THE L_q NORM OF THE RUDIN-SHAPIRO POLYNOMIALS ON SUBARCS OF THE UNIT CIRCLE

Tamás Erdélyi

June 12, 2023

Abstract. Littlewood polynomials are polynomials with each of their coefficients in $\{-1,1\}$. A sequence of Littlewood polynomials that satisfies a remarkable flatness property on the unit circle of the complex plane is given by the Rudin-Shapiro polynomials. Let P_k and Q_k denote the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$. For polynomials S we define

$$M_q(S,[\alpha,\beta]) := \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \left|S(e^{it})\right|^q\,dt\right)^{1/q}, \qquad q>0\,.$$

Let $\gamma := \sin^2(\pi/8)$. We prove that

$$\frac{\gamma}{4\pi}(\gamma n)^{q/2} \leq M_q(P_k,[\alpha,\beta])^q \leq (2n)^{q/2}$$

for every $q>0$ and $32\pi/n \leq \beta-\alpha$. The same estimates hold for P_k replaced by Q_k.

1. Introduction and Notation

Let $\alpha<\beta$ be real numbers. The Mahler measure $M_0(S,[\alpha,\beta])$ is defined for polynomials S as

$$M_0(S,[\alpha,\beta]) := \exp\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \log|S(e^{it})|\,dt\right)\,.$$

It is well known, see [17] for instance, that

$$M_0(S,[\alpha,\beta]) = \lim_{q\to 0+} M_q(S,[\alpha,\beta])\,,$$

where

$$M_q(S,[\alpha,\beta]) := \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \left|S(e^{it})\right|^q\,dt\right)^{1/q}, \qquad q>0\,.$$

Key words and phrases. Rudin-Shapiro polynomials, L_q norms, Mahler measure, subarcs of the unit circle.

2020 Mathematics Subject Classifications. 11C08, 41A17

It is a simple consequence of the Jensen formula that

$$M_0(S,[0,2\pi]) = |c| \prod_{k=1}^{n} \max\{1, |z_k|\}$$

for every polynomial of the form

$$S(z) = c \prod_{k=1}^{n} (z - z_k)\,, \qquad c, z_k \in \mathbb{C}\,.$$

See [3 p. 271] or [2 p. 3] for instance. Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk of the complex plane. Let $\partial D := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle of the complex plane. Littlewood polynomials are polynomials with each of their coefficients in $\{-1, 1\}$. A special sequence of Littlewood polynomials is the sequence the Rudin-Shapiro polynomials, They appear in Harold Shapiro's 1951 thesis [21] at MIT and are sometimes called just the Shapiro polynomials. They also arise independently in Golay's paper [16]. They are remarkably simple to construct recursively as follows. Let

$$P_0(z) := 1\,, \qquad Q_0(z) := 1\,,$$

and

$$P_{k+1}(z) := P_k(z) + z^{2^k} Q_k(z)\,,$$
$$Q_{k+1}(z) := P_k(z) - z^{2^k} Q_k(z)\,,$$

for $k = 0, 1, 2, \ldots$. Note that both P_k and Q_k are polynomials of degree $n-1$ with $n := 2^k$ having each of their coefficients in $\{-1, 1\}$. In what follows P_k and Q_k denote the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$. It is well known, and easy to check by using the parallelogram law, that

$$|P_{k+1}(z)|^2 + |Q_{k+1}(z)|^2 = 2(|P_k(z)|^2 + |Q_k(z)|^2)\,, \qquad z \in \partial D\,.$$

Hence

(1.1) $$|P_k(z)|^2 + |Q_k(z)|^2 = 2^{k+1} = 2n\,, \qquad z \in \partial D\,.$$

It is also well known, see Section 4 of [2] or [6] for instance, that

(1.2) $$Q_k(z) = (-1)^{k+1} P_k^*(-z)\,, \qquad z \in \partial D\,,$$

where $P_k^*(z) := z^{n-1} P_k(1/z)$. Hence

(1.3) $$|Q_k(z)| = |P_k(-z)|\,, \qquad z \in \partial D\,.$$

Peter Borwein's book [2] presents a few more basic results on the Rudin-Shapiro polynomials. Cyclotomic properties of the Rudin-Shapiro polynomials are discussed in [6]. Obviously $M_2(P_k, [0, 2\pi]) = 2^{k/2}$ by the Parseval formula. In 1968 Littlewood [19] showed that $M_4(P_k, [0, 2\pi]) \sim (4^{k+1}/3)^{1/4}$. Here, and in what follows, $a_k \sim b_k$ means that $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$. Rudin-Shapiro like polynomials in L_4 on the unit circle ∂D of the complex plane are studied in [4]. Let $K := \mathbb{R} \pmod{2\pi}$. Let $m(A)$ denote the one-dimensional Lebesgue measure of $A \subset K$. In 1980 Saffari conjectured the following result. He did not publish this conjecture himself, and it first appeared in print in the work of Doche and Habsieger [9].

Theorem 1.1. *We have*

$$M_q(P_k,[0,2\pi]) = M_q(Q_k,[0,2\pi]) \sim \frac{2^{(k+1)/2}}{(q/2+1)^{1/q}} = \frac{(2n)^{1/2}}{(q/2+1)^{1/q}}$$

for all real exponents $q>0$*. Equivalently, we have*

$$\lim_{k\to\infty} m\left(\left\{t\in K : \left|\frac{P_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [\alpha,\beta]\right\}\right)$$
$$= \lim_{k\to\infty} m\left(\left\{t\in K : \left|\frac{Q_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [\alpha,\beta]\right\}\right) = 2\pi(\beta-\alpha)$$

whenever $0\le\alpha<\beta\le 1$*.*

Theorem 1.1 was proved for all even values of $q \le 52$ by Doche [8] and Doche and Habsieger [9]. Rodgers [20] proved Theorem 1.1 for all $q>0$. See also [10]. An application of Theorem 1.1 may be found in [15]. An extension of Saffari's conjecture is Montgomery's conjecture below proved by Rodgers [20] as well.

Theorem 1.2. *We have*

$$\lim_{k\to\infty} m\left(\left\{t\in K : \frac{P_k(e^{it})}{\sqrt{2^{k+1}}} \in E\right\}\right)$$
$$= \lim_{k\to\infty} m\left(\left\{t\in K : \frac{Q_k(e^{it})}{\sqrt{2^{k+1}}} \in E\right\}\right) = 2m(E)$$

for any rectangle $E\subset D := \{z\in\mathbb{C} : |z|<1\}$*.*

In [11] we proved the following lower bound for the Mahler measure of the Rudin-Shapiro polynomials on subarcs of the unit circle ∂D.

Theorem 1.3. *There is an absolute constant* $c>0$ *such that*

$$M_0(P_k,[\alpha,\beta]) \ge cn^{1/2}$$

for all $k\in\mathbb{N}$ *and for all* $\alpha,\beta\in\mathbb{R}$ *such that*

$$\frac{32\pi}{n} \le \frac{(\log n)^{3/2}}{n^{1/2}} \le \beta-\alpha \le 2\pi\,.$$

The same lower bound holds for $M_0(P_k,[\alpha,\beta])$ *replaced by* $M_0(Q_k,[\alpha,\beta])$*.*

It looks plausible that Theorem 1.3 holds whenever $32\pi/n \le \beta-\alpha$, but we have not been able to handle the case $32\pi/n \le \beta-\alpha \le (\log n)^{3/2}n^{-1/2}$. Nevertheless our Theorem 2.2 gives a lower bound for the values $M_q(P_k,[\alpha,\beta])$ and $M_q(Q_k,[\alpha,\beta])$ for every $q>0$ and $32\pi/n\le\beta-\alpha$. See also [7] on sums of monomials with large Mahler measure on subarcs of the unit circle ∂D. In [13] the asymptotic values of $M_0(P_k,[0,2\pi])$ and $M_0(Q_k,[0,2\pi])$, conjectured by Saffari, have been found. Namely in [13] we showed the following.

Theorem 1.4. *We have*

$$\lim_{n\to\infty} \frac{M_0(P_k,[0,2\pi])}{n^{1/2}} = \lim_{n\to\infty} \frac{M_0(Q_k,[0,2\pi])}{n^{1/2}} = \left(\frac{2}{e}\right)^{1/2}.$$

Properties of the Rudin Shapiro polynomials have played a a central role in [1] as well as in [14] to prove a longstanding conjecture of Littlewood on the existence of flat Littlewood polynomials S_n of degree n satisfying the inequalities

$$c_1 n^{1/2} \le |S_n(e^{it})| \le c_2 n^{1/2}, \qquad t \in \mathbb{R},$$

with absolute constants $c_1 > 0$ and $c_2 > 0$.

New Results

Let $\gamma := \sin^2(\pi/8)$ and $n := 2^k$ The Lebesgue measure of a set $E \subset \mathbb{R}$ is denoted by $m(E)$.

Theorem 2.1. *Let* $E := \{t \in [\alpha,\beta] : |P_k(t)| \ge \gamma n\}$. *We have*

$$m(E) \ge \frac{(\beta-\alpha)\gamma}{4\pi}$$

for every $32\pi/n \le \beta - \alpha$. *The same estimate holds for* P_k *replaced by* Q_k.

Theorem 2.2. *We have*

$$\frac{\gamma}{4\pi}(\gamma n)^{q/2} \le M_q(P_k,[\alpha,\beta])^q \le (2n)^{q/2}$$

for every $q > 0$ *and* $32\pi/n \le \beta - \alpha$. *The same estimate holds for* P_k *replaced by* Q_k.

3. Lemmas

Let $n := 2^k$, $\gamma := \sin^2(\pi/8)$, $z_j := e^{it_j}$, $t_j := 2\pi j/n$, $j \in \mathbb{Z}$.

Lemma 3.1. *We have*

$$\max\{|P_k(z_j)|^2, |P_k(z_{j+r})|^2\} \ge \gamma 2^{k+1} = 2\gamma n, \quad r \in \{-1,1\},$$

for every $j = 2u$, $u \in \mathbb{Z}$. *The same estimate holds for* P_k *replaced by* Q_k.

Lemma 3.1 tells us that the modulus of the Rudin-Shapiro polynomials P_k is certainly not smaller than $(2\gamma n)^{1/2}$ at least at one of any two consecutive n-th root of unity, where $n := 2^k$. This is a crucial observation proved in [11] and plays a key role in [12], [13], [14] and [15] as well. Our Lemma 3.2 below follows from Lemma 3.1 reasonably simply.

Lemma 3.2. *We have*

$$|P_k(e^{it})|^2 \geq \gamma n\,, \qquad t \in [t_j - \gamma/n, t_j + \gamma/n]\,,$$

for every $j \in \mathbb{Z}$ such that

$$|P_k(z_j)|^2 \geq \gamma 2^{k+1} = 2\gamma n\,. \tag{3.1}$$

The same estimate holds with P_k replaced by Q_k.

Proof of Lemma 3.2. By (1.3) it is sufficient to prove the lemma only for P_k. The proof of the lemma is a simple combination of the Mean Value Theorem and Bernstein's inequality applied to the nonnegative trigonometric polynomial R_k of degree $n-1$ with $n = 2^k$ defined by $R_k(t) := P_k(e^{it})P_k(e^{-it})$. Recall that (1.1) implies $0 \leq R_k(t) = |P_k(e^{it})|^2 \leq 2n$ for every $t \in \mathbb{R}$. Note also that the Bernstein factor is $n/2$ rather than n for the class of nonnegative trigonometric polynomials of degree at most n, see Lemma 3.3 below. Suppose $j \in \mathbb{Z}$ satisfies (3.1) and $t \in \mathbb{R}$ satisfies $|t - t_j| \leq \gamma/n$. Then by the Mean Value Theorem there is a ξ between t_j and t such that

$$R_k(t_j) - R_k(t) \leq |R_k(t_j) - R_k(t)| = |t_j - t||R_k'(\xi)| \leq \frac{\gamma}{n}\frac{n}{2}\max_{\tau \in K}\{R_k(\tau) \leq \frac{\gamma}{n}\frac{n}{2}2n = \gamma n\,.$$

Therefore, recalling (3.1), we get

$$R_k(t) \geq R_k(t_j) - \gamma n = 2\gamma n - \gamma n = \gamma n\,, \qquad t \in [t_j - \gamma/n\,, t_j + \gamma/n]\,.$$

□

Let $K := \mathbb{R} \pmod{2\pi}$, as before.

Lemma 3.3. *We have*

$$\max_{\tau \in K}|T'(\tau)| \leq \frac{n}{2}\max_{\tau \in K}T(\tau)$$

for every trigonometric polynomial T of degree at most n that is nonnegative on $\mathbb{R}$.

Proof of Lemma 3.3. Suppose T is a trigonometric polynomial of degree at most n that is nonnegative on $\mathbb{R}$. The Bernstein inequality, see [3] for instance, asserts that

$$\max_{\tau \in K}|Q'(\tau)| \leq n \max_{\tau \in K}|Q(\tau)|$$

for every real trigonometric polynomial Q of degree at most n. Applying the Bernstein inequality to the real trigonometric polynomial $Q := T - M$ of degree at most n with $M := \frac{1}{2}\max_{\tau \in K}|Q(\tau)|$ gives the lemma. □

4. Proof of the theorems

Proof of Theorem 2.1. By (1.3) it is sufficient to prove the theorem only for P_k. Observe that Lemmas 3.1 and 3.2 imply that E contains at least $\dfrac{(\beta-\alpha)n}{4\pi} - 4$ disjoint intervals of length at least $2\gamma/n$, hence

$$m(E) \geq \left(\frac{(\beta-\alpha)n}{4\pi} - 4\right)\frac{2\gamma}{n} \geq \frac{(\beta-\alpha)n}{8\pi}\,\frac{2\gamma}{n} = \frac{(\beta-\alpha)\gamma}{4\pi}$$

whenever $32\pi/n \leq \beta - \alpha$. $\square$

Proof of Theorem 2.2. By (1.2) it is sufficient to prove Theorem 2.1 for P_k. The upper bound of the theorem follows immediately from (1.1). Now we prove the lower bound of the theorem. Using Theorem 2.1 we have

$$\begin{aligned} M_q(P_k,[\alpha,\beta])^q &:= \frac{1}{\beta-\alpha}\int_\alpha^\beta |P_k(t)|^q\,dt \geq \frac{1}{\beta-\alpha}\int_E |P_k(t)|^q\,dt \\ &\geq \frac{1}{\beta-\alpha} m(E)(\gamma n)^{q/2} \geq \frac{1}{\beta-\alpha}\,\frac{(\beta-\alpha)\gamma}{4\pi}(\gamma n)^{q/2} \\ &\geq \frac{\gamma}{4\pi}(\gamma n)^{q/2} \end{aligned}$$

whenever $32\pi/n \leq \beta - \alpha$. $\square$

5. More observations and problems

Let P_k and Q_k be the usual Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$.

As for $k \geq 1$ both P_k and Q_k have odd degree $n - 1 = 2^k - 1$, both P_k and Q_k have at least one real zero. The fact that for $k \geq 1$ both P_k and Q_k have exactly one real zero was proved by Brillhart in [5]. Another interesting observation made in [6] is the fact that P_k and Q_k cannot vanish at any roots of unity different from -1 and 1. In [12] we proved that the Rudin-Shapiro polynomials P_k and Q_k have only $o(n)$ zeros on the unit circle ∂D. Observe, see [6] for instance, that

$$P_k(1) = 2^{[(k+1)/2]}, \qquad Q_k(-1) = (-1)^{k+1}2^{[(k+1)/2]},$$

and

$$P_k(-1) = Q_k(1) = \frac{1}{2}(1 + (-1)^k)2^{[k/2]},$$

where $[x]$ denotes the integer part of a real number x.

Problem 5.1. *Is it true that if k is odd then P_k has a zero on the unit circle partialD only at -1 and Q_k has a zero on the unit circle ∂D only at 1, while if k is even then neither P_k nor Q_k has a zero on the unit circle ∂D?*

Combining (1.2) with the observation that the Rudin-Shapiro polynomials P_k and Q_k of degree $n - 1$ with $n := 2^k$ have only $o(n)$ zeros on the unit circle ∂D, we can deduce that the products P_kQ_k have $n - o(n)$ zeros in the open unit disk D, where $o(n)$ denotes real numbers such that $o(n)/n$ converges to 0 as n tends to ∞.

Problem 5.2. *Is there an absolute constant $c > 0$ such that both of the Rudin-Shapiro polynomials P_k and Q_k have at least cn zeros in the open unit disk D?*

Problem 5.3. *Is it true that both of the Rudin-Shapiro polynomials P_k and Q_k have $n/2 - o(n)$, zeros in the open unit disk D?*

Problem 5.4. *Is it true that Theorem 1.3 remains valid for all $32\pi/n \leq \beta - \alpha \leq 2\pi$?*

Problem 5.5. *Is there an absolute constant $c > 0$ such that*

$$M_0(|P_k|^2 - n, [0, 2\pi]) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left||P_k(e^{it})|^2 - n\right| dt\right) \geq cn^{1/2}?$$

6. A connection to sew-reciprocal polynomials

A polynomial S of the form

$$S(z) = \sum_{j=0}^{2m} a_j z^j\,, \qquad a_j \in \mathbb{R}\,, \quad a_{2m} \neq 0\,,$$

is called skew-reciprocal if

$$a_{m-j} = (-1)^j a_{m+j}\,, \qquad j = 1, 2, \ldots, m\,. \tag{6.2)}$$

A beautiful observation of Mercer [18] states the following.

Theorem 6.1. *Skew-reciprocal Littlewood polynomials do not have any zeros on the unit circle ∂D.*

The Rudin-Shapiro polynomials P_k and Q_k of degree $n-1$ with $n := 2^k$ are quite close to be skew-reciprocal. However, as the degrees of P_k and Q_k are odd, Theorem 6.1 does not apply to the Rudin-Shapiro polynomials. Having a middle term in the polynomial S in the proof below is crucial.

Proof of Theorem 6.1. Let S be a skew-reciprocal Littlewood polynomial of the form

$$S(z) = \sum_{j=0}^{2m} a_j z^j, \quad a_j \in \{-1, 1\}\,, \quad j = 0, 1, \ldots, 2m\,, \quad a_{2m} \neq 0\,,$$

with

$$a_{m-j} = (-1)^j a_{m+j}\,, \qquad j = 1, 2, \ldots, m\,.$$

For notational convenience we assume that $m = 2\mu$ is even; the proof in the case when $m = 2\mu - 1$ is odd can be handled similarly. We have $z^{-m}S(z) = A(z) + B(z)$, where the function

$$A(z) := \sum_{j=0}^{\mu} a_{m+2j}(z^{2j} + z^{-2j})\,, \qquad z \in \partial D\,,$$

takes purely real values on the unit circle ∂D, and the function

$$B(z) := \sum_{j=1}^{\mu} a_{m+2j-1}(z^{2j-1} - z^{-2j-1}), \qquad z \in \partial D,$$

takes purely imaginary values on the unit circle ∂D. Suppose to the contrary that S vanishes at a point z_0 on the unit circle ∂D. Then z_0 is a common zero of A and B. We study the greatest common divisor of the polynomials $\widetilde{A}(z) := z^m A(z)$ and $\widetilde{B}(z) := z^m B(z)$ over the field $\mathbf{F}_2$. We have

$$\widetilde{A}(z) - z\widetilde{B}(z) = \sum_{j=0}^{m} z^{2j} - z\sum_{j=1}^{m} z^{2j-1} = 1$$

over the field $\mathbf{F}_2$, showing that the greatest common divisor of the polynomials $\widetilde{A}$ and $\widetilde{B}$ over the field $\mathbf{F}_2$ is 1. Hence $A(z)$ and $B(z)$ cannot have a common zero on the unit circle ∂D, a contradiction. $\square$

Note that the same approach works to prove that skew-reciprocal polynomials with only odd coefficients do not have any zeros on the unit circle ∂D.

References

1. Balister, B. Bollobás, R. Morris, J. Sahasrabudhe, and M. Tiba, *Flat Littlewood polynomials exist*, Ann. of Math. **192** (2020), no. 997-1003.
2. P. Borwein, *Computational Excursions in Analysis and Number Theory*, Springer, New York, 2002.
3. P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer, New York, 1995.
4. P. Borwein and M.J. Mossinghoff, *Rudin-Shapiro like polynomials in L_4*, Math. Comp. **69** (2000), 1157–1166.
5. J. Brillhart, *On the Rudin-Shapiro polynomials*, Duke Math. J. **40** (1973), no. 2, 335–353.
6. J. Brillhart, J.S. Lemont, and P. Morton, *Cyclotomic properties of the Rudin-Shapiro polynomials*, J. Reine Angew. Math. (Crelle's J.) **288** (1976), 37–65.
7. K.-K. S. Choi and T. Erdélyi, *Sums of monomials with large Mahler measure*, J. Approx. Theory **197** (2015), 49–61.
8. Ch. Doche, *Even moments of generalized Rudin-Shapiro polynomials*, Math. Comp. **74** (2005), no. 252, 1923–1935.
9. Ch. Doche and L. Habsieger, *Moments of the Rudin-Shapiro polynomials*, J. Fourier Anal. Appl. **10** (2004), no. 5, 497–505.
10. S.B. Ekhad and D. Zeilberger, *Integrals involving Rudin-Shapiro polynomials and sketch of a proof of Saffari's conjecture*, in Analytic number theory, modular forms and q-hypergeometric series **221** (2017), Springer Proc. Math. Stat., Springer, Cham, 253–265.
11. T. Erdélyi, *The Mahler measure of the Rudin-Shapiro polynomials*, Constr. Approx. **43** (2016), no. 3, 357–369.

12. T. Erdélyi, *On the oscillation of the modulus of the Rudin-Shapiro polynomials on the unit circle*, Mathematika **66** (2020), no. 1, 144–160.

13. T. Erdélyi, *The asymptotic value of the Mahler measure of the Rudin-Shapiro polynomials*, J. Anal. Math. **142** (2020), no. 2, 521–537.

14. T. Erdélyi, *Do flat skew-reciprocal Littlewood polynomials exist?*, Constr. Approx. **56** (2022), no. 3, 537–554.

15. T. Erdélyi, *Improved results on the oscillation of the modulus of the Rudin-Shapiro polynomials on the unit circle*, Proc. Amer. Math. Soc. **151** (2023), 2733–2740.

16. M.J. Golay, *Static multislit spectrometry and its application to the panoramic display of infrared spectra*, J. Opt. Soc. America **41** (1951), 468–472.

17. G.H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Univ. Press, London, 1952.

18. I.D. Mercer, *Unimodular roots of special Littlewood polynomials*, Canad. Math. Bull. **49** (2006), no. 3, 438–447.

19. J.E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968.

20. B. Rodgers, *On the distribution of Rudin-Shapiro polynomials and lacunary walks on $SU(2)$*, Adv. Math. **320** (2017), 993–1008.

21. H.S. Shapiro, *Master thesis*, MIT, 1951.

Department of Mathematics, Texas A&M University College Station, Texas 77843
E-mail address: terdelyi@tamu.edu