LOWER BOUNDS FOR THE MERIT FACTORS OF TRIGONOMETRIC POLYNOMIALS FROM LITTLEWOOD CLASSES

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Abstract. With the notation $K := \mathbb{R} \pmod{2\pi}$,

$$\|p\|_{L_\lambda(K)} := \left(\int_K |p(t)|^\lambda \, dt\right)^{1/\lambda} \qquad \text{and} \qquad M_\lambda(p) := \left(\frac{1}{2\pi}\int_K |p(t)|^\lambda \, dt\right)^{1/\lambda}$$

we prove the following result.

Theorem 1. *Assume that p is a trigonometric polynomial of degree at most n with real coefficients that satisfies*

$$\|p\|_{L_2(K)} \leq An^{1/2} \qquad \textit{and} \qquad \|p'\|_{L_2(K)} \geq Bn^{3/2}\,.$$

Then

$$M_4(p) - M_2(p) \geq \varepsilon M_2(p)$$

with

$$\varepsilon := \left(\frac{1}{111}\right)\left(\frac{B}{A}\right)^{12}\,.$$

We also prove that

$$M_\infty(1+2p) - M_2(1+2p) \geq (\sqrt{4/3}-1)M_2(1+2p)$$

and

$$M_2(p) - M_1(p) \geq 10^{-31} M_2(p)$$

for every $p \in \mathcal{A}_n$, where $\mathcal{A}_n$ denotes the collection of all trigonometric polynomials of the form

$$p(t) := p_n(t) := \sum_{j=1}^n a_j \cos(jt + \alpha_j)\,, \qquad a_j = \pm 1\,, \quad \alpha_j \in \mathbb{R}\,.$$

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Introduction

We give shorter and more direct proofs of some of the main results from Littlewood's papers [Li-61], [Li-62], [Li-66a], [Li-66b], and [Li-68]. There are two reasons for doing this. First our approaches are, we believe, much easier, and secondly they lead to explicit constants. Littlewood himself remarks that his methods were "extremely indirect." Motivation and discussion of these types of results may be found in [Bo-02]. Kahane's paper [Ka-85] is also central among those related to the subject of this paper.

2. New Results

We use the notation $K := \mathbb{R} \pmod{2\pi}$. Let

$$\|p\|_{L_\lambda(K)} := \left(\int_K |p(t)|^\lambda \, dt\right)^{1/\lambda} \quad \text{and} \quad M_\lambda(p) := \left(\frac{1}{2\pi}\int_K |p(t)|^\lambda \, dt\right)^{1/\lambda}.$$

Theorem 1. *Assume that p is a trigonometric polynomial of degree at most n with real coefficients that satisfies*

$$\|p\|_{L_2(K)} \le An^{1/2} \tag{1}$$

and

$$\|p'\|_{L_2(K)} \ge Bn^{3/2}\,. \tag{2}$$

Then

$$M_4(p) - M_2(p) \ge \varepsilon M_2(p)$$

with

$$\varepsilon := \left(\frac{1}{111}\right)\left(\frac{B}{A}\right)^{12}.$$

Let the Littlewood class $\mathcal{A}_n$ be the collection of all trigonometric polynomials of the form

$$p(t) := p_n(t) := \sum_{j=1}^{n} a_j \cos(jt + \alpha_j)\,, \qquad a_j = \pm 1\,, \quad \alpha_j \in \mathbb{R}.$$

Note that for the Littlewood class $\mathcal{A}_n$ we have

$$\left(\frac{B}{A}\right)^{12} = 3^{-6}\,.$$

Corollary 2. *We have*

$$M_4(p) - M_2(p) \ge \frac{M_2(p)}{80920}$$

for every $p \in \mathcal{A}_n$. The merit factor

$$\left(\frac{M_4^4(p)}{M_2^4(p)} - 1 \right)^{-1}$$

is bounded above by 20230 *for every $p \in \mathcal{A}_n$.*

If Q_n is a polynomial of degree n of the form

$$Q_n(z) = \sum_{k=0}^{n} a_k z^k \,, \qquad a_k \in \mathbb{C} \,,$$

and the coefficients a_k of Q_n satisfy

$$a_k = \overline{a}_{n-k} \,, \qquad k = 0, 1, \dots n \,,$$

then we call Q_n a *conjugate-reciprocal* polynomial of degree n. We say that the polynomial Q_n is unimodular, if $|a_k| = 1$ for each $k = 0, 1, 2, \dots, n$. Note that if $p \in \mathcal{A}_n$, then

$$1 + 2p(t) = e^{int} Q_{2n}(e^{it})$$

with a conjugate-reciprocal unimodular polynomial Q_{2n} of degree exactly $2n$. One can ask how flat a conjugate reciprocal unimodular polynomial can be. Here we reprove a result of Erdős [Er-62]. His proof is much longer and his constant $\varepsilon > 0$ is unspecified. This result has already been recorded in [Er-01].

Theorem 3. *Let ∂D denote the unit circle. Let P be a conjugate reciprocal unimodular polynomial of degree n. Then*

$$\max_{z \in \partial D} |P(z)| \ge (1 + \varepsilon)\sqrt{n+1}$$

with $\varepsilon := \sqrt{4/3} - 1$. As a consequence, we have

$$M_\infty(1 + 2p) - M_2(1 + 2p) \ge (\sqrt{4/3} - 1) M_2(1 + 2p)$$

for every $p \in \mathcal{A}_n$.

In our next theorem we give the numerical value of an unspecified constant appearing in another main result of Littlewood. In the proof we will need to refer to only a two-page-long (very clever) piece of Littlewood's paper [Li-66a].

Theorem 4. *We have*

$$M_2(p) - M_1(p) \ge 10^{-31} M_2(p)$$

for every $p \in \mathcal{A}_n$.

Based on the fact that for a fixed trigonometric polynomial p the function

$$\lambda \to \lambda \log(M_\lambda(p))$$

is a convex function on $[0, \infty)$, we can state explicit numerical values of certain unspecified constants in some other related Littlewood results. For example, as a consequence of Theorem 4, we have

Theorem 5. *We have*

$$\log(M_\lambda(p)) - \log(M_2(p)) \geq \frac{\lambda-2}{\lambda} \log\left(\frac{1}{1-10^{-31}}\right), \qquad \lambda > 2,$$

and

$$\log(M_2(p)) - \log(M_\lambda(p)) \geq \frac{2-\lambda}{\lambda} \log\left(\frac{1}{1-10^{-31}}\right), \qquad 1 \leq \lambda < 2,$$

for every $p \in \mathcal{A}_n$.

3. Proofs

Proof of Theorem 1. For the sake of brevity let $\mu_n := \mu_n(p) = M_2(p)$. Note that Bernstein's inequality in $L_2(K)$ implies $B \leq A$. Without loss of generality we may assume that

$$\|p\|^4_{L_4(K)} \leq 2\pi \frac{33}{32} \mu_n^4 . \tag{3}$$

Then by the Bernstein Inequality for trigonometric polynomials in $L_4(K)$ we can deduce that

$$\|p'\|_{L_4(K)} \leq n\|p\|_{L_4(K)} \leq n\left(\frac{33}{32}\right)^{1/4} (2\pi)^{1/4}\mu_n \leq \pi^{-1/4}\left(\frac{33}{64}\right)^{1/4} An^{3/2} .$$

Hence, combining this with (2) and Hölder's Inequality, we obtain

$$B^2n^3 \leq \int_K |p'(t)|^2\,dt \leq \|p'\|^{2/3}_{L_1(K)}\|p'\|^{4/3}_{L_4(K)} \leq \|p'\|^{2/3}_{L_1(K)}\pi^{-1/3}\left(\frac{33}{64}\right)^{1/3} A^{4/3}n^2 .$$

Therefore

$$\frac{\pi^{1/3}\left(\frac{64}{33}\right)^{1/3} B^2}{A^{4/3}} n \leq \|p'\|^{2/3}_{L_1(K)} ,$$

that is

$$\frac{\pi^{1/2}\left(\frac{64}{33}\right)^{1/2} B^3}{A^2} n^{3/2} \leq \|p'\|_{L_1(K)} .$$

Combining this with (1), we have

$$\gamma n\mu_n \leq \gamma n \frac{A}{(2\pi)^{1/2}}\sqrt{n} \leq \frac{\pi^{1/2}\left(\frac{64}{33}\right)^{1/2} B^3}{A^2} n^{3/2} \leq \|p'\|_{L_1(K)}$$

with

$$\gamma := \frac{\left(\frac{128}{33}\right)^{1/2}\pi B^3}{A^3} .$$

Now let

$$E := E(n,p,\gamma) := \left\{t \in [0,2\pi) : (|p(t)| - \mu_n)^2 \geq \left(\frac{\gamma\mu_n}{16}\right)^2\right\} .$$

Then using Hölder's Inequality and then Bernstein's Inequality for trigonometric polynomials in $L_2(K)$, we can deduce that

$$\begin{aligned}\gamma n\mu_n &\leq \int_0^{2\pi} |p'(t)|\,dt \leq \int_{[0,2\pi]\setminus E} |p'(t)|\,dt + \int_E |p'(t)|\,dt \\ &\leq 2\cdot(2n)\cdot\frac{2\gamma}{16}\mu_n + \int_E |p'(t)|\,dt \leq \frac{\gamma}{2}n\mu_n + \sqrt{m(E)}\left(\int_E |p'(t)|^2\,dt\right)^{1/2} \\ &\leq \frac{\gamma}{2}n\mu_n + \sqrt{m(E)}\; n\left(\int_0^{2\pi} |p(t)|^2\,dt\right)^{1/2} \leq \frac{\gamma}{2}n\mu_n + \sqrt{m(E)}\; n(2\pi)^{1/2}\mu_n\,.\end{aligned}$$

Hence

$$\frac{\gamma}{2}n\mu_n \leq \sqrt{m(E)}n(2\pi)^{1/2}\mu_n\,,$$

that is

$$\beta := \frac{\gamma^2}{8\pi} \leq m(E)\,. \tag{4}$$

So we have

$$\begin{aligned}2\pi(M_4(p)^4 - M_2(p)^4) &= \|p\|_{L_4(K)}^4 - 2\pi\mu_n^4 \\ &= \int_0^{2\pi} (p(t)^2 - \mu_n^2)^2\,dt \\ &\geq m(E)\left(\frac{\gamma\mu_n}{16}\right)^2\mu_n^2 \geq \frac{\gamma^2}{8\pi}\left(\frac{\gamma\mu_n}{16}\right)^2\mu_n^2 \\ &= \frac{1}{2^{11}\pi}\left(\frac{128}{33}\right)^2\pi^4\left(\frac{B}{A}\right)^{12}\mu_n^4\,.\end{aligned}$$

Combining this with (3) we obtain

$$M_4(p) - M_2(p) \geq 2^{-14}\left(\frac{B}{A}\right)^{12}\left(\frac{128}{33}\right)^2\pi^2 M_2(p)\,,$$

and the theorem is proved Here we used

$$M_4(p)^4 - M_2(p)^4 \geq (M_4(p) - M_2(p))4M_2(p)^3\,,$$

which is a consequence of the Mean Value Theorem. Note that

$$\frac{1}{111} \leq 2^{-14}\left(\frac{128}{33}\right)^2\pi^2 \leq \frac{1}{110}\,.$$

□

Proof of Theorem 3. Let P be a conjugate reciprocal unimodular polynomial of degree n. To prove the statement, observe that Malik's inequality [MMR, p. 676] gives

$$\max_{z\in\partial D}|P'(z)| \leq \frac{n}{2}\max_{z\in\partial D}|P(z)|\,.$$

(Note that the fact that P is conjugate reciprocal improves the Bernstein factor for P on ∂D from n to $n/2$.) Using the fact that each coefficient of P is of modulus 1, then applying Parseval's formula and Malik's inequality, we obtain

$$2\pi\,\frac{n^2(n+1)}{3} \leq 2\pi\,\frac{n(n+1)(2n+1)}{6} = \int_{\partial D} |P'(z)|^2\,|dz| \leq 2\pi\left(\frac{n}{2}\right)^2 \max_{z\in\partial D} |P(z)|^2\,,$$

and

$$\max_{z\in\partial D} |P(z)| \geq \sqrt{4/3}\sqrt{n+1}$$

follows. $\square$

Proof of Theorem 4. Let $p \in \mathcal{A}_n$. For the sake of brevity let $\mu_n := \mu_n(p) = M_2(p)$. Let $N(p,v)$ be the number of real roots of $p - v\mu_n = 0$ in $(-\pi,\pi)$. Littlewood proves (see Theorem 1 (i) of [Li-66a]) that if $p \in \mathcal{A}_n$ and

$$\frac{1}{2\pi}\int_0^{2\pi} |p(t)|\,dt = c\mu_n\,,$$

then

$$N(p,v) \geq 2^{-16}c^{11}n\,, \qquad |v| \leq 2^{-5}c^3\,.$$

The reader may wish to find this lower bound hidden in the proof of Theorem 1 (i) of Littlewood's paper [Li-66a]. Hence, by estimating the total variation in the usual way, $\gamma n\mu_n \leq \|p'\|_{L_1(K)}$ with $\gamma := 2^{-20}c^{14}$. If $c \leq 2^{-1/14}$, then the proof of the theorem is finished. If $c \geq 2^{-1/14}$, then $\gamma \geq 2^{-21}$, so in the sequel we may assume that $\gamma \geq 2^{-21}$ holds. Now let

$$E := E(n,p,\gamma) := \left\{t \in [0,2\pi) : (|p(t)| - \mu_n)^2 \geq \left(\frac{\gamma\mu_n}{16}\right)^2\right\}\,.$$

Estimating the total variation of p on $[0,2\pi]\setminus E$, hen using Hölder's Inequality and then Bernstein's Inequality for trigonometric polynomials in $L_2(K)$, we can deduce that

$$\begin{aligned}\gamma n\mu_n &\leq \int_0^{2\pi} |p'(t)|\,dt \leq \int_{[0,2\pi]\setminus E} |p'(t)|\,dt + \int_E |p'(t)|\,dt\\ &\leq 2\cdot(2n)\cdot\frac{2\gamma}{16}\mu_n + \int_E |p'(t)|\,dt \leq \frac{\gamma}{2}n\mu_n + \sqrt{m(E)}\left(\int_E |p'(t)|^2\,dt\right)^{1/2}\\ &\leq \frac{\gamma}{2}n\mu_n + \sqrt{m(E)}\;n\left(\int_0^{2\pi} |p(t)|^2\,dt\right)^{1/2} \leq \frac{\gamma}{2}n\mu_n + \sqrt{m(E)}\;n(2\pi)^{1/2}\mu_n\,.\end{aligned}$$

Hence

$$\frac{\gamma}{2}n\mu_n \leq \sqrt{m(E)}n(2\pi)^{1/2}\mu_n\,,$$

that is

$$\beta := \frac{\gamma^2}{8\pi} \leq m(E)\,.$$

So we have

$$\begin{aligned}4\pi((M_2(p))^2 - M_2(p)M_1(p)) &= \int_0^{2\pi} (|p(t)| - \mu_n)^2 \, dt \\ &\geq m(E) \left(\frac{\gamma\mu_n}{16}\right)^2 \geq \frac{\gamma^2}{8\pi} \left(\frac{\gamma\mu_n}{16}\right)^2 \\ &= \frac{\gamma^4}{2^{11}\pi}\mu_n^2 \geq 2^{-95}\pi^{-1}\mu_n^2 \,.\end{aligned}$$

This implies

$$M_2(p) - M_1(p) \geq 2^{-97}\pi^{-2} M_2(p) \,,$$

and the theorem is proved. □

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