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A Markov-Type Inequality for the Derivatives of Constrained Polynomials

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Markov's inequality asserts that

$$\max_{-1\leqslant x\leqslant 1}|p'(x)|\leqslant n^2\max_{-1\leqslant x\leqslant 1}|p(x)| \tag{1}$$

for every polynomial of degree at most n. The magnitude of

$$\sup_{p\in S}\frac{\max_{-1\leqslant x\leqslant 1}|p'(x)|}{\max_{-1\leqslant x\leqslant 1}|p(x)|} \tag{2}$$

was examined by several authors for certain subclasses S of Π_n. In this paper we introduce $S=S_n^m(r)$ $(0\leqslant m\leqslant n,\ 0<r\leqslant 1)$, the set of those polynomials from Π_n which have all but at most m zeros outside the circle with center 0 and radius r, and establish the exact order of the above expression up to a multiplicative constant depending only on m.

1. INTRODUCTION, NOTATIONS

Denote the set of all real algebraic polynomials of degree at most n by Π_n. Let $P_n^m(r)$ $(0\leqslant m\leqslant n,\ r>0)$ be the set of those polynomials from Π_n which have only real zeros, at most m of which are in $(-r,r)$. In 1940 P. Erdős [5] proved that

$$\max_{-1\leqslant x\leqslant 1}|p'(x)|\leqslant\frac{e}{2}n\max_{-1\leqslant x\leqslant 1}|p(x)| \tag{3}$$

for every polynomial from $P_n^0(1)$. Let $K(r)=\{z\in\mathbf{C}\colon |z|<r\}$ and denote by $S_n^m(r)$ $(0\leqslant m\leqslant n,\ 0<r\leqslant 1)$ the set of those polynomials from Π_n which

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have all but at most m zeros outside $K(r)$. In 1963 G. G. Lorentz [6] defined the class

$$P_n(a, b) = \left\{ p\colon p(x) = \sum_{j=1}^{n} a_j (x-a)^j (b-x)^{n-j} \text{ with all } a_j \geqslant 0 \right\}$$

and proved that

$$\max_{-1 \leqslant x \leqslant 1} |p^{(k)}(x)| \leqslant c(k)\, n^k \max_{-1 \leqslant x \leqslant 1} |p(x)| \qquad (p \in P_n(-1, 1)) \tag{4}$$

with a constant depending only on k. He observed the relation $S_n^0(1) \subset P_n(-1, 1)$ as well. For the first derivative J. T. Scheick [7] extended Erdős' inequality for polynomials from $P_n(-1, 1)$ with the best possible constant $e/2$. In [3], T. Erdélyi proved the sharp inequality

$$\max_{-1 \leqslant x \leqslant 1} |p^{(k)}(x)| \leqslant c(k) \min\{n^2, nr^{-1/2}\}^k \max_{-1 \leqslant x \leqslant 1} |p(x)| \tag{5}$$

for polynomials of degree at most n having no zeros in the union of the circles with diameters $[-1, -1+2r]$ and $[1-2r, 1]$, respectively $(0 < r < 1)$. In this paper we examine the magnitude of

$$\sup_{p \in S} \frac{\max_{-1 \leqslant x \leqslant 1} |p'(x)|}{\max_{-1 \leqslant x \leqslant 1} |p(x)|},$$

where $S = S_n^m(r)$ $(0 \leqslant m \leqslant n,\ 0 < r \leqslant 1)$ and establish the exact order up to a multiplicative constant depending only on m. The theorem we prove is a common generalization of Markov's inequality $(r = 0,\ m = 0)$ and Lorentz's result $(r = 1,\ m = 0)$.

2. New Result

THEOREM. *For every $0 < r \leqslant 1$ and $0 \leqslant m \leqslant n$ we have*

$$c_1(m)(n + (1-r)n^2) \leqslant \sup_{p \in S_n^m(r)} \frac{\max_{-1 \leqslant x \leqslant 1} |p'(x)|}{\max_{-1 \leqslant x \leqslant 1} |p(x)|} \leqslant c_2(m)(n + (1-r)n^2),$$

where $c_1(m)$ and $c_2(m)$ depend only on m.

3. Lemmas for the Theorem

To prove our theorem we need several lemmas. First we deal with the upper bound. The crux of the proof is to give the desired upper bound for $|p'(1)|$, from this we will deduce the right hand side inequality easily. Our first lemma guarantees the existence of an extremal polynomial with some additional properties. Let

$$\tilde{K}(r) = \{z \in \mathbf{C} : |z - r/2| < r/2\}$$

and denote by $\tilde{S}_n^m(r)$ $(0 \leqslant m \leqslant n,\ 0 < r \leqslant 1)$ the set of those polynomials from Π_n which have all but at most m zeros outside $\tilde{K}(r)$.

Lemma 1. *Let* $0 < r \leqslant 1$ *and* $0 \leqslant m \leqslant n$. *There exists a polynomial* $Q_n \in \tilde{S}_n^m(r)$ *with the following properties*:

(i) $|Q_n'(1)|/\max_{0 \leqslant x \leqslant 1} |Q_n(x)| = \sup_{p \in \tilde{S}_n^m(r)} (|p'(1)|/\max_{0 \leqslant x \leqslant 1} |p(x)|)$.

(ii) Q_n *has all but at most m zeros in the set* $\{z \in \mathbf{C} : |z - r/2| = r/2\} \cup [r, 1]$, *and the remaining at most m zeros are in* $(0, r)$.

To formulate our next lemma we need to introduce a number of notations. According to Lemma 1, Q_n is of the form

$$Q_n(x) = cx^\alpha \prod_{j=1}^{\beta} (x - z_j)(x - \bar{z}_j) \prod_{j=1}^{\gamma} (x - x_j) \prod_{j=1}^{\delta} (x - y_j), \tag{6}$$

where

$$|z_j - r/2| = r/2, \qquad z_j \notin \mathbf{R}\ (1 \leqslant j \leqslant \beta), \tag{7}$$

$$x_j \in [r, 1] \qquad (1 \leqslant j \leqslant \gamma), \tag{8}$$

$$y_j \in (0, r) \qquad (1 \leqslant j \leqslant \delta \leqslant m), \tag{9}$$

$$s := \alpha + 2\beta + \gamma + \delta \leqslant n. \tag{10}$$

Observe that (7) implies

$$(x - z_j)(x - \bar{z}_j) = \mu_j x^2 + v_j (r - x)^2 \qquad (\mu_j, v_j \geqslant 0,\ 1 \leqslant j \leqslant \beta),$$

from which we deduce

$$\prod_{j=1}^{\beta} (x - z_j)(x - \bar{z}_j) = \sum_{j=0}^{\beta} a_{2j} x^{2j} (r - x)^{2\beta - 2j} \tag{11}$$

with

$$a_{2j} \geqslant 0 \qquad (0 \leqslant j \leqslant \beta). \tag{12}$$

From (6) and (11) with $c=1$ we obtain

$$Q_n(x) = \sum_{j=0}^{\beta} a_{2j} q_j(x) \tag{13}$$

with

$$q_j(x) = x^{\alpha+2j}(r-x)^{2\beta-2j} \prod_{j=1}^{\gamma} (x-x_j) \prod_{j=1}^{\delta} (x-y_j) \qquad (0 \leqslant j \leqslant \beta). \tag{14}$$

For the sake of brevity let

$$\gamma_j := 2\beta - 2j + \gamma. \tag{15}$$

Further we introduce

$$r^* := \min\left\{r, 1 - \frac{1}{10s}\right\} \qquad (0 < r \leqslant 1), \tag{16}$$

where $s \geqslant 1$ is defined by (10). Choose a

$$z^* \in [1 - 5(1-r^*), 1 - 4(1-r^*)] \tag{17}$$

such that

$$|z^* - y_j| \geqslant \frac{1-r^*}{2(m+1)} \qquad (1 \leqslant j \leqslant \delta \leqslant m). \tag{18}$$

From (17) and (18) we easily deduce

$$|z^* - y_j| \geqslant \frac{1-r^*}{2(m+1)} \geqslant \frac{1-z^*}{10(m+1)} \qquad (1 \leqslant j \leqslant \delta)$$

which gives

$$|z^* - y_j| \geqslant \frac{1-y_j}{10m+11} \qquad (1 \leqslant j \leqslant \delta). \tag{19}$$

Using the notations introduced in (6)–(19) we can establish

LEMMA 2. *Let* $9/10 \leqslant r \leqslant 1$. *If an index* $0 \leqslant j \leqslant \beta$ *satisfies*

$$\gamma_j \geqslant 20s(1-r^*), \tag{20}$$

then we have

$$|q_j'(1)| \leqslant c_3(m) s \, |q_j(z^*)|,$$

where $c_3(m)$ *is a constant depending only on* m.

Our following lemma is a slight extension [2, Corollary 3.1] of a deep theorem of Borwein [1]. We will not prove it in this paper.

LEMMA 3. *If* $p \in \Pi_n$ *has at most* k $(0 \leqslant k \leqslant n)$ *zeros in the open circle with centre and radius* $1/2$, *then*

$$\max_{0 \leqslant x \leqslant 1} |p'(x)| \leqslant 18n(k+1) \max_{0 \leqslant x \leqslant 1} |p(x)|,$$

and this inequality is sharp up to the constant 18.

This result was conjectured by J. Szabados [8], and he showed it would be sharp. P. Borwein [1] proved Lemma 3 under the additional assumption that p has only real zeros. In [2, Corollary 1.3] Lemma 3 was shown without this additional assumption.

Remark. In [4], Lemma 4 was generalized for higher derivatives. Namely the sharp inequality

$$\max_{0 \leqslant x \leqslant 1} |p^{(j)}(x)| \leqslant c(j)(n(k+1))^j \max_{0 \leqslant x \leqslant 1} |p(x)|$$

holds for every polynomial $p \in \Pi_n$ which has at most k $(0 \leqslant k \leqslant n)$ zeros in the open circle with centre and radius $1/2$. This result does not follow from Lemma 3 by a simple induction on j.

4. PROOF OF THE LEMMAS

Proof of Lemma 1. Let $0 < \eta < 1$ be fixed. We first consider the corresponding extremal problem for the uniform norm on $[0, \eta]$,

$$\frac{|Q_{n,\eta}'(1)|}{\max_{0 \leqslant x \leqslant \eta} |Q_{n,\eta}(x)|} = \sup_{p \in \tilde{S}_n^m(r)} \frac{|p'(1)|}{\max_{0 \leqslant x \leqslant \eta} |p(x)|}. \tag{21}$$

The subset of polynomials in $\tilde{S}_n^m(r)$ whose uniform norm on $[0, \eta]$ is bounded by 1 is compact and the operator $p \to p'(1)$ is continuous on this subset. This guarantees the existence of maximal $Q_{n,\eta}$ in (21). To prove that (ii) holds for $Q_{n,\eta}$, first we show that $Q_{n,\eta}(z_1) = 0$, $z_1 \notin \mathbf{R}$ imply

$|z_1 - r/2| = r/2$. Suppose indirectly that $Q_{n,\eta}(z_1) = 0$, $z_1 \notin \mathbf{R}$, and $|z_1 - r/2| \neq r/2$. Then the polynomial

$$p_\varepsilon(x) = Q_{n,\eta}(x) - \frac{\varepsilon(x-1)^2}{(x-z_1)(x-\bar{z}_1)} Q_{n,\eta}(x)$$

with a sufficiently small $\varepsilon > 0$ contradicts the maximality of $Q_{n,\eta}$. Now we prove that $Q_{n,\eta}(z_1) = 0$, $z_1 \in \mathbf{R} \setminus (0, r)$ imply either $z_1 = 0$ or $z_1 \in [r, 1]$. By the just proved part of the lemma, $Q_{n,\eta}$ is of the form

$$Q_{n,\eta}(x) = c \prod_{j=1}^{\beta} (x - z_j)(x - \bar{z}_j) \prod_{j=1}^{\gamma} (x - x_j) \prod_{j=1}^{\delta} (x - y_j), \tag{22}$$

where

$$|z_j - r/2| = r/2; \qquad z_j \notin \mathbf{R}\ (1 \leqslant j \leqslant \beta); \qquad x_1 \leqslant x_2 \leqslant \cdots \leqslant x_\gamma \in \mathbf{R} \setminus (0, r);$$
$$y_j \in (0, r); \qquad (1 \leqslant j \leqslant \delta \leqslant m); \qquad 2\beta + \gamma + \delta \leqslant n; \qquad c \neq 0.$$

To finish the proof of the lemma we show that $x_1 < 0$ or $x_\gamma > 1$ contradicts the maximality of $Q_{n,\eta}$. To see this we distinguish three cases.

Case 1. $Q_{n,\eta}$ has at least two zeros (counting multiplicities) in $\mathbf{R} \setminus [0, 1]$. Denote these (not necessarily different) two zeros by α_1 and α_2. Then the polynomial

$$P_\varepsilon(x) = Q_{n,\eta}(x) - \varepsilon \operatorname{sign}(\alpha_1 \alpha_2) \frac{(x-1)^2}{(x-\alpha_1)(x-\alpha_2)} Q_{n,\eta}(x)$$

with a sufficiently small $\varepsilon > 0$ contradicts the maximality of $Q_{n,\eta}$.

Case 2. $x_\gamma > 1$, $x_{\gamma-1} \leqslant 1$. Then we can choose a $u \geqslant 1$ such that

$$\left(\frac{x-u}{x-x_\gamma} Q_{n,\eta}(x) \right) (1) = 0.$$

To see this we introduce the polynomial $f_u(x) = ((x-u)/(x-x_\gamma))\, Q_{n,\eta}(x)$. If $x_{\gamma-1} = 1$, then $u = 1$ is suitable. If $x_{\gamma-1} < 1$, then

$$\frac{f'_u(1)}{f_u(1)} = \left\{ \sum_{j=1}^{\beta} \left(\frac{1}{1-z_j} + \frac{1}{1-\bar{z}_j} \right) + \sum_{j=1}^{\delta} \frac{1}{1-y_j} + \sum_{j=1}^{\gamma-1} \frac{1}{1-x_j} \right\} + \frac{1}{1-u}.$$

Since $\{\cdot\}$ is positive, $u > 1$ can be chosen so that $f'_u(1)/f_u(1) = 0$ holds.

Therefore the polynomial

$$P_\varepsilon(x) = Q_{n,\eta}(x) - \varepsilon \frac{x-u}{x-x_\gamma} Q_{n,\eta}(x)$$

with a sufficiently small $\varepsilon > 0$ contradicts the maximality of $Q_{n,\eta}$.

Case 3. $x_1 < 0$, $x_2 \geqslant 0$, $x_\gamma \leqslant 1$. Then similarly to Case 2, we can choose a $u \geqslant 1$ such that

$$\left(\frac{x-u}{x-x_1} Q_{n,\eta}(x)\right)' (1) = 0,$$

therefore the polynomial

$$P_\varepsilon(x) = Q_{n,\eta}(x) + \varepsilon \frac{x-u}{x-x_1} Q_{n,\eta}(x)$$

with sufficiently small $\varepsilon > 0$ contradicts the maximality of $Q_{n,\eta}$. ∎

Proof of Lemma 2. Recalling (14), (8), (9), and (16) we easily get

$$|q_j'(1)| \leqslant \Bigg[(\alpha + 2j)(1 - r^*)^{\gamma_j} + \gamma_j (1 - r^*)^{\gamma_j - 1} + (1 - r^*)^{\gamma_j} \sum_{i=1}^{\delta} \frac{1}{1 - y_i} \Bigg] \prod_{i=1}^{\delta} (1 - y_i) \qquad (0 \leqslant j \leqslant \beta). \tag{23}$$

Further, using (14), (17), (16), (8), (9), $9/10 \leqslant r^* \leqslant 1$, $1 - x \geqslant e^{-2x}$ $(0 \leqslant x \leqslant 0{,}7)$, (19), (20), (10), and $\delta \leqslant m$, we obtain

$$\begin{aligned} |q_j(z^*)| &\geqslant (1 - 5(1 - r^*))^{\alpha + 2j} (3(1 - r^*))^{\gamma_j} \prod_{i=1}^{\delta} |z^* - y_i| \\ &\geqslant \exp(-10(\alpha + 2j)(1 - r^*))\, 3^{\gamma_j} (1 - r^*)^{\gamma_j} \\ &\quad \times \left(\frac{1}{10m + 11}\right)^{\delta} \prod_{i=1}^{\delta} (1 - y_i) \\ &\geqslant \left(\frac{3}{e}\right)^{10(\alpha + 2j)(1 - r^*)} 3^{\gamma_j/2} (1 - r^*)^{\gamma_j} c_4(m) \prod_{i=1}^{\delta} (1 - y_i) \\ &\geqslant c_4(m)\, 3^{\gamma_j/2} (1 - r^*)^{\gamma_j} \prod_{i=1}^{\delta} (1 - y_i). \end{aligned} \tag{24}$$

Thus (23), (24), (10), $\delta \leqslant m$, and (16) yield

$$\frac{|q_j'(1)|}{|q_j(z^*)|} \leqslant \frac{3^{-\gamma_j/2}}{c_4(m)} \left(\alpha + 2j + \frac{\gamma_j}{1-r^*} + \frac{\delta}{1-r^*} \right)$$
$$\leqslant c(m)(s + 10s + 10sm)\, c_3(m) s,$$

thus the Lemma is proved. ∎

5. Proof of the Upper Estimate of the Theorem

If $0 < r \leqslant 9/10$ the Markov inequality (1) gives the desired result without exploiting any information on the zeros. Therefore in the sequel we assume that

$$\tfrac{9}{10} < r \leqslant 1. \tag{25}$$

First we give the desired upper bound for $|Q_n'(1)|$ where Q_n is the extremal polynomial defined by Lemma 1. Recalling the representation (10) we split the sum in (13) as

$$Q_n(x) = p_1(x) + p_2(x), \tag{26}$$

where

$$p_1(x) = \sum_{\substack{j=0 \\ \gamma_j \geqslant 20s(1-r^*)}}^{\beta} a_{2j} q_j(x) \tag{27}$$

and

$$p_2(x) = \sum_{\substack{j=0 \\ \gamma_j < 20s(1-r^*)}}^{\beta} a_{2j} q_j(x). \tag{28}$$

By Lemma 3, (27), (26), (12), (16), (17), and (25) we easily deduce

$$|p_1'(1)| \leqslant c_3(m)\, s |p_1(z^*)| \leqslant c_3(m) s |Q_n(z^*)|$$
$$\leqslant c_3(m) s \max_{0 \leqslant x \leqslant 1} |Q_n(x)|. \tag{29}$$

Now observe that $p_2(x)$ is a polynomial of degree at most n, which has all but at most $[20s(1-r^*)] + m$ zeros at 0 (see (28), (14), (15), (9), and (10)), so using Lemma 3 and (16), we obtain

$$|p_2'(1)| \leqslant 18s(20s(1-r^*) + m + 1) \max_{0 \leqslant x \leqslant 1} |p_2(x)|$$
$$\leqslant c_5(m)(s + (1-r)\, s^2) \max_{0 \leqslant x \leqslant 1} |p_2(x)|. \tag{30}$$

It is easy to see that (26), (27), (28), and (12) imply

$$\max_{0 \leq x \leq 1} |p_2(x)| \leq \max_{0 \leq x \leq 1} |Q_n(x)|,$$

hence (30) yields

$$|p_2'(1)| \leq c_5(m)(s + (1-r)\, s^2) \max_{0 \leq x \leq 1} |Q_n(x)|. \tag{31}$$

From (26), (29), (31), and (10) we conclude

$$|Q_n'(1)| \leq c_6(m)(n + (1-r)\, n^2) \max_{0 \leq x \leq 1} |Q_n(x)|, \tag{32}$$

therefore by the maximality of Q_n we have

$$|p'(1)| \leq c_6(m)(n + (1-r)\, n^2) \max_{0 \leq x \leq 1} |p(x)| \qquad (p \in \tilde{S}_n^m(r)). \tag{33}$$

From (33), by a linear transformation we easily deduce

$$|p'(y)| \leq c_7(m)(n + (1-r)\, n^2) \max_{0 \leq x \leq 1} |p(x)| \qquad (p \in \tilde{S}_n^m(r),\ r < y \leq 1). \tag{34}$$

Furthermore, after a linear transformation Lemma 3 yields

$$\begin{aligned} |p'(y)| &\leq \frac{18}{r}\, n(m+1) \max_{0 \leq x \leq r} |p(x)| \\ &< 20(m+1)n \max_{0 \leq x \leq r} |p(x)| \\ &\qquad (p \in \tilde{S}_n^m(r),\ 9/10 < r \leq 1,\ 0 \leq y \leq r). \end{aligned} \tag{35}$$

Now (34) and (35) show that

$$\max_{0 \leq x \leq 1} |p'(x) \leq c_8(m)(n + (1-r)\, n^2) \max_{0 \leq x \leq 1} |p(x)| \qquad (p \in S_n^m(r) \subset \tilde{S}_n^m(r)) \tag{36}$$

and by reason of symmetry this gives the upper estimate of the theorem when $9/10 < r < 1$. ∎

6. Remark on the Higher Derivatives

Observing that $p \in P_n^m(r)$ $(0 \leqslant m \leqslant n-1,\ 0 < r \leqslant 1)$ implies $p \in P_n^{m+1}(r)$, from the result of Section 5, by induction on m we obtain

Corollary 1. *We have*

$$\max_{-1 \leqslant x \leqslant 1} |p^m(x)| \leqslant c_9(m)(n+(1-r)\,n^2)^m \max_{-1 \leqslant x \leqslant 1} |p(x)|$$

for every $p \in P_n^0(r)$.

7. Proof of the Lower Estimate of the Theorem and the Sharpness of Corollary 1

In this section we prove that

$$\sup_{p \in P_n^0(r)} \frac{\max_{-1 \leqslant x \leqslant 1} |p^{(m)}(x)|}{\max_{-1 \leqslant x \leqslant 1} |p(x)|} \geqslant c_{10}(m)(n+(1-r)\,n^2)^m. \tag{37}$$

To show this we distinguish two cases.

Case 1. $0 < r < 1 - 8m/n$. Then we can choose an integer $m \leqslant k \leqslant n$ such that

$$1 - \frac{8(k+1)}{n} \leqslant r < 1 - \frac{8k}{n} \tag{38}$$

and let

$$T_k(x) = \frac{1}{2^{k-1}} \cos(k \operatorname{arc\,cos} x) = \prod_{j=1}^{k} (x - v_j) \tag{39}$$

$$-1 < v_1 < v_2 < \cdots < v_k < 1.$$

We introduce the polynomial

$$q(x) = \left(x + \frac{n-2k}{2k}\right)^{n-k} T_k(x). \tag{40}$$

Observe that

$$|T_k(x)| \leqslant \max_{-1 \leqslant y \leqslant 1} |T_k(y)| = \frac{1}{2^{k-1}} < 2(1-x)^k \qquad (-1 \leqslant x \leqslant 1/2), \tag{41}$$

and from (39) we deduce

$$|T_k(x)| \leqslant (1-x)^k \qquad (x < -1). \tag{42}$$

Since $x = 1/2$ is the only point in $((2k-n)/(2k), 1)$ where

$$\frac{d}{dx}\left(\left(x+\frac{n-2k}{2k}\right)^{n-k}(1-x)^k\right)$$

vanishes, recalling (40), (41), and (42), we conclude

$$\begin{aligned} |q(x)| &\leqslant 2\left(x+\frac{n-2k}{2k}\right)^{n-k}(1-x)^k \\ &\leqslant \left(\frac{1}{2}+\frac{n-2k}{2k}\right)^{n-k}\left(\frac{1}{2}\right)^k \\ &\leqslant q(1) \qquad \left(\frac{2k-n}{2k} \leqslant x \leqslant \frac{1}{2}\right). \end{aligned} \tag{43}$$

Apparently

$$|q(x)| \leqslant q(1) \qquad (\tfrac{1}{2} \leqslant x \leqslant 1) \tag{44}$$

and this together with (43) yields

$$\max_{(2k-n)/(2k) \leqslant y \leqslant 1} |q(x)| = q(1). \tag{45}$$

Now let

$$p(y) = q\left(\frac{n}{4k}y + \frac{4k-n}{4k}\right) \tag{46}$$

and

$$\eta_j = \frac{4k}{n}\cos\frac{(k-j)\pi}{k} + \frac{n-4k}{n} \qquad (0 \leqslant j \leqslant k). \tag{47}$$

From (45), (46), (39), and (40) we easily deduce that

$$\max_{-1 \leqslant y \leqslant 1} |p(y)| = p(1) \tag{48}$$

and p has all its zeros at -1 or in $(1-8k/n, 1)$, hence by (38) we get

$$p \in P_n^0(r). \tag{49}$$

By (40), (46), (47), (48), $\cos x \geqslant 1 - x^2/2$, and $8k < n$ it is easy to see that

$$\begin{aligned}|p(\eta_j)| &= \left(\cos\frac{(k-j)\pi}{k} + \frac{n-2k}{2k}\right)^{n-k}\left(\frac{1}{2}\right)^{k-1}\\ &\geqslant c_{11}(m)\left(1+\frac{n-2k}{2k}\right)^{n-k}\left(\frac{1}{2}\right)^{k-1} = c_{11}(m)\,p(1)\\ &= c_{11}(m)\max_{-1\leqslant y\leqslant 1}|p(y)| \qquad (k-m\leqslant j\leqslant k) \end{aligned}\tag{50}$$

and

$$\operatorname{sgn} p(\eta_j) = -\operatorname{sgn} p(\eta_{j+1}) \qquad (k-m\leqslant j\leqslant k-1). \tag{51}$$

By (47) and $\cos x \geqslant 1 - x^2/2$ we obtain

$$1-\eta_j \leqslant \frac{c_{12}(m)}{nk} \qquad (k-m\leqslant j\leqslant k). \tag{52}$$

Let $\Omega(x) = \prod_{j=k-m}^{k}(x-\eta_j)$, then (52) implies

$$|\Omega'(\eta_j)| \leqslant \left(\frac{c_{12}(m)}{nk}\right)^m \qquad (k-m\leqslant j\leqslant k) \tag{53}$$

and obviously

$$\operatorname{sgn}\Omega'(\eta_j) = -\operatorname{sgn}\Omega'(\eta_{j+1}) \qquad (k-m\leqslant j\leqslant k-1). \tag{54}$$

Using (50), (51), (53), and (54), by a well-known relation for the mth order divided differences, we obtain that there exists a suitable $\xi \in [\eta_{k-m}, 1]$ such that

$$\begin{aligned}|p^{(m)}(\xi)| &= m!\left|\sum_{j=k-m}^{k}\frac{p(\eta_j)}{\Omega'(\eta_j)}\right| = m!\sum_{j=k-m}^{k}\left|\frac{p(\eta_j)}{\Omega'(\eta_j)}\right|\\ &\geqslant c_{13}(m)(nk)^m \max_{-1\leqslant y\leqslant 1}|p(y)|\\ &\geqslant c_{14}(m)(n+(1-r)\,n^2)^m \max_{-1\leqslant y\leqslant 1}|p(y)|,\end{aligned}$$

which together with (49) proves (37).

Case 2. $1 - 8m/n < r \leqslant 1$. Then (37) holds obviously by taking the polynomials $(1+x)^n$. This completes the proof of the theorem. ∎

8. Remark on the Case $r > 1$

Here we discuss what happens in the Theorem when $r > 1$. This turns out to be much easier than the case $0 < r \leqslant 1$.

PROPOSITION. *For $1 \leqslant m \leqslant n$ and $r > 1$ we have*

$$c_{15}(m)\left(\frac{n}{r}\right)^m \leqslant \sup_{p \in S_n^0(r)} \frac{\max_{-1 \leqslant x \leqslant 1} |p^{(m)}(x)|}{\max_{-1 \leqslant x \leqslant 1} |p(x)|} \leqslant c_{16}(m)\left(\frac{n}{r}\right)^m,$$

where $c_{15}(m)$ and $c_{16}(m)$ depend only on m.

Proof of the Proposition. The left hand side inequality can be obtained by taking the polynomials $(x+r)^n$. When $1 < r \leqslant 2$ the right hand side inequality follows from Lorentz's Theorem (see Theorem B in [3]) and the observation that a polynomial $p \in S_n^0(1)$ has the representation

$$p(x) = \sum_{j=1}^{n} a_j(1-x)^j (1+x)^{n-j} \qquad \text{with all } a_j \geqslant 0 \text{ or all } a_j \leqslant 0.$$

Now let $r > 2$. Observe that $p \in S_n^0(r)$ can be written as

$$p(x) = \sum_{j=1}^{n} a_j q_{n,j} \qquad \text{with all } a_j \geqslant 0 \text{ or all } a_j \leqslant 0, \tag{55}$$

where

$$q_{n,j}(x) = (r-x)^j (r+x)^{n-j}. \tag{56}$$

By Rolle's Theorem $q_{n,j}^{(m)}$ $(1 \leqslant m \leqslant n)$ has all its zeros in $[-r, r]$, so a simple calculation shows

$$\frac{|q_{n,j}^{(m)}(x)|}{|q_{n,j}(x)|} \leqslant \frac{n!}{(n-m)!} \frac{(r+1)^m}{(r-1)^m} \leqslant c_{17}(m)\left(\frac{n}{r}\right)^m \qquad (|x| \leqslant 1,\ 1 \leqslant m \leqslant n,\ r > 2). \tag{57}$$

Thus (55), (56), and (57) yield

$$\frac{|p^{(m)}(x)|}{\max_{-1 \leqslant x \leqslant 1} |p(x)|} \leqslant \frac{|p^{(m)}(x)|}{|p(x)|} \leqslant c_{17}(m)\left(\frac{n}{r}\right)^m \qquad (|x| \leqslant 1,\ 1 \leqslant m \leqslant n,\ r > 2)$$

which gives the Proposition. ∎

9. Further Problems

Our theorem does not show any improvement to the Markov inequality if $r > 0$ is small, e.g., $0 < r < 1/2$. P. Erdős raised the following

Conjecture. For all $0 < r < 1$ there exists a constant $c(r) < 1$ depending only on r such that

$$\max_{-1 \leqslant x \leqslant 1} |p'(x)| \leqslant c(r)\, n^2 \max_{-1 \leqslant x \leqslant 1} |p(x)| \qquad (p \in S_n^0(r),\ n \geqslant n_0(r)).$$

At present this is an open problem.

Problem. Let S be the collection of those polynomials of degree at most n, which have no zeros in the region bounded by the lines $y = \pm x \pm 1$. The order of

$$\sup_{p \in S} \frac{\max_{-1 \leqslant x \leqslant 1} |p'(x)|}{\max_{-1 \leqslant x \leqslant 1} |p(x)|} \tag{58}$$

is obviously between $O(n)$ and n^2. What is the exact order of (58)? The author was not able to prove even that the order of (58) is $o(n^2)$ but conjectures that it is $O(n)$.

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