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Nikolskii-Type Inequalities for Generalized Polynomials and Zeros of Orthogonal Polynomials

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Generalized polynomials are defined as products of polynomials raised to positive real powers. The generalized degree can be defined in a natural way. Relying on Remez-type inequalities on the size of generalized polynomials, we estimate the supremum norm of a generalized polynomial by its weighted L_1 norm. Based on such Nikolskii-type inequalities we give sharp upper bounds for the distance of the consecutive zeros of orthogonal polynomials associated with weight functions from rather wide classes. The estimates contain some old results as special cases. © 1991 Academic Press, Inc.

1. INTRODUCTION

How large can the modulus of an algebraic polynomial be on $[-1, 1]$ if it is less than 1 on a subset of $[-1, 1]$ with prescribed measure? This question was answered by Chebyshev when the subset is an interval, but his elegant method based on zero counting fails to work when we do not have this additional information. The proof of the general case (due to Remez [7]) and an application in the theory of orthogonal polynomials can be found in [4]; a simpler proof is given in [2]. Remez-type inequalities for generalized polynomials in the trigonometric and the pointwise algebraic cases were established in [1]. We summarize these results in Section 3. We will use them to estimate the supremum norm of a generalized polynomial by its weighted L_1 norm. Such estimates are called (special) Nikolskii-type inequalities, which are interesting in themselves. Improving an old technique from [8, p. 112–115], we will apply our Nikolskii-type inequalities to obtain sharp upper bounds for the distance of the adjacent zeros of orthogonal polynomials associated with weight functions from rather wide classes beyond the Szegő class.

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2. Generalized Polynomials: Definitions and Notations

Denote by Π_n the set of all real algebraic polynomials of degree at most n. The set of all real trigonometric polynomials of degree at most n will be denoted by T_n. The function

$$f(z) = c \prod_{j=1}^{k} (z - z_j)^{r_j} \qquad (0 \neq c \in \mathbf{C},\ z_j \in \mathbf{C},\ r_j > 0 \text{ are real}) \tag{1}$$

will be called a generalized complex algebraic polynomial of (generalized) degree

$$N = \sum_{j=1}^{k} r_j. \tag{2}$$

To be precise, in this paper we will use the definition

$$z^r = \exp(r \log |z| + ir \arg z) \qquad (z \in \mathbf{C},\ r \in \mathbf{R},\ -\pi \leqslant \arg z < \pi).$$

Obviously

$$|f(z)| = |c| \prod_{j=1}^{k} |z - z_j|^{r_j}.$$

Denote by GCAP_N the set of all generalized complex algebraic polynomials of degree at most N. In the trigonometric case let

$$f(z) = c \prod_{j=1}^{k} (\sin((z - z_j)/2))^{r_j} \qquad (0 \neq c \in \mathbf{C},\ z_j \in \mathbf{C},\ r_j > 0 \text{ are real}). \tag{3}$$

We say the function f is a generalized complex trigonometric polynomial of degree

$$N = \frac{1}{2} \sum_{j=1}^{k} r_j. \tag{4}$$

We have

$$|f(z)| = |c| \prod_{j=1}^{k} |\sin((z - z_j)/2)|^{r_j}.$$

Denote by GCTP_N the set of all generalized complex trigonometric polynomials of degree at most N. To express our information on the measure of the subset, where the modulus of a generalized is not greater than 1, we introduce the notations

$$\mathrm{GCAP}_N(s) = \{f \in \mathrm{GCAP}_N : m(\{x \in [-1, 1] : |f(x)| \leqslant 1\}) \geqslant 2 - s\} \qquad (0 < s < 2)$$

and

$$\mathrm{GCTP}_N(s) = \{f \in \mathrm{GCTP}_N : m(\{t \in [-\pi, \pi) : |f(t)| \leqslant 1\}) \geqslant 2\pi - s\} \qquad (0 < s < 2\pi).$$

Throughout this paper c_i will denote positive absolute constants. If A is a subset of the real line, then $m(A)$ will denote its one-dimensional Lebesgue measure.

3. Remez-Type Inequalities

Remez-type inequalities play a significant role in this paper. Remez's theorem [4, pp. 119–121] states that

$$\max_{-1 \leqslant x \leqslant 1} |p(x)| \leqslant Q_n(4/(2-s) - 1) \tag{5}$$

for every $p \in \mathrm{GCAP}_n(s) \cap \Pi_n$, where $Q_n(x) = \cos(n \arccos x)$ $(-1 \leqslant x \leqslant 1)$ is the Chebyshev polynomial of degree n. The proof of this theorem can be found in [4, pp. 119–121] and a simpler proof is given in [2]. In trying to establish a similar inequality for generalized algebraic polynomials it seems hard to tell what we should put in place of the Chebyshev polynomial in (5). We can, however, prove an equally useful version of (5) for generalized algebraic polynomials which preserves the best possible order of magnitude. This generalization is given by

Theorem 1. *We have*

$$\max_{-1 \leqslant x \leqslant 1} |f(x)| \leqslant \exp(5N\sqrt{s}) \qquad (0 < s \leqslant 1)$$

and

$$\max_{-1 \leqslant x \leqslant 1} |f(x)| \leqslant \exp\left(\frac{8N}{2-s}\right) \qquad (1 < s < 2)$$

for every $f \in \mathrm{GCAP}_N(s)$.

Theorem 1 is proved in [1]. Because of the periodicity we may expect a significant improvement in the trigonometric case. By establishing the best possible order of magnitude, this is confirmed by

THEOREM 2. *There is an absolute constant* c_1 *such that*

$$\max_{-\pi \leqslant t \leqslant \pi} |f(t)| \leqslant \exp(c_1 Ns) \qquad (0 < s \leqslant \pi/2)$$

for every $f \in \mathrm{GCTP}_N(s)$.

Theorem 2 is proved in [1] as well. Though we will not use it in this paper, we mention that a sharp pointwise algebraic Remez-type inequality is also obtained in [1].

It is easy to check that Theorems 1 and 2 imply the following theorems.

THEOREM 1*. *There is an absolute constant* $0 < c_2 < 1$ *such that*

$$m(\{y \in [-1, 1] : |f(y)| \geqslant \exp(-N\sqrt{s}) \max_{-1 \leqslant x \leqslant 1} |f(x)|\}) \geqslant c_2 s$$

for every $f \in \mathrm{GCAP}_N$ *and* $0 < s < 2$.

THEOREM 2*. *There is an absolute constant* $0 < c_3 < 1$ *such that*

$$m(\{t \in [-\pi, \pi) : |f(t)| \geqslant \exp(-Ns) \max_{-\pi \leqslant \tau \leqslant \pi} |f(\tau)|\}) \geqslant c_3 s$$

for every $f \in \mathrm{GCTP}_N$ *and* $0 < s < 2\pi$.

The advantage of Theorems 1* and 2* is to have an inequality for every s between the natural bounds ($0 < s < 2$ in the algebraic case, and $0 < s < 2\pi$ in the trigonometric case, respectively).

4. New Results

If g is a measurable function on the interval $[a, b]$, and for every $\lambda > 0$ there is a constant $K = K(g)$ depending only on g such that

$$m(\{x \in [a, b] : |g(x)| \geqslant \lambda\}) \leqslant K(g)\, \lambda^{-p}, \tag{6}$$

then we say g is in weak $L_p(a, b)$, and we will use the notation $g \in WL_p(a, b)$. It is well known that if g is in $L_p(a, b)$, then g is in $WL_p(a, b)$. In what follows w will denote a non-negative weight function from $L_1(-1, 1)$.

4.1. *Nikolskii-Type Inequalities*

Working with large families of weight functions, we estimate the supremum norm of generalized complex algebraic polynomials of degree at most N by their weighted L_1 norm. We will need these inequalities in Section 6 to give upper bounds for the distance of the consecutive zeros of orthogonal polynomials. Since for $q>0$ the qth power of a generalized polynomial is also a generalized polynomial, we can easily derive $L_\infty \to L_q(w)$ inequalities from our $L_\infty \to L_1(w)$ inequalities. Though we will not apply them in this paper, for completeness we establish $L_p(w) \to L_q(w)$ $(0<q<p<\infty)$ Nikolskii-type inequalities as well. In our theorems we will use the function $\log^-(x) = \min\{\log x, 0\}$.

THEOREM 3. *Let* $0<\alpha<1$, $p=2/\alpha-2$, *and* $\log^-(w(x)) \in WL_p(-1,1)$. *Then*

$$\max_{-1\leqslant x\leqslant 1} |f(x)| \leqslant \exp(c(\alpha, K)(1+N)^\alpha) \int_{-1}^{1} |f(x)|\, w(x)\, dx$$

for every $f \in \mathrm{GCAP}_N$, *where* $K = K(\log^-(w))$ *is defined by* (6), *and* $c(\alpha, K)$ *depends only on* α *and* K.

THEOREM 3*. *Let* $0<\alpha<1$, $p=2/\alpha-2$, *and* $\log^-(w(x)) \in WL_p(-1,1)$. *Then*

$$\left(\int_{-1}^{1} |f(x)|^p\, w(x)\, dx\right)^{1/p}$$
$$\leqslant (\exp(c(\alpha, K)(1+qN)^\alpha))^{1/q-1/p} \left(\int_{-1}^{1} |f(x)|^q\, w(x)\, dx\right)^{1/q}$$

for every $f \in \mathrm{GCAP}_N$ *and* $0<q<p<\infty$, *where* $K = K(\log^-(w))$ *is defined by* (6), *and* $c(\alpha, K)$ *depends only on* α *and* K.

In our Theorems 4 and 4* we take only half as large p as in Theorems 3 and 3*, but we assume that $\log^-(w(\cos\theta))$ is in $WL_p(-\pi, \pi)$, and we obtain the same conclusion.

THEOREM 4. *Let* $0<\alpha<1$, $p=1/\alpha-1$, *and* $\log^-(w(\cos\theta)) \in WL_p(-\pi,\pi)$. *Then*

$$\max_{-1\leqslant x\leqslant 1} |f(x)| \leqslant \exp(c(\alpha, K)(1+N)^\alpha) \int_{-1}^{1} |f(x)|\, w(x)\, dx$$

for every $f \in \mathrm{GCAP}_N$, *where* $K = K(\log^-(w(\cos\theta)))$ *is defined by* (6), *and* $c(\alpha, K)$ *is a constant depending only on* α *and* K.

THEOREM 4* *Let* $0<\alpha<1, p=1/\alpha-1$, *and* $\log^{-}(w(\cos\theta))\in WL_p(-\pi,\pi)$. *Then*

$$\left(\int_{-1}^{1}|f(x)|^p w(x)\,dx\right)^{1/p} \leqslant (\exp(c(\alpha,K)(1+qN)^{\alpha}))^{1/q-1/p}\left(\int_{-1}^{1}|f(x)|^q w(x)\,dx\right)^{1/q}$$

for every $f\in \mathrm{GCAP}_N$ *and* $0<q<p<\infty$, *where* $K=K(\log^{-}(w(\cos\theta)))$ *is defined by* (6), *and* $c(\alpha, K)$ *is a constant depending only on* α *and* K.

We remark that in the case of $\frac{1}{2}\leqslant\alpha<1$ $(0<p\leqslant 1)$ the Szegő class is properly contained in the classes of Theorems 4 and 4*. The Nikolskii-type inequalities of our Theorems 5 and 5* give better upper bounds for less wide classes.

THEOREM 5. *Let* $w^{-\varepsilon}\in WL_1(-1,1)$ *for some* $\varepsilon>0$. *Then*

$$\max_{-1\leqslant x\leqslant 1}|f(x)|\leqslant c(\varepsilon,K)(1+N)^M\int_{-1}^{1}|f(x)|\,w(x)\,dx$$

for every $f\in\mathrm{GCAP}_N$, *where* $M=2/\varepsilon+2$, $K=K(w^{-\varepsilon})$ *is defined by* (6), *and* $c(\varepsilon,K)$ *depends only on* ε *and* K.

THEOREM 5*. *Let* $w^{-\varepsilon}\in WL_1(-1,1)$ *for some* $\varepsilon>0$. *Then*

$$\left(\int_{-1}^{1}|f(x)|^p w(x)\,dx\right)^{1/p} \leqslant (c(\varepsilon,K)(1+qN)^{M})^{1/q-1/p}\left(\int_{-1}^{1}|f(x)|^q w(x)\,dx\right)^{1/q}$$

for every $f\in\mathrm{GCAP}_N$ *and* $0<q<p<\infty$, *where* $M=2/\varepsilon+2$, $K=K(w^{-\varepsilon})$ *is defined by* (6), *and* $c(\varepsilon,K)$ *depends only on* ε *and* K.

4.2. *Applications: Zeros of Orthogonal Polynomials*

Let w be an integrable weight function on $[-1,1]$. Denote the zeros of the associated orthonormal polynomials $p_n(x)$ by $x_{1,n}>x_{2,n}>\cdots>x_{n,n}$. Let $x_{\nu,n}=\cos\theta_{\nu,n}$, where $0<\theta_{\nu,n}<\pi$, $\theta_{0,n}=0$ and $\theta_{n+1,n}=\pi$. We give upper bounds for the distance of the consecutive zeros of orthogonal polynomials associated with weight functions from the classes for which we established Nikolskii-type inequality. In our first zero estimate the conditions are the same as in Theorem 3.

THEOREM 6. *Let* $0<\alpha<1$, $p=2/\alpha-2$, *and* $\log\ (w(x))\in WL_p(-1,1)$. *Then*

$$\theta_{v+1,n}-\theta_{v,n}\leqslant c(\alpha,K)\,n^{\alpha-1}\qquad(0\leqslant v\leqslant n),$$

where $K=K(\log^{-}(w))$ *is defined by* (6) *and* $c(\alpha,K)$ *depends only on* α, K, *and* $\int_{-1}^{1}w(x)\,dx$.

Our next theorem shows that the same zero estimates can be established under the conditions of Theorem 4. When $\log^{-}(w(\cos\theta))$ is in $L_1(-\pi,\pi)$ (thus w is in the Szegő class), the theorem was proved by Nevai in [6, pp. 157–158], but only for $x_{v,n}$ instead of $\theta_{v,n}$, and even in this special case Theorem 7 assumes w to be in the wider "weak" Szegő class.

THEOREM 7. *Let* $0<\alpha<1$, $p=1/\alpha-1$, *and* $\log^{-}(w(\cos\theta))\in WL_p(-\pi,\pi)$. *Then for all* $0\leqslant v\leqslant n$ *we have*

$$\theta_{v+1,n}-\theta_{v,n}\leqslant(\alpha,K)\,n^{\alpha-1}\qquad(n\geqslant 1),$$

where $K=K(\log^{-}(w(\cos\theta)))$ *is defined by* (6) *and* $c(\alpha,K)$ *is a constant depending only on* α, K, *and* $\int_{-1}^{1}w(x)\,dx$.

The following theorem is due to Erdős and Turán [3] when w^{-1} is in $L_1(-1,1)$. A generalization, when $w^{-\varepsilon}$ is in $L_1(-1,1)$ for some $\varepsilon>0$, was established by Nevai [6, p. 158], but he works with $x_{v,n}$ instead of $\theta_{v,n}$.

THEOREM 8. *Let* $w^{-\varepsilon}\in WL_1(-1,1)$ *for some* $\varepsilon>0$. *Then for all* $0\leqslant v\leqslant n$ *we have*

$$\theta_{v+1,n}-\theta_{v,n}\leqslant c(\varepsilon,K)(\log n)/n\qquad(n\geqslant 2),$$

where $K=K(w^{-\varepsilon})$ *is defined by* (6) *and* $c(\varepsilon,K)$ *depends only on* ε, K, *and* $\int_{-1}^{1}w(x)\,dx$.

4.3. *The Sharpness of Our Zero Estimates*

To see the sharpness of Theorems 6 and 7 we introduce the generalized Pollaczek weight functions by

$$w_\beta(x)=\exp(-(1-x^2)^{-\beta})\qquad(0\leqslant\beta<\infty).$$

A result of Lubinsky and Saff announced in [5, p. 411, (16)] implies that for the above weight functions we have

$$\theta_{n+1,n}-\theta_{n,n}=\pi-\theta_{n,n}\geqslant c(\beta)\,n^{-1/(2\beta+1)}\qquad(0\leqslant\beta<\infty),\tag{7}$$

where $c(\beta)$ depends only on β. If $\beta=(2/\alpha-2)^{-1}$ $(0<\alpha<1)$, then $\log^-(w_\beta(x))$ is in $WL_p(-1,1)$ with $p=2/\alpha-2$, and $\log^-(w_\beta(\cos\theta))$ is in $WL_p(-\pi,\pi)$ with $p=1/\alpha-1$. Since $n^{-1/(2\beta+1)}=n^{\alpha-1}$, (7) shows the sharpness of Theorems 6 and 7.

5. Proof of Theorems 3, 5, and 5

To prove Theorems 3, 4, and 5 we use our Remez-type inequalities discussed in Section 3. Our idea is to integrate only on a "sufficiently large" subset, where both the generalized polynomial (compared with its supremum norm) and the weight function are "sufficiently large," and to balance in an optimal way.

Proof of Theorem 3. For an $f\in \mathrm{GCAP}_N$ we introduce

$$D=D_{f,N,\alpha}=\{y\in[-1,1]:|f(y)|\geqslant\exp(-(1+N)^\alpha)\max_{-1\leqslant x\leqslant 1}|f(x)|\}$$

By Theorem 1* we obtain

$$m(D)\geqslant c_2(1+N)^{2\alpha-2} \tag{8}$$

Let

$$F=F_{w,N,c,\alpha}=\{y\in[-1,1]:w(y)\leqslant\exp(-(c+cN)^\alpha)\},$$

where $c>0$ will be chosen later. Since $\log^-(w(x))\in WL_p(-1,1)$ and $p=2/\alpha-2$, we have

$$m(F)\leqslant K(c+cN)^{-\alpha p}=Kc^{2\alpha-2}(1+N)^{2\alpha-2}\leqslant\frac{c_2}{2}(1+N)^{2\alpha-2}$$
$$\text{with}\quad c=(c_2/(2K))^{1/(2\alpha-2)} \tag{9}$$

Now we define the set

$$G=\left\{y\in[-1,1]\;\middle|\;\begin{array}{l}|f(y)|\geqslant\exp(-(1+N)^\alpha)\max\limits_{-1\leqslant x\leqslant 1}|f(x)|;\\ w(y)\geqslant\exp((c+cN)^\alpha)\end{array}\right\}$$

From (8) and (9) we deduce

$$m(G)\geqslant\frac{c_2}{2}(1+N)^{2\alpha-2} \tag{10}$$

This yields

$$\int_{-1}^{1} |f|\,w \geqslant \int_G |f|\,w$$
$$\geqslant \frac{c_2}{2}(1+N)^{2\alpha-2}\exp(-(c^\alpha+1)(1+N)^\alpha) \max_{-1\leqslant x\leqslant 1} |f(x)|,$$

which gives the desired result. ∎

Proof of Theorem 4. For an $f \in \mathrm{GCAP}_N$ we define

$$D = D_{f,N,\alpha}$$
$$= \{\theta \in [-\pi, \pi) : |f(\cos\theta)| \geqslant \exp(-(1+N)^\alpha) \max_{-\pi\leqslant t\leqslant\pi} |f(\cos t)|\}.$$

Observe that $f \in \mathrm{GCAP}_N$ implies $g(t) = f(\cos t) \in \mathrm{GCTP}_N$ (this follows easily from the observation that the range of the function $\cos z$ is the whole complex plane), hence Theorem 2* yields

$$m(D) \geqslant c_3(1+N)^{\alpha-1} \tag{11}$$

Let

$$F = F_{w,N,c,\alpha} = \{\theta \in [-\pi, \pi] : w(\cos\theta) \leqslant \exp(-(c+cN)^\alpha)\},$$

where $c > 0$ will be chosen suitably later. Since $\log^-(w(\cos\theta))$ is in $WL_1(-\pi, \pi)$ and $p = 1/\alpha - 1$, we have

$$m(F) \leqslant K(c+cN)^{-\alpha p} = Kc^{\alpha-1}(1+N)^{\alpha-1} \leqslant \frac{c_3}{2}(1+N)^{\alpha-1}$$
$$\text{with} \quad c = (c_3/(2K))^{1/(\alpha-1)}. \tag{12}$$

We introduce

$$\delta = \frac{c_3}{16}(1+N)^{\alpha-1}, \tag{13}$$

$$K_\delta = \{\theta \in [-\pi, \pi), \pi - \theta \geqslant \delta, \pi + \theta \geqslant \delta, |\theta| \geqslant \delta\} \tag{14}$$

and

$$G_\delta = \left\{ \theta \in K_\delta \;\middle|\; \begin{array}{l} f(\cos\theta) \leqslant \exp(-(1+N)^\alpha) \max\limits_{-\pi\leqslant t\leqslant\pi} |f(x)|; \\ w(\cos\theta) \geqslant \exp((c+cN)^\alpha) \end{array} \right\}. \tag{15}$$

From (11)–(15) we deduce

$$m(G_\delta) \geqslant \frac{c_3}{2}(1+N)^{\alpha-1} - \frac{4c_3}{16}(1+N)^{\alpha-1} \geqslant \frac{c_3}{4}(1+N)^{\alpha-1}. \tag{16}$$

With the notation

$$\tilde{G}_\delta = \{x : x = \cos\theta, \theta \in G_\delta\},$$

we obtain from (16) that

$$\begin{aligned}\int_{\tilde{G}_\delta} |f(x)|\, w(x)(1-x^2)^{-1/2}\, dx &= \frac{1}{2}\int_{G_\delta} |f(\cos\theta)|\, w(\cos\theta)\, d\theta \\ &\geqslant \frac{c_3}{4}(1+N)^{\alpha-1} \exp(-(c^\alpha+1)(1+N)^\alpha) \max_{-1\leqslant x\leqslant 1} |f(x)|. \end{aligned} \tag{17}$$

Since $(1-x^2)^{-1/2} \leqslant c_4(1+N)^{1-\alpha}$ on $\tilde{G}_\delta$, (17) gives the desired result. Thus Theorem 4 is proved. ∎

Proof of Theorem 5. For an $f \in \mathrm{GCAP}_N$ let

$$D = D_{f,N} = \{y \in [-1, 1] : |f(y)| \geqslant e^{-1} \max_{-1\leqslant x\leqslant 1} |f(x)|\}.$$

By Theorem 1 we have

$$m(D) \geqslant c_2(1+N)^{-2} \tag{18}$$

Let

$$F = F_{w,N,\varepsilon,c} = \{x \in [-1, 1] : w(x) \leqslant (c(1+N))^{-2/\varepsilon}\},$$

where $c > 0$ will be chosen later. Since $w^{-\varepsilon}$ is in $WL_1(-1, 1)$, we have

$$m(F) \leqslant Kc^{-2}(1+N)^{-2} \leqslant \frac{c_2}{2}(1+N)^{-2} \qquad \text{with} \quad c = (2K/c_2)^{1/2}. \tag{19}$$

We introduce the set

$$G = \{y \in [-1, 1] : |f(y)| \geqslant e^{-1} \max_{-1\leqslant x\leqslant 1} |f(x)|;\, w(y) \geqslant (c(1+N))^{-2/\varepsilon}\}.$$

From (18) and (19) we deduce

$$m(G) \geqslant \frac{c_2}{2}(1+N)^{-2} \tag{20}$$

Therefore

$$\int_{-1}^{1} |f|\, w \geqslant \int_{G} |f|\, w \geqslant \frac{c_2}{2}(1+N)^{-2} e^{-1}(c(1+N))^{-2/\varepsilon} \max_{-1 \leqslant x \leqslant 1} |f(x)|,$$

which gives the desired result. ∎

Proof of Theorems 3*, 4*, *and* 5*. We show how Theorem 3* follows from Theorem 3; the proof of Theorem 4* and 5* is identical. Observe that $f \in \mathrm{GCAP}_N$ and $q > 0$ imply $f^q \in \mathrm{GCAP}_{qN}$, hence by Theorem 3 we obtain

$$\max_{-1 \leqslant x \leqslant 1} |f(x)|^q \leqslant \exp(c(\alpha, K)(1+qN)^\alpha) \int_{-1}^{1} |f(x)|^q\, w(x)\, dx$$

for every $f \in \mathrm{GCAP}_N$. Therefore

$$\begin{aligned}
&\int_{-1}^{1} |f(x)|^p\, w(x)\, dx \\
&\qquad \leqslant \max_{-1 \leqslant x \leqslant 1} |f(x)|^{p-q} \int_{-1}^{1} |f(x)|^q\, w(x)\, dx \\
&\qquad \leqslant \left(\exp(c(\alpha, K)(1+qN)^\alpha) \int_{-1}^{1} |f(x)|^q\, w(x)\, dx\right)^{(p-q)/q} \\
&\qquad\quad \times \int_{-1}^{1} |f(x)|^q\, w(x)\, dx,
\end{aligned}$$

and taking the pth root of both sides, we obtain Theorem 3* immediately. ∎

6. Proof of Theorems 6, 7, and 8

We verify the zero estimates of Section 4.2 by improving a method of B. Lengyel [8, pp. 112–115]. Our improvement is based on the results of Section 4.1. Though we proved our Nikolskii-type inequalities for generalized polynomials, in this section we need them to apply only for ordinary polynomials. Let $0 \leqslant \nu \leqslant n$ be a fixed integer and $\gamma = (\theta_{\nu+1,n} + \theta_{\nu,n})/2$. We define $\rho(x) = \rho(\cos\theta)$ by

$$2\rho(x) = \left(\frac{\sin(N(\gamma+\theta)/2)}{N\sin((\gamma+\theta)/2)}\right)^{2m} + \left(\frac{\sin(N(\gamma-\theta)/2)}{N\sin((\gamma-\theta)/2)}\right)^{2m},$$

where N and m are certain positive integers. Then ρ is an algebraic polynomial of degree $m(N-1)$; see [8, 6.11.3]. By [8, 6.11.7] we have

$$\rho(x_{k,n}) \leqslant (N\sin((\theta_{\nu+1,n} - \theta_{\nu,n})/4))^{-2m} \qquad (k = 1, 2, ..., n),$$

so by the Gaussian quadrature formula

$$\int_{-1}^{1} \rho(x)\, w(x)\, dx \leqslant (N \sin((\theta_{\nu+1,n} - \theta_{\nu,n})/4))^{-2m} \int_{-1}^{1} w(x)\, dx \tag{21}$$

for $m(N-1) \leqslant 2n-1$. Further $\rho(y) \geqslant \frac{1}{2}$, hence

$$\max_{-1 \leqslant x \leqslant 1} |\rho(x)| \geqslant \tfrac{1}{2}. \tag{22}$$

In the case of Theorems 6 and 7, Theorems 3 and 4, together with (21) and (22) give

$$\tfrac{1}{2} \leqslant \exp(c(\alpha, K)\, n^{\alpha-1})(N \sin((\theta_{\nu+1,n} - \theta_{\nu,n})/4))^{-2m} \int_{-1}^{1} w(x)\, dx,$$

thus

$$(\theta_{\nu+1,n} - \theta_{\nu,n})/(2\pi) \leqslant N^{-1} \exp(c(\alpha, K)\, n^{\alpha}/m) \left(2 \int_{-1}^{1} w(x)\, dx\right)^{1/(2m)}$$

From here choosing $m = [n^{\alpha}]$ and $N = [n^{1-\alpha}]$, we obtain the desired result.

In the case of Theorem 8, Theorem 3 together with (16) and (17) gives

$$\tfrac{1}{2} \leqslant c(\varepsilon, K) \exp(M \log n)(N \sin((\theta_{\nu+1,n} - \theta_{\nu,n})/4))^{-2m} \int_{-1}^{1} w(x)\, dx;$$

therefore

$$(\theta_{\nu+1,n} - \theta_{\nu,n})/(2\pi) \leqslant c(\varepsilon, K)\, N^{-1} \exp(M \log n/m) \left(2 \int_{-1}^{1} w(x)\, dx\right)^{1/(2m)}$$

From here the choices $m = [\log n]$ and $N = [n/\log n]$ give the desired result. ∎

We remark that our method of proving Theorems 6, 7, and 8 can be used to give upper bounds for the distance of the consecutive zeros associated with weight functions from even wider classes.

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References

1. T. Erdélyi, Remez-type inequalities on the size of generalized polynomials, *J. London Math. Soc.*, to appear.
2. T. Erdélyi, The Remez inequality on the size of polynomials, *in* "Approximation Theory VI, Vol. 1" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 243–246, Academic Press, Boston, 1989.
3. P. Erdős and P. Turán, On interpolation. II. On the distribution of the fundamental points of Lagrange and Hermite interpolation, *Ann. of Math.* **39** (1938), 703–724.
4. G. Freud, "Orthogonal Polynomials," Pergamon, Oxford, 1971.
5. D. S. Lubinsky and E. B. Saff, Szegő asymptotics for non-Szegő weights on $[-1, 1]$, *in* "Approximation Theory VI, Vol. 2" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 409–412, Academic Press, Boston, 1989.
6. P. Nevai, "Orthogonal Polynomials," Mem. Amer. Math. Soc., Vol. 213, Amer. Math. Soc., Providence, RI, 1979.
7. E. J. Remez, Sur une propriété des polynômes de Tchebycheff, *Comm. Inst. Sci. Kharkow* **13** (1936), 93–95.
8. G. Szegő, "Orthogonal Polynomials," Amer. Math. Soc. Colloquium Publications, Vol. 23, 4th ed., Amer. Math. Soc., Providence, RI, 1975.