m} q_m^{L-u}\right) = 0.$$

Now we are ready to take the limit in 4.2 when $|q_m| \to \infty$. We obtain

$$\frac{\mu_L}{L!}\frac{d^L}{d\beta^L}\left(\frac{1}{(1+\beta^2)^{3/2}}\right) + 0 = 0.$$

□

4.3 Type I domain, Possible Asymptotes for $X = 0$

Proposition 4.2. *Let $\delta \in (0, 1/2)$ be fixed. If*

$$X(p_m, q_m) = 0, \quad \left|\frac{p_m}{q_m}\right| < 1-\delta, \quad q_m \ne 0, \quad \lim_{|q_m|\to\infty}\frac{p_m}{q_m} = \beta,$$

then

$$\frac{d^L}{d\beta^L}\left(\frac{\beta}{(1+\beta^2)^{3/2}}\right) = 0.$$

Proof. We have

$$\begin{aligned}
0 =&\ q_m^{L+1}|q_m|X(p_m, q_m) = q_m^{L-1}\sum_{j=1}^{M}a_j(p_m - x_j)\left(1+\left(\frac{p_m - x_j}{q_m}\right)^2\right)^{-3/2} \\
=&\ q_m^{L-1}\sum_{j=1}^{M}a_j\sum_{n=0}^{\infty}\binom{-3/2}{n}\frac{(p_m - x_j)^{2n+1}}{q_m^{2n}}
\end{aligned}$$

for all sufficiently large m. Note that we have adjusted with the factor $q_m^{L+1}|q_m|$. Replacing $2n$ by $2n+1$ in 4.1, we also have

$$\sum_{j=1}^{M} a_j (p_m - x_j)^{2n+1} = \sum_{u=L}^{2n+1} \binom{2n+1}{u} \mu_u p_m^{2n+1-u}\,,$$

and hence

$$\begin{aligned} 0 &= q_m^{L-1} \sum_{j=1}^{M} a_j \sum_{n=0}^{\infty} \binom{-3/2}{n} \frac{(p_m - x_j)^{2n+1}}{q_m^{2n}} = q_m^{L-1} \sum_{n=0}^{\infty} \binom{-3/2}{n} \sum_{j=1}^{M} \frac{(p_m - x_j)^{2n+1}}{q_m^{2n}} \\ &= \sum_{n=0}^{\infty} \binom{-3/2}{n} \sum_{u=L}^{2n+1} \binom{2n+1}{u} \mu_u \left(\frac{p_m}{q_m}\right)^{2n+1-u} q_m^{L-u}\,. \end{aligned}$$

for all sufficiently large m. Separating the term for which $u = L$, we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} \binom{-3/2}{n} \binom{2n+1}{L} \mu_L \left(\frac{p_m}{q_m}\right)^{2n+1-L} + \sum_{n=0}^{\infty} \binom{-3/2}{n} \sum_{u=L+1}^{2n+1} \binom{2n+1}{u} \mu_u \left(\frac{p_m}{q_m}\right)^{2n+1-u} q_m^{L-u} \\ &= \frac{\mu_L}{L!} \frac{d^L}{dz^L} \left(z \left(1+z^2\right)^{-3/2} \right) \Big|_{z=p_m/q_m} + \sum_{u=L+1}^{\infty} \frac{\mu_u}{u!} \frac{d^u}{dz^u} \left(z \left(1+z^2\right)^{-3/2} \right) \Big|_{z=p_m/q_m} q_m^{L-u} \end{aligned} \tag{4.7}$$

vanishes for all sufficiently large m. Here we interchanged the order of summations, which is legitimate as one can easily check that under our conditions the double sum converges absolutely. See Section 4.6. The Cauchy Integral Formula and $|p_m/q_m| \le 1 - \delta$ imply that

$$\left| \frac{1}{u!} \frac{d^u}{dz^u} \left(z \left(1+z^2\right)^{-3/2} \right) \Big|_{z=p_m/q_m} \right| \le \frac{\delta}{2} \max_{|z|=1-\delta/2} \left| \left(1+z^2\right)^{-3/2} \right| \left(\frac{\delta}{2}\right)^{-(u+1)} \le \left(\frac{2}{\delta}\right)^{u+2}. \tag{4.8}$$

Combining 4.3, 4.8, and 4.5, we obtain

$$\begin{aligned} &\left| \sum_{u=L+1}^{\infty} \frac{(-1)^u \mu_u}{u!} \frac{d^u}{dz^u} \left(z \left(1+z^2\right)^{-3/2} \right) \Big|_{z=p_m/q_m} q_m^{L-u} \right| \\ &\le \sum_{u=L+1}^{\infty} |\mu_u| \left| \frac{1}{u!} \frac{d^u}{dz^u} \left(z \left(1+z^2\right)^{-3/2} \right) \Big|_{z=p_m/q_m} \right| |q_m|^{L-u} \\ &\le \sum_{u=L+1}^{\infty} A x_M^u \left(\frac{2}{\delta}\right)^{u+2} |q_m|^{L-u} \le |q_m|^{-1/2} \sum_{u=L+1}^{\infty} A x_M^u \left(\frac{2}{\delta}\right)^{u+2} |q_m|^{-u/(2L+2)} \\ &\le |q_m|^{-1/2} A \left(\frac{2}{\delta}\right)^2 \sum_{u=L+1}^{\infty} \left(\frac{2x_M}{\delta |q_m|^{1/(2L+2)}}\right)^u. \end{aligned} \tag{4.9}$$

As $|q_m| \to \infty$, 4.9 implies that

$$\lim_{|q_m|\to\infty} \left(\sum_{u=L+1}^{\infty} \frac{\mu_u}{u!} \frac{d^u}{dz^u} \left(z \left(1+z^2\right)^{-3/2} \right) \Big|_{z=p_m/q_m} q_m^{L-u} \right) = 0\,.$$

Now we are ready to take the limit in 4.7 when $|q_m| \to \infty$. We obtain

$$\frac{\mu_L}{L!} \frac{d^L}{d\beta^L} \left(\frac{\beta}{(1+\beta^2)^{3/2}} \right) + 0 = 0\,.$$

□

4.4 Type II domain, Possible Asymptotes for $Y = 0$

Proposition 4.3. *Let $\delta \in (0, 1/4)$ be fixed. If*

$$Y(p_m, q_m) = 0, \quad |p_m| > x_M, \quad q_m \neq 0, \quad \left|\frac{q_m}{p_m}\right| < 1 - 2\delta, \quad \lim_{|p_m| \to \infty} \frac{q_m}{p_m} = \alpha\,,$$

then $\alpha \neq 0$ and

$$\frac{d^L}{d\alpha^L}\left(\frac{\alpha^{L+2}}{(1+\alpha^2)^{3/2}}\right) = 0\,.$$

Proof. The Binomial Theorem gives

$$\frac{1}{(x - x_j)^{2n+3}} = \frac{1}{x^{2n+3}}\left(1 - \frac{x_j}{x}\right)^{-(2n+3)} = \frac{1}{x^{2n+3}} \sum_{u=0}^{\infty} \binom{-(2n+3)}{u} \left(\frac{-x_j}{x}\right)^u, \qquad |x| > x_M\,,$$

for each $j = 1, 2, \ldots, M$, and hence

$$\begin{aligned}
\sum_{j=1}^{M} \frac{a_j}{(x - x_j)^{2n+3}} &= \sum_{j=1}^{M} \frac{a_j}{x^{2n+3}} \sum_{u=0}^{\infty} \binom{-(2n+3)}{u} \left(\frac{-x_j}{x}\right)^u \\
&= \sum_{j=1}^{M} \frac{a_j}{x^{2n+3}} \left(\sum_{u=0}^{L} \binom{-(2n+3)}{u} \left(\frac{-x_j}{x}\right)^u + R_{L,n}\left(\frac{-x_j}{x}\right)\right) \qquad (4.10)\\
&= \mu_L \binom{-(2n+3)}{L} x^{-(2n+3)-L} + A_{L,n}(x)\,, \qquad |x| > |x_M|\,,
\end{aligned}$$

with

$$A_{L,n}(x) := x^{-(2n+3)} \sum_{j=1}^{M} a_j R_{L,n}\left(\frac{-x_j}{x}\right),$$

where $R_{L,n}(x)$ is the Lth remainder term in the Taylor series expansion of the function $f(t) := (1+t)^{-(2n+3)}$ centered at 0, that is,

$$(1+t)^{-(2n+3)} = T_{L,n}(t) + R_{L,n}(t)\,, \qquad |t| < 1\,,$$

where $T_{L,n}(t)$ it the Lth Taylor polynomial centered at 0 associated with the function $f(t) := (1+t)^{-(2n+3)}$. Estimating by using the Cauchy form of the remainder term $R_{L,n}(t)$ in the Taylor series expansion of the function $f(t) := (1+t)^{-(2n+3)}$, we obtain

$$|R_{L,n}(t)| \leq (L+1)\left|\binom{-(2n+3)}{L+1}\right| (1 - |t|)^{-(2n+4)} |t|^{L+1}\,, \qquad |t| < 1\,,$$

and hence with $A := \sum_{j=1}^{M} |a_j|$ we have

$$\begin{aligned}
|A_{L,n}(x)| &= \left| x^{-(2n+3)} \sum_{j=1}^{M} a_j R_{L,n}\left(\frac{-x_j}{x}\right) \right| \\
&\leq AM|x|^{-(2n+3)}(L+1)\left|\binom{-(2n+3)}{L+1}\right| \left(1 - \left|\frac{x_M}{x}\right|\right)^{-(2n+4)} \left|\frac{x_M}{x}\right|^{L+1}, \qquad (4.11)\\
&\leq AM(L+1)x_M^{L+1} \left|\binom{-(2n+3)}{L+1}\right| (|x| - x_M)^{-(2n+4)} |x|^{-L}\,, \qquad |x| > x_M\,.
\end{aligned}$$

A simple algebra shows that

$$\frac{1}{y}Y(x,y) = \sum_{j=1}^{M} \frac{a_j}{((x-x_j)^2+y^2)^{3/2}} = \sum_{j=1}^{M} \frac{a_j}{|x-x_j|^3}\left(1+\left(\frac{y}{x-x_j}\right)^2\right)^{-3/2}, \quad |x|>x_M,\ y\neq 0\,.$$

Observe that $\left|\dfrac{q_m}{p_m-x_j}\right| \leq 1-\delta$ for all sufficiently large m, and hence the Binomial Theorem gives

$$\begin{aligned}
0 = \pm\frac{p_m^{L+3}}{q_m}Y(p_m,q_m) &= p_m^{L+3}\sum_{j=1}^{M}\frac{a_j}{(p_m-x_j)^3}\left(1+\left(\frac{q_m}{p_m-x_j}\right)^2\right)^{-3/2} \\
&= p_m^{L+3}\sum_{j=1}^{M}\frac{a_j}{(p_m-x_j)^3}\sum_{n=0}^{\infty}\binom{-3/2}{n}\left(\frac{q_m}{p_m-x_j}\right)^{2n}
\end{aligned}$$

for all sufficiently large m. Note that we have adjusted with the factor p_m^{L+3}. Combining this with 4.10, we obtain

$$\begin{aligned}
0 &= p_m^{L+3}\sum_{j=1}^{M}\frac{a_j}{(p_m-x_j)^3}\sum_{n=0}^{\infty}\binom{-3/2}{n}\left(\frac{q_m}{p_m-x_j}\right)^{2n} \\
&= p_m^{L+3}\sum_{n=0}^{\infty}\sum_{j=1}^{M}\binom{-3/2}{n}\frac{a_jq_m^{2n}}{(p_m-x_j)^{2n+3}} \\
&= \sum_{n=0}^{\infty}\binom{-3/2}{n}q_m^{2n}\left(\mu_L\binom{-(2n+3)}{L}p_m^{-2n}+p_m^{L+3}A_{L,n}(p_m)\right) \\
&= \sum_{n=0}^{\infty}\binom{-3/2}{n}\left(\mu_L\binom{-(2n+3)}{L}\left(\frac{q_m}{p_m}\right)^{2n}+q_m^{2n}p_m^{L+3}A_{L,n}(p_m)\right)
\end{aligned} \tag{4.12}$$

for all sufficiently large m. It follows from 4.11 that

$$\begin{aligned}
&\left|\sum_{n=0}^{\infty}\binom{-3/2}{n}q_m^{2n}p_m^{L+3}A_{L,n}(p_m)\right| \\
&\leq AM(L+1)x_M^{L+1}\sum_{n=0}^{\infty}\left|\binom{-3/2}{n}\binom{-(2n+3)}{L+1}\right|\left(\frac{|q_m|}{|p_m|-x_M}\right)^{2n}\left(\frac{|p_m|}{|p_m|-x_M}\right)^3\frac{1}{|p_m|-x_M} \\
&\leq AM(L+1)x_M^{L+1}\sum_{n=0}^{\infty}\left|\binom{-3/2}{n}\binom{-(2n+3)}{L+1}\right|(|\alpha|+\delta)^{2n}\left(\frac{|p_m|}{|p_m|-x_M}\right)^3\frac{1}{|p_m|-x_M}
\end{aligned}$$

for all sufficiently large m, where the right-hand side converges to

$$AM(L+1)x_M^{L+1}\left(\sum_{n=0}^{\infty}\left|\binom{-3/2}{n}\binom{-(2n+3)}{L+1}\right|(|\alpha|+\delta)^{2n}\right)\left(\lim_{|p_m|\to\infty}\frac{|p_m|^3}{(|p_m|-x_M)^4}\right)=0\,.$$

Therefore taking the limit in 4.12 as $|p_m|\to\infty$ gives

$$\sum_{n=0}^{\infty}\binom{-3/2}{n}\mu_L\binom{-(2n+3)}{L}\alpha^{2n}=0\,.$$

That is,

$$\sum_{n=0}^{\infty} \binom{-3/2}{n} \mu_L \binom{-(2n+L+2)}{L} \alpha^{2n} = 0\,.$$

As $|\alpha| \leq 1-\delta$ we conclude that $\alpha \neq 0$ and

$$\frac{\mu_L}{L!} \frac{d^L}{d\alpha^L} \left(\frac{\alpha^{L+2}}{(1+\alpha^2)^{3/2}} \right) = 0\,.$$

□

4.5 Type II domain, Possible Asymptotes for $X = 0$

Proposition 4.4. *Let $\delta \in (0, 1/4)$ be fixed. If*

$$Y(p_m, q_m) = 0, \quad |p_m| > x_M, \quad \left| \frac{q_m}{p_m} \right| < 1 - 2\delta, \quad \lim_{|p_m| \to \infty} \frac{q_m}{p_m} = \alpha\,,$$

then $\alpha \neq 0$ and

$$\frac{d^L}{d\alpha^L} \left(\frac{\alpha^{L+1}}{(1+\alpha^2)^{3/2}} \right) = 0\,.$$

Proof. Replacing $2n+3$ by $2n+2$ in 4.10, we obtain

$$\sum_{j=1}^{M} \frac{a_j}{(x-x_j)^{2n+2}} = \mu_L \binom{-(2n+2)}{L} x^{-(2n+3)-L} + A_{L,n}(x)\,, \qquad |x| > |x_M|\,, \tag{4.13}$$

with

$$A_{L,n}(x) := x^{-(2n+2)} \sum_{j=1}^{M} a_j R_{L,n} \left(\frac{-x_j}{x} \right),$$

where $R_{L,n}(x)$ is the Lth remainder term in the Taylor series expansion of the function $f(t) := (1+t)^{-(2n+2)}$ centered at 0, that is,

$$(1+t)^{-(2n+2)} = T_{L,n}(t) + R_{L,n}(t)\,, \qquad |t| < 1\,,$$

where $T_{L,n}(t)$ it the Lth Taylor polynomial centered at 0 associated with the function $f(t) := (1+t)^{-(2n+2)}$. Estimating by using the Cauchy form of the remainder term $R_{L,n}(t)$ in the Taylor series expansion of the function $f(t) := (1+t)^{-(2n+3)}$, we obtain

$$|R_{L,n}(t)| \leq\ (L+1) \left| \binom{-(2n+2)}{L+1} \right| (1-|t|)^{-(2n+3)} |t|^{L+1}\,, \qquad |t| < 1\,,$$

and hence, replacing $2n+3$ with $2n+2$ in 4.11, we have

$$|A_{L,n}(x)| \leq AM(L+1) x_M^{L+1} \left| \binom{-(2n+2)}{L+1} \right| (|x| - x_M)^{-(2n+3)} |x|^{-L}\,, \qquad |x| > x_M\,. \tag{4.14}$$

A simple algebra shows that

$$X(x,y) = \sum_{j=1}^{M} \frac{a_j(x-x_j)}{((x-x_j)^2+y^2)^{3/2}} = \sum_{j=1}^{M} \frac{a_j(x-x_j)}{|x-x_j|^3} \left(1 + \left(\frac{y}{x-x_j} \right)^2 \right)^{-3/2}, \quad |x| > x_M,\ y \neq 0\,.$$

Observe that

$$\left|\frac{q_m}{p_m - x_j}\right| \le 1-\delta$$

for all sufficiently large m, and hence the Binomial Theorem gives

$$\begin{aligned}
0 = \ &\pm p_m^{L+2} X(p_m, q_m) = p_m^{L+2} \sum_{j=1}^{M} \frac{a_j(p_m - x_j)}{(p_m - x_j)^3} \left(1 + \left(\frac{q_m}{p_m - x_j}\right)^2\right)^{-3/2} \\
&= p_m^{L+2} \sum_{j=1}^{M} \frac{a_j}{(p_m - x_j)^2} \sum_{n=0}^{\infty} \binom{-3/2}{n} \left(\frac{q_m}{p_m - x_j}\right)^{2n}
\end{aligned}$$

for all sufficiently large m. Note that we have adjusted with the factor p_m^{L+2}. Hence, similarly to 4.12, we get

$$\begin{aligned}
0 = \ & p_m^{L+2} \sum_{j=1}^{M} \frac{a_j}{(p_m - x_j)^2} \sum_{n=0}^{\infty} \binom{-3/2}{n} \left(\frac{q_m}{p_m - x_j}\right)^{2n} \\
&= \sum_{n=0}^{\infty} \binom{-3/2}{n} \left(\mu_L \binom{-(2n+2)}{L} \left(\frac{q_m}{p_m}\right)^{2n} + q_m^{2n} p_m^{L+2} A_{L,n}(p_m)\right)
\end{aligned} \tag{4.15}$$

for all sufficiently large m. It follows from 4.14 that

$$\begin{aligned}
&\left|\sum_{n=0}^{\infty} \binom{-3/2}{n} q_m^{2n} p_m^{L+2} A_{L,n}(p_m)\right| \\
\le & AM(L+1)x_M^{L+1} \sum_{n=0}^{\infty} \left|\binom{-3/2}{n}\binom{-(2n+2)}{L+1}\right| \left(\frac{|q_m|}{|p_m| - x_M}\right)^{2n} \left(\frac{|p_m|}{|p_m| - x_M}\right)^2 \frac{1}{|p_m| - x_M} \\
\le & AM(L+1)x_M^{L+1} \sum_{n=0}^{\infty} \left|\binom{-3/2}{n}\binom{-(2n+2)}{L+1}\right| (|\alpha| + \delta)^{2n} \left(\frac{|p_m|}{|p_m| - x_M}\right)^2 \frac{1}{|p_m| - x_M}
\end{aligned}$$

for all sufficiently large m, where the right-hand side converges to

$$AM(L+1)x_M^{L+1} \left(\sum_{n=0}^{\infty} \left|\binom{-3/2}{n}\binom{-(2n+2)}{L+1}\right| (|\alpha| + \delta)^{2n}\right) \left(\lim_{|p_m|\to\infty} \frac{|p_m|^2}{(|p_m| - x_M)^3}\right) = 0\,.$$

Therefore taking the limit in 4.15 as $|p_m| \to \infty$ gives

$$\sum_{n=0}^{\infty} \binom{-3/2}{n} \mu_L \binom{-(2n+2)}{L} \alpha^{2n} = 0\,.$$

That is,

$$\sum_{n=0}^{\infty} \binom{-3/2}{n} \mu_L \binom{2n+L+1}{L} \alpha^{2n} = 0\,.$$

As $|\alpha| \le 1 - 2\delta$ we conclude that $\alpha \neq 0$ and

$$\frac{\mu_L}{L!} \frac{d^L}{d\alpha^L} \left(\frac{\alpha^{L+1}}{(1+\alpha^2)^{3/2}}\right) = 0\,.$$

□

4.6 Interchanging the Order of Summation

In the analysis in Section 4.2 to find the possible asymptotes for $\{Y = 0\}$ in the Type I domain we interchanged the order of summations in

$$\sum_{n=0}^{\infty} \binom{-3/2}{n} \sum_{u=L}^{2n} \binom{2n}{u} \mu_u \left(\frac{p_m}{q_m}\right)^{2n-u} q_m^{L-u} = \sum_{u=L}^{\infty} \sum_{n=0}^{\infty} \binom{2n}{u} \mu_u \left(\frac{p_m}{q_m}\right)^{2n-u} q_m^{L-u}$$

for all sufficiently large m, or equivalently

$$\sum_{n=0}^{\infty} \binom{-3/2}{n} \sum_{u=L}^{2n} \binom{2n}{u} \mu_u \left(\frac{p_m}{q_m}\right)^{2n-u} q_m^{-u} = \sum_{u=L}^{\infty} \sum_{n=0}^{\infty} \binom{2n}{u} \mu_u \left(\frac{p_m}{q_m}\right)^{2n-u} q_m^{-u}$$

for all sufficiently large m. To see that this is legitimate, recall that $\mu_u := (-1)^u \sum_{j=1}^{u} a_j x_j^u$ and $\mu_u = 0$ for each $u = 0, 1, \dots, L-1$, and hence we have

$$\sum_{n=0}^{\infty} \sum_{u=L}^{2n} \left| \binom{-3/2}{n} \binom{2n}{u} \mu_u \left(\frac{p_m}{q_m}\right)^{2n-u} q_m^{-u} \right| = \sum_{n=0}^{\infty} \sum_{u=0}^{2n} \left| \binom{-3/2}{n} \binom{2n}{u} \mu_u \left(\frac{p_m}{q_m}\right)^{2n-u} q_m^{-u} \right|$$

$$= \sum_{n=0}^{\infty} \sum_{u=0}^{2n} \left| \binom{-3/2}{n} \binom{2n}{u} \sum_{j=1}^{M} a_j (-x_j)^u \left(\frac{p_m}{q_m}\right)^{2n-u} q_m^{-u} \right| \leq \sum_{j=1}^{M} \sum_{n=0}^{\infty} |a_j| \left| \binom{-3/2}{n} \right| \left(\frac{|p_m| + x_j}{|q_m|} \right)^{2n}$$

$$\leq \sum_{j=1}^{M} \sum_{n=0}^{\infty} |a_j| \left| \binom{-3/2}{n} \right| (|\beta| + \delta/2)^{2n} < \infty$$

for all sufficiently large m, as $|\beta| \leq 1-\delta$ and the power series $\sum_{n=0}^{\infty} \binom{-3/2}{n} z^{2n}$ converges absolutely for all $z \in (-1, 1)$.

A similar argument shows that interchanging the order of summation in our work in Section 4.2 to find the possible asymptotes for $\{X = 0\}$ in the Type I domain is legitimate.

4.7 Interlacing properties of zeros of polynomials I

We now address how we can conclude that the asymptotic directions of the zero sets of Y and X are distinct. Proving this would show that there are no zeros for $F = (X, Y)$ outside some large disc, in the Special Case. From the analysis in Section 3, we already know this because the zero set is finite. But our argument may have independent value because it gives interlacing properties for a class of functions that has not previously been considered. In addition, this interlacing is needing for applications of the General Case that we will address later. Moreover, this interlacing sets up a context in which some of the challenges in the General Case can be understood better. We thank Vilmos Totik who early on gave us methods for getting results like the ones in this section.

We define

$$A_0(x) := \frac{x}{(1+x^2)^{3/2}}, \qquad B_0(x) := \frac{1}{(1+x^2)^{3/2}},$$

$$A_L(x) := \frac{d^L A_0}{dx^L}, \qquad B_L(x) := \frac{d^L B_0}{dx^L}, \qquad L = 1, 2, \dots .$$

It is easy to see that there are polynomials P_L of degree $L+1$ and Q_L of degree L such that

$$A_L(x) = \frac{P_L(x)}{(1+x^2)^{(2L+3)/2}} = \frac{(1+x^2)P'_{L-1}(x) - (2L+1)xP_{L-1}(x)}{(1+x^2)^{(2L+3)/2}}$$

and

$$B_L(x) = \frac{Q_L(x)}{(1+x^2)^{(2L+3)/2}} = \frac{(1+x^2)Q'_{L-1}(x) - (2L+1)xQ_{L-1}(x)}{(1+x^2)^{(2L+3)/2}}.$$

As the degree $L+1$ of the polynomial P_L and the degree L of the polynomial Q_L are less than $(2L+3)/2$ we have

$$\lim_{x\to\pm\infty} \frac{P_L(x)}{(1+x^2)^{(L+3)/2}} = \lim_{x\to\pm\infty} \frac{Q_L(x)}{(1+x^2)^{(L+3)/2}} = 0 \tag{4.16}$$

Proposition 4.5. *Let $L \geq 1$ be an integer.*

(a) The polynomial P_L of degree $L+1$ has $L+1$ distinct real zeros, and the zeros of P_L and P_{L-1} strictly interlace.

(b) The polynomial Q_L of degree L has L distinct real zeros, and the zeros of Q_L and Q_{L-1} strictly interlace.

Proof. We prove only (a), the proof of (b) is identical. The proof is a simple induction on L. The statements is obviously true for $L = 1$. Assume that statement (a) is true for some integer $L-1 \geq 1$. Note that the zeros of P_{L-1} and A_{L-1} are the same. Let us denote the distinct real zeros of P_{L-1} and A_L by $x_1 < x_2 < \cdots < x_L$. The limit relations 4.16 and Rolle's theorem imply that $A_{L+1} = A'_L$ has zeros $y_j \in (x_j, x_{j+1})$, $j = 0, 1, \ldots, L$, where $x_0 := -\infty$ and $x_{L+1} := \infty$. However, the zeros of A_L are the same as the zeros of P_L, which means that statement (a) is true for L. $\square$

Proposition 4.6. *Let $L \geq 1$ be an integer.*

(a) The zeros of zeros of P_L and P'_L strictly interlace.

(b) The zeros of Q_L and Q'_L strictly interlace.

Proof. Suppose a polynomial R of degree L has L distinct zeros $x_1 < x_2 < \cdots < x_L$. Then Rolle's Theorem implies that R' has zeros $y_j \in (x_j, x_{j+1})$, $j = 1, 2, \ldots, L-1$. However, the degree of R' is $L-1$, so R' has $L-1$ distinct zeros. $\square$

Proposition 4.7. *Let $L \geq 1$ be an integer. The zeros of A_L and B_L strictly interlace. Equivalently, the zeros of P_L and Q_L strictly interlace.*

Proof. Observe that

$$A_L(\beta) = \frac{d^L A_0}{d\beta^L} \qquad \text{and} \qquad A_0(\beta) = \beta B_0(\beta)\,,$$

so using the Leibniz formula we obtain

$$A_L(\beta) = L B_{L-1}(\beta) + \beta B_L(\beta)\,,$$

or equivalently

$$P_L(\beta) = L Q_{L-1}(\beta)(1+\beta^2) + \beta Q_L(\beta) \tag{4.17}$$

The fact that P_L and Q_L have no common zeros already follows from it simply, as any such common zero of P_L and Q_L is a common zero of Q_L and Q_{L-1} which is impossible by Proposition 1, as the zeros of Q_L and Q'_L strictly interlace. It is somewhat more subtle to see that the zeros P_L and the zeros of Q_L interlace Denoting the zeros of Q_L by $\beta_1 < \beta_2 < \cdots < \beta_L$, we can deduce from 4.17 that

$$P_L(\beta_{j-1})P_L(\beta_j) < 0\,, \qquad j = 2, 3, \ldots, L\,,$$

since by Proposition 1.1 we already know that Q_{L-1} has exactly one zero in (β_{j-1}, β_j). Hence P_L has at least one zero in each of the intervals (β_{j-1}, β_j), $j = 2, 3, \ldots, L$. Also, it is easy to see by induction that the sign of the leading coefficient of Q_L is $(-1)^L$, hence

$$\lim_{\beta \to \pm\infty} \frac{Q_L(\beta)}{\beta^L} = (-1)^L . \tag{4.18}$$

It follows from 4.17 and 4.18 that P_L has a sign change, and hence at least one zero in (β_L, ∞), and P_L has a sign change, and hence at least one zero in $(-\infty, \beta_1)$. Finally recall that the degree of P_L is $L+1$ and the degree of Q_L is L. In conclusion, there is exactly one zero of Q_L strictly between any two consecutive real zeros of P_L. $\square$

4.8 The Inversion Formula and Interlacing Properties of Polynomials

There is an interesting connection between the values of β in the Type I domains, and the values of α in the Type II domains. This connection is not at first evident because of constraints that need to be made in order that we have convergence of the power series that we consider. But we can observe this now.

Let $L \geq 0$ be an integer. We define

$$\begin{aligned}
A_0(x) &:= \frac{x}{(x^2+1)^{3/2}}, &\qquad B_0(x) &:= \frac{1}{(x^2+1)^{3/2}}, \\
A_L(x) &:= \frac{d^L}{dx^L}\frac{x}{(x^2+1)^{3/2}}, &\qquad B_L(x) &:= \frac{d^L}{dx^L}\frac{1}{(x^2+1)^{3/2}}, \\
C_L(x) &:= \frac{d^L}{dx^L}\frac{x^{L+2}}{(x^2+1)^{3/2}}, &\qquad D_L(x) &:= \frac{d^L}{dx^L}\frac{x^{L+1}}{(x^2+1)^{3/2}}.
\end{aligned}$$

Proposition 4.8. *(Inversion Formula) We have*

$$\operatorname{sgn}(x) x^{L+1} B_L(x) = (-1)^L C_L(1/x), \qquad x \in \mathbb{R} \setminus \{0\}, \qquad L = 0, 1, \ldots .$$

Proof. As both sides are even functions of $x \in \mathbb{R} \setminus \{0\}$, without loss of generality we may assume that $x > 0$. For an integer $L \geq 0$ we define

$$\widetilde{B}_L(x) := x^{L+1} B_L(x), \qquad C^*_L(x) := C_L(1/x), \qquad \widetilde{C}_L(x) := (-1)^L C^*_L(x).$$

Observe that for $L \geq 1$ we have

$$\begin{aligned}
\widetilde{B}_L(x) &= x^{L+1} B_L(x) = x x^L B'_{L-1}(x) \\
&= x(x^L B'_{L-1}(x)) + x(L x^{L-1} B_{L-1}(x)) - L x^L B_{L-1}(x) \\
&= x \widetilde{B}'_{L-1}(x) - L \widetilde{B}_{L-1}(x),
\end{aligned}$$

that is, the functions $\widetilde{B}_L$ satisfy the recursion

$$\widetilde{B}_0(x) = \frac{x}{(1+x^2)^{3/2}}, \tag{4.19}$$

$$\widetilde{B}_L(x) = x \widetilde{B}'_{L-1}(x) - L \widetilde{B}_{L-1}(x), \qquad L = 1, 2, \ldots . \tag{4.20}$$

Observe that

$$\widetilde{C}_0(x) = \frac{x}{(x^2+1)^{3/2}}, \tag{4.21}$$

and the Leibniz formula yields

$$C_L(x) = \frac{d^L}{dx^L}\frac{x^{L+1}x}{(x^2+1)^{3/2}} = xC'_{L-1}(x) + LC_{L-1}(x)\,, \qquad L = 1, 2, \dots\,. \tag{4.22}$$

Replacing x by $1/x$ we get

$$C_L(1/x) = (1/x)C'_{L-1}(1/x) + LC_{L-1}(1/x)\,, \qquad L = 1, 2, \dots\,,$$

that is,

$$C^*_L(x) = -xC^{*\prime}_{L-1}(x) + LC^*_{L-1}(x)\,, \qquad L = 1, 2, \dots\,,$$

and hence

$$\widetilde{C}_L(x) = x\widetilde{C}'_{L-1}(x) - L\widetilde{C}_{L-1}(x)\,, \qquad L = 1, 2, \dots\,. \tag{4.23}$$

Now observe that by 4.19, 4.20, 4.21, and 4.23 the functions $\widetilde{B}_L$ and $\widetilde{C}_L$ satisfy the same recursion. In conclusion

$$\widetilde{B}_L(x) = \widetilde{C}_L(x)\,, \qquad L = 0, 1, \dots\,,$$

and the lemma follows. □

Proposition 4.9. *Let $L \geq 0$ be an integer. The functions C_L and D_L do not have a common zero different from* 0.

Proof. The statement is obvious for $L = 0$. Let $L \geq 1$ be an integer. Observe that

$$D_L(x) = C'_{L-1}(x)\,,$$

and hence 4.22 can be written as

$$C_L(x) = xD_L(x) + LC_{L-1}(x)\,, \qquad L = 1, 2, \dots\,.$$

Therefore if $\alpha \neq 0$ is a common zero of C_L and D_L, then $\alpha \neq 0$ is a common zero of C_L and C_{L-1}, and hence $\alpha \neq 0$ is a common zero of C_{L-1} and C'_{L-1}. By the Inversion Formula this means that a is a common zero of

$$\operatorname{sgn}(x)\widetilde{B}_{L-1}(1/x) = x^{-L}B_{L-1}(1/x)$$

and

$$\frac{d}{dx}\widetilde{B}_{L-1}(1/x) = -x^{-L-2}B'_{L-1}(1/x) + (-L)x^{-L-1}B_{L-1}(1/x)\,.$$

This means that $1/\alpha$ is a common zero of B_{L-1} and B_L. However, this is impossible. Indeed, as as we have seen before, the Leibniz formula implies that

$$A_L(x) = LB_{L-1}(x) + xB_L(x)\,,$$

and hence any common zero of B_{L-1} and B_L is a common zero of A_L and B_L. However, the zeros of A_L and B_L strictly interlace by Proposition 1.3 of the previous section. □

4.9 The Boundary Case $y = \pm x$

In a forthcoming paper we may be able to prove that $y = \pm x$ cannot be a common asymptotic direction to both of the zero sets $\{X = 0\}$ and $\{Y = 0\}$. Originally we needed this to argue that the zero set $\{X = Y = 0\}$ is bounded. However, Proposition 3.4 already implies that in the Special Case the zero set $\{X = Y = 0\}$ is bounded, as it is a finite set of points.

4.10 Final Conclusion

Combining the results in Section 4.2 through Section 4.9, we conclude that in the Special Case the set of possible asymptotic directions of the zero set $\{(x,y) : X(x,y) = 0\}$ and the set of possible asymptotic directions of the zero set $\{(x,y) : Y(x,y) = 0\}$ are distinct, except possibly the lines $y = \pm x$. Whether or not the lines $y = \pm x$ can be a common asymptotic direction to both of the zero sets $\{X = 0\}$ and $\{Y = 0\}$ remains open in this paper.

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Authors:

T. Erdélyi, Department of Mathematics, Texas A & M University, College Station, TX, terdelyi@math.tamu.edu

J. Rosenblatt, Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL, rosnbltt@illinois.edu

R. Rosenblatt, Education and Human Resources at the National Science Foundation, Alexandria, VA,
rrosenbl@nsf.gov

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