ON THE OSCILLATION OF THE MODULUS OF THE RUDIN-SHAPIRO POLYNOMIALS ON THE UNIT CIRCLE

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Abstract. In signal processing the Rudin-Shapiro polynomials have good autocorrelation properties and their values on the unit circle are small. Binary sequences with low autocorrelation coefficients are of interest in radar, sonar, and communication systems. In this paper we study the oscillation of the modulus of the Rudin-Shapiro polynomials on the unit circle. We also show that the Rudin-Shapiro polynomials P_k and Q_k of degree $n-1$ with $n := 2^k$ have $o(n)$ zeros on the unit circle. This should be compared with a result of B. Conrey, A. Granville, B. Poonen, and K. Soundararajan stating that for odd primes p the Fekete polynomials f_p of degree $p-1$ have asymptotically $\kappa_0 p$ zeros on the unit circle, where $0.500813 > \kappa_0 > 0.500668$. Our approach is based heavily on the Saffari and Montgomery conjectures proved recently by B. Rodgers. We also prove that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that the k-th Rudin-Shapiro polynomials P_k and Q_k of degree $n-1 = 2^k - 1$ have at least $c_2 n$ zeros in the annulus

$$\left\{ z \in \mathbb{C} : 1 - \frac{c_1}{n} < |z| < 1 + \frac{c_1}{n} \right\} .$$

1. Introduction and Notation

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk of the complex plane. Let $\partial D := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle of the complex plane. Let $K := \mathbb{R} \pmod{2\pi}$. The Mahler measure $M_0(f)$ is defined for bounded measurable functions f on ∂D by

$$M_0(f) := \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| \, dt \right) .$$

It is well known, see [HL-52], for instance, that

$$M_0(f) = \lim_{q \to 0+} M_q(f) \, ,$$

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where

$$M_q(f) := \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(e^{it})\right|^q \, dt\right)^{1/q}, \qquad q > 0\,.$$

It is also well known that for a function f continuous on ∂D we have

$$M_\infty(f) := \max_{t\in[0,2\pi]} |f(e^{it})| = \lim_{q\to\infty} M_q(f)\,.$$

It is a simple consequence of the Jensen formula that

$$M_0(f) = |c| \prod_{j=1}^n \max\{1, |z_j|\}$$

for every polynomial of the form

$$f(z) = c\prod_{j=1}^n (z - z_j)\,, \qquad c, z_j \in \mathbb{C}\,.$$

See [BE-95, p. 271] or [B-02, p. 3], for instance. It will be convenient for us to introduce the notation

$$M_q(S) := M_q(f)\,, \qquad 0 \le q \le \infty\,,$$

for functions S defined on the period $K := \mathbb{R} \pmod{2\pi}$ by $S(t) := f(e^{it})$, where f is a bounded measurable functions f on ∂D.

Let $\mathcal{P}_n^c$ be the set of all algebraic polynomials of degree at most n with complex coefficients. Let

$$\mathcal{T}_n := \left\{ a_0 + \sum_{j=1}^n (a_j \cos(jt) + b_j \sin(jt))\,, \quad a_j, b_j \in \mathbb{R} \right\}$$

be the set of all real trigonometric polynomials of degree at most n. Finding polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The classes

$$\mathcal{L}_n := \left\{ f: \;\; f(z) = \sum_{j=0}^n a_j z^j\,, \quad a_j \in \{-1, 1\} \right\}$$

of Littlewood polynomials and the classes

$$\mathcal{K}_n := \left\{ f: \;\; f(z) = \sum_{j=0}^n a_j z^j\,, \quad a_j \in \mathbb{C},\;\; |a_j| = 1 \right\}$$

of unimodular polynomials are two of the most important classes considered. Observe that $\mathcal{L}_n \subset \mathcal{K}_n$ and

$$M_0(f) \le M_2(f) = \sqrt{n+1}$$

for every $f \in \mathcal{K}_n$. Beller and Newman [BN-73] constructed unimodular polynomials $f_n \in \mathcal{K}_n$ whose Mahler measure $M_0(f_n)$ is at least $\sqrt{n} - c/\log n$, where $c > 0$ is an absolute constant.

Section 4 of [B-02] is devoted to the study of Rudin-Shapiro polynomials. Littlewood asked if there were polynomials $f_{n_k} \in \mathcal{L}_{n_k}$ satisfying

$$c_1\sqrt{n_k+1} \le |f_{n_k}(z)| \le c_2\sqrt{n_k+1}\,, \qquad z \in \partial D\,,$$

with some absolute constants $c_1 > 0$ and $c_2 > 0$, see [B-02, p. 27] for a reference to this problem of Littlewood. To satisfy just the lower bound, by itself, seems very hard, and no such sequence (f_{n_k}) of Littlewood polynomials $f_{n_k} \in \mathcal{L}_{n_k}$ is known. A sequence of Littlewood polynomials that satisfies just the upper bound is given by the Rudin-Shapiro polynomials. The Rudin-Shapiro polynomials appear in Harold Shapiro's 1951 thesis [S-51] at MIT and are sometimes called just Shapiro polynomials. They also arise independently in Golay's paper [G-51]. They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures. The Rudin-Shapiro polynomials are defined recursively as follows:

$$\begin{aligned} P_0(z) &:= 1\,, \qquad Q_0(z) := 1\,, \\ P_{k+1}(z) &:= P_k(z) + z^{2^k} Q_k(z)\,, \\ Q_{k+1}(z) &:= P_k(z) - z^{2^k} Q_k(z)\,, \end{aligned}$$

for $k = 0, 1, 2, \ldots$. Note that both P_k and Q_k are polynomials of degree $n-1$ with $n := 2^k$ having each of their coefficients in $\{-1, 1\}$. In signal processing, the Rudin-Shapiro polynomials have good autocorrelation properties and their values on the unit circle are small. Binary sequences with low autocorrelation coefficients are of interest in radar, sonar, and communication systems. It is well known and easy to check by using the parallelogram law that

$$|P_{k+1}(z)|^2 + |Q_{k+1}(z)|^2 = 2(|P_k(z)|^2 + |Q_k(z)|^2)\,, \qquad z \in \partial D\,.$$

Hence

$$|P_k(z)|^2 + |Q_k(z)|^2 = 2^{k+1} = 2n\,, \qquad z \in \partial D\,. \tag{1.1}$$

It is also well known (see Section 4 of [B-02], for instance), that

$$Q_k(-z) = P_k^*(z) := z^{n-1} P_k(1/z)\,, \qquad z \in \mathbb{C} \setminus \{0\}\,,$$

and hence

$$|Q_k(-z)| = |P_k(z)|\,, \qquad z \in \partial D\,. \tag{1.2}$$

P. Borwein's book [B-02] presents a few more basic results on the Rudin-Shapiro polynomials. Various properties of the Rudin-Shapiro polynomials are discussed in [B-73] by

Brillhart and in [BL-76] by Brillhart, Lemont, and Morton. Obviously $M_2(P_k) = 2^{k/2}$ by the Parseval formula. In 1968 Littlewood [L-68] evaluated $M_4(P_k)$ and found that $M_4(P_k) \sim (4^{k+1}/3)^{1/4}$. The M_4 norm of Rudin-Shapiro like polynomials on ∂D are studied in [BM-00]. P. Borwein and Lockhart [BL-01] investigated the asymptotic behavior of the mean value of normalized M_q norms of Littlewood polynomials for arbitrary $q > 0$. They proved that

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f\in\mathcal{L}_n} \frac{(M_q(f))^q}{n^{q/2}} = \Gamma\left(1+\frac{q}{2}\right),$$

where Γ is the usual Gamma function. In [C-15c] we proved that

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f\in\mathcal{L}_n} \frac{M_q(f)}{n^{1/2}} = \left(\Gamma\left(1+\frac{q}{2}\right)\right)^{1/q}$$

for every $q > 0$. In [CE-15c] we also proved the following result on the average Mahler measure of Littlewood polynomials. We have

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{f\in\mathcal{L}_n} \frac{M_0(f)}{n^{1/2}} = e^{-\gamma/2} = 0.749306\ldots,$$

where

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) = 0.577215\ldots$$

is the Euler constant. These are analogues of the results proved earlier by Choi and Mossinghoff [CM-11] for polynomials in $\mathcal{K}_n$. Let $m(A)$ denote the one-dimensional Lebesgue measure of $A \subset K := \mathbb{R} \pmod{2\pi}$. In 1980 Saffari conjectured the following.

Conjecture 1.1. *Let P_k and Q_k be the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$. We have*

$$M_q(P_k) = M_q(Q_k) \sim \frac{2^{k+1)/2}}{(q/2+1)^{1/q}}$$

for all real exponents $q > 0$. Equivalently, we have

$$\lim_{k\to\infty} m\left(\left\{t \in K : \left|\frac{P_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [\alpha,\beta]\right\}\right)$$
$$= \lim_{k\to\infty} m\left(\left\{t \in K : \left|\frac{Q_k(e^{it})}{\sqrt{2^{k+1}}}\right|^2 \in [\alpha,\beta]\right\}\right) = 2\pi(\beta-\alpha)$$

whenever $0 \le \alpha < \beta \le 1$.

This conjecture was proved for all even values of $q \le 52$ by Doche [D-05] and Doche and Habsieger [DH-04]. Recently B. Rodgers [R-16] proved Saffari's Conjecture 1.1 for all $q > 0$. See also [EZ-17]. An extension of Saffari's conjecture is Montgomery's conjecture below.

Conjecture 1.2. *Let P_k and Q_k be the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$. We have*

$$\begin{aligned}&\lim_{k\to\infty} m\left(\left\{t \in K : \frac{P_k(e^{it})}{\sqrt{2^{k+1}}} \in E\right\}\right)\\ = &\lim_{k\to\infty} m\left(\left\{t \in K : \frac{Q_k(e^{it})}{\sqrt{2^{k+1}}} \in E\right\}\right) = 2m(E)\end{aligned}$$

for any rectangle $E \subset D := \{z \in \mathbb{C} : |z| < 1\}$.

B. Rodgers [R-16] proved Montgomery's Conjecture 1.2 as well.

Despite the simplicity of their definition not much is known about the Rudin-Shapiro polynomials. It has been shown in [E-16c] fairly recently that the Mahler measure (M_0 norm) and the M_∞ norm of the Rudin-Shapiro polynomials P_k and Q_k of degree $n-1$ with $n := 2^k$ on the unit circle of the complex plane have the same size, that is, the Mahler measure of the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$ is bounded from below by $cn^{1/2}$, where $c > 0$ is an absolute constant.

It is shown in this paper that the Rudin-Shapiro polynomials P_k and Q_k of degree $n-1$ with $n := 2^k$ have $o(n)$ zeros on the unit circle. We also prove that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that the k-th Rudin-Shapiro polynomials P_k and Q_k of degree $n-1 = 2^k - 1$ have at least $c_2 n$ zeros in the annulus

$$\left\{z \in \mathbb{C} : 1 - \frac{c_1}{n} < |z| < 1 + \frac{c_1}{n}\right\},$$

while there is an absolute constant $c > 0$ such that each of the functions $\mathrm{Re}(P_k)$, $\mathrm{Re}(Q_k)$, $\mathrm{Im}(P_k)$, and $\mathrm{Im}(Q_k)$ has at least cn zeros on the unit circle. The oscillation of $R_k(t) := |P_k(e^{it})|^2$ and $R_k(t) := |Q_k(e^{it})|^2$ on the period $[0, 2\pi)$ is also studied.

For a prime number p the p-th Fekete polynomial is defined as

$$f_p(z) := \sum_{j=1}^{p-1} \left(\frac{j}{p}\right) z^j,$$

where

$$\left(\frac{j}{p}\right) = \begin{cases} 1, & \text{if } \ x^2 \equiv j \pmod{p} \ \text{ has a nonzero solution,} \\ 0, & \text{if } \ p \ \text{ divides } \ j, \\ -1, & \text{otherwise} \end{cases}$$

is the usual Legendre symbol. Since f_p has constant coefficient 0, it is not a Littlewood polynomial, but g_p defined by $g_p(z) := f_p(z)/z$ is a Littlewood polynomial of degree $p-2$. Fekete polynomials are examined in detail in [B-02], [CG-00], [E-11], [E-12], [E-18], [EL-07], and [M-80]. In [CE-15a] and [CE-15b] the authors examined the maximal size of the Mahler measure and the L_p norms of sums of n monomials on the unit circle as well as on subarcs of the unit circles. In the constructions appearing in [CE-15a] properties of the Fekete polynomials f_p turned out to be quite useful. In [CG-00] B. Conrey, A. Granville, B. Poonen, and K. Soundararajan proved that for an odd prime p the Fekete polynomial

$f_p(z) = \sum_{j=0}^{p-1} \left(\frac{j}{p}\right) z^j$ (the coefficients are Legendre symbols) has $\sim \kappa_0 p$ zeros on the unit circle, where $0.500813 > \kappa_0 > 0.500668$. So Fekete polynomials are far from having only $o(p)$ zeros on the unit circle.

Mercer [M-06a] proved that if a Littlewood polynomial $P \in \mathcal{L}_n$ of the form $P(z) = \sum_{j=0}^{n} a_j z^j$ is skew-reciprocal, that is, $a_j = (-1)^j a_{n-j}$ for each $j = 0, 1, \dots, n$, then it has no zeros on the unit circle. However, by using different elementary methods it was observed in both [E-01] and [M-06a] that if a Littlewood polynomial P of the form $P(z) = \sum_{j=0}^{n} a_j z^j$ is self-reciprocal, that is, $a_j = a_{n-j}$ for each $j = 0, 1, \dots, n$, $n \geq 1$, then it has at least one zero on the unit circle. It is proved in [BE-97] that every every polynomial P of the from

$$P(z) = \sum_{j=0}^{n} a_j z^j \,, \qquad |a_0| = 1, \ \ |a_j| \leq 1 \,, \ \ a_j \in \mathbb{C} \,,$$

has at most $c\sqrt{n}$ zeros inside any polygon with vertices on the unit circle ∂D, where c depends only on the polygon. One of the main results of [BE-08b] gives explicit estimates for the number and location of zeros of polynomials with bounded coefficients. Namely if

$$\delta_n := 33\pi \frac{\log n}{\sqrt{n}} \leq 1 \,,$$

then every polynomial P of the from

$$P(z) = \sum_{j=0}^{n} a_j z^j \,, \qquad |a_0| = |a_n| = 1, \ \ |a_j| \leq 1 \,, \ \ a_j \in \mathbb{C} \,,$$

has at least $8\sqrt{n} \log n$ zeros in any disk with center on the unit circle and radius δ_n. More on Littlewood polynomials may be found in [B-02], [E-02], [M-17], and [O-18], for example.

There are many other papers on the zeros of constrained polynomials. Some of them are [BP-32], [BE-01], [BE-07], [BE-08a], [BE-99], [BE-13], [B-97], [D-08], [E-08a], [E-08b], [E-16a], [E-16b], [L-61], [L-64], [L-66a], [L-66b], [L-68], [M-06b], [Sch-32], [Sch-33], [Sz-34], and [TV-07].

2. New Results

Let P_k and Q_k be the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$. Let either $R_k(t) := |P_k(e^{it})|^2$ or $R_k(t) := |Q_k(e^{it})|^2$. Let $\gamma := \sin^2(\pi/8)$. We use the notation

$$\|g\|_A := \sup_{x \in A} |g(x)|$$

for a complex-valued function g defined on a set $A \subset \mathbb{R}$. Let $K := \mathbb{R} \pmod{2\pi}$.

Theorem 2.1. *P_k and Q_k have $o(n)$ zeros on the unit circle.*

The proof of Theorem 2.1 will follow by combining the recently proved Saffari's conjecture stated as Conjecture 1.1 and the theorem below.

Theorem 2.2. *If $S \in \mathcal{T}_n$ is of the form $S(t) = |f(e^{it})|^2$, where $f \in \mathcal{P}_n^c$, and f has at least u zeros (counted with multiplicities) in K, then*

$$m(\{t \in K : |S(t)| \leq \alpha \|S\|_K\}) \geq \frac{\sqrt{\alpha}}{e\sqrt{2}} \frac{u}{n}$$

for every $\alpha \in (0,1)$, where $m(A)$ denotes the one-dimensional Lebesgue measure of a measurable set $A \subset K$.

Theorem 2.3. *There is an absolute constant $c_1 > 0$ such that each of the functions $\mathrm{Re}(P_k)$, $\mathrm{Re}(Q_k)$, $\mathrm{Im}(P_k)$, and $\mathrm{Im}(Q_k)$ has at least $c_1 n$ zeros on the unit circle for every $n = 2^k - 1 \geq 1$.*

Theorem 2.4. *There is an absolute constant $c_2 > 0$ such that the equation $R_k(t) = \eta n$ has at most $c_2 \eta^{1/2} n$ solutions (counted with multiplicities) in K for every $\eta \in (0,1]$ and sufficiently large $k \geq k_\eta$, while the equation $R_k(t) = \eta n$ has at most $c_2(2-\eta)^{1/2} n$ solutions (counted with multiplicities) in K for every $\eta \in [1,2)$ and sufficiently large $k \geq k_\eta$.*

Theorem 2.5. *The equation $R_k(t) = \eta n$ has at least $(1-\varepsilon)\eta n/2$ distinct solutions in K for every $\eta \in (0, 2\gamma)$, $\varepsilon > 0$, and sufficiently large $k \geq k_{\eta,\varepsilon}$, while the equation $R_k(t) = \eta n$ has at least $(1-\varepsilon)(2-\eta)n/2$ distinct solutions in K for every $\eta \in (2-2\gamma, 2)$, $\varepsilon > 0$, and sufficiently large $k \geq k_{\eta,\varepsilon}$.*

Theorem 2.6. *There is an absolute constants $c_3 > 0$ such that the equation $R_k(t) = (1+\eta)n$ has at least $c_3 n^{0.36}$ distinct solutions in K whenever η is real and $|\eta| < 2^{-8}$.*

Theorem 2.7. *There are absolute constants $c_4 > 0$ and $c_5 > 0$ such that P_k and Q_k have at least $c_5 n$ zeros in the annulus*

$$\left\{ z \in \mathbb{C} : 1 - \frac{c_4}{n} < |z| < 1 + \frac{c_4}{n} \right\}.$$

We note that for every $c_6 \in (0,1)$ there is an absolute constant $c_7 > 0$ depending only on c_6 such that every $U_n \in \mathcal{P}_n^c$ of the form

$$U_n(z) = \sum_{j=0}^{n} a_j z^j, \qquad |a_0| = |a_n| = 1, \quad a_j \in \mathbb{C}, \quad |a_j| \leq 1,$$

has at least $c_6 n$ zeros in the annulus

$$\left\{ z \in \mathbb{C} : 1 - \frac{c_7 \log n}{n} < |z| < 1 + \frac{c_7 \log n}{n} \right\}. \tag{2.1}$$

See Theorem 2.1 in [E-01].

On the other hand, there is an absolute constant $c_7 > 0$ such that for every $n \in \mathbb{N}$ there is a polynomial $U_n \in \mathcal{K}_n$ having no zeros in the annulus (2.1). See Theorem 2.3 in [E-01]. So in the proof of Theorem 2.7 some special properties, in addition to being Littlewood polynomials, of the Rudin-Shapiro polynomials must be exploited.

A key to the proof of Theorem 2.7 is the result below.

Theorem 2.8. *Let $t_0 \in K$. There is an absolute constant $c_8 > 0$ depending only on $c_9 > 0$ such that P_k has at least one zero in the disk*

$$\left\{ z \in \mathbb{C} : |z - e^{it_0}| < \frac{c_8}{n} \right\} ,$$

whenever

$$T_k'(t_0) \geq c_9 n^2 \,, \qquad T_k(t) = P_k(e^{it}) P_k(e^{-it}) \,.$$

Problem 2.9. *Is there an absolute constant $c > 0$ such that the equation $R_k(t) = \eta n$ has at least $c\eta n$ distinct solutions in K for every $\eta \in (0,1)$ and sufficiently large $k \geq n_\eta$? In other words, can Theorem 2.5 be extended to all $\eta \in (0,1)$?*

We note that it follows from $Q_k(z) = P_k^*(-z)$, $z \in \mathbb{C}$, that the products $P_k Q_k$ have at least $n-1$ zeros in the closed unit disk and at least $n-1$ zeros outside the open unit disk. So in the light of Theorem 2.1 the products $P_k Q_k$ have asymptotically n zeros in the open unit disk. However, as far as we know, the following questions are open.

Problem 2.10. *Is there an absolute constant $c > 0$ such that P_k has at least cn zeros in the open unit disk?*

Problem 2.11. *Is there an absolute constant $c > 0$ such that Q_k has at least cn zeros in the open unit disk?*

Problem 2.12. *Is it true that both P_k and Q_k have asymptotically half of their zeros in the open unit disk?*

Problem 2.13. *Is it true that if n is odd then P_k has a zero on the unit circle ∂D only at -1 and Q_k has a zero on the unit circle ∂D only at 1, while if n is even then neither P_k nor Q_k has a zero on the unit circle?*

As for $k \geq 1$ both P_k and Q_k have odd degree, both P_k and Q_k have at least one real zero. The fact that for $k \geq 1$ both P_k and Q_k have exactly one real zero was proved in [B-73].

3. Lemmas

Let P_k and Q_k be the Rudin-Shapiro polynomials of degree $n-1$ with $n := 2^k$. Let

$$D(a, r) := \{ z \in \mathbb{C} : |z - a| < r \}$$

denote the open disk of the complex plane centered at $a \in \mathbb{C}$ of radius $r > 0$. Let $K := \mathbb{R} \pmod{2\pi}$. To prove Theorem 2.1 we need the lemma below that is proved in [BE-95, E.11 of Section 5.1 on pages 236–237].

Lemma 3.1. *If $S \in \mathcal{T}_n$, $t_0 \in K$, and $r > 0$, then S has at most $\dfrac{enr\|S\|_K}{|S(t_0)|}$ zeros in the interval $[t_0 - r, t_0 + r]$.*

Our next lemma is stated as Lemma 3.5 in [E-16c], where its proof may also be found.

Lemma 3.2. *If* $\gamma := \sin^2(\pi/8)$ *and*

$$z_j := e^{it_j}\,, \quad t_j := \frac{2\pi j}{n}\,, \qquad j \in \mathbb{Z}\,,$$

then

$$\max\{|P_k(z_j)|^2, |P_k(z_{j+r})|^2\} \geq \gamma 2^{k+1} = 2\gamma n\,, \quad r \in \{-1, 1\}\,,$$

for every $j = 2u$, $u \in \mathbb{Z}$.

By Lemma 3.2, for every $n = 2^k$ there are

$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} := \tau_1 + 2\pi$$

such that

$$\tau_j - \tau_{j-1} = \frac{2\pi l}{n}\,, \qquad l \in \{1, 2\}\,,$$

and with

$$a_j := e^{i\tau_j}, \qquad j = 1, 2, \ldots, m+1, \tag{3.4}$$

we have

$$|P_k(a_j)|^2 \geq 2\gamma n\,, \qquad j = 1, 2, \ldots, m+1\,.$$

(Moreover, each a_j is an n-th root of unity.)

Our next lemma is stated and proved as Lemma 3.4 in [E-19].

Lemma 3.3. *There is an absolute constant* $c_{10} > 0$ *such that*

$$\mu := \left| \left\{ j \in \{2, 3, \ldots, m+1\} : \min_{t \in [\tau_{j-1}, \tau_j]} R_k(t) \leq \varepsilon n \right\} \right| \leq c_{10} n \varepsilon^{1/2}$$

for every sufficiently large $n = 2^k \geq n_\varepsilon$, $k = 1, 2, \ldots,$ *and* $\varepsilon > 0$.

Our next lemma is based on the work of M. Taghavi in [T-96] and [T-97], and gives an upper bound for the so-called autocorrelation coefficients of the Rudin-Shapiro polynomials.

Lemma 3.4. *If*

$$|P_k(z)|^2 = P_k(z)P_k(1/z) = \sum_{j=-n+1}^{n-1} a_j z^j\,, \qquad z \in \partial D\,,$$

then

$$\max_{1 \leq j \leq n-1} |a_j| \leq c_{11} n^{0.8190}$$

with an absolute constant $c_{11} > 0$, *while obviously* $a_0 = n$, $a_j = a_{-j}$, $j = 1, 2, \ldots, n-1$.

In fact, Taghavi [T-97] claimed

$$\max_{1\le j\le n-1} |a_j| \le (3.2134)n^{0.7303}\,.$$

However, as Allouche and Saffari observed, in his proof Taghavi used an incorrect statement saying that the spectral radius of the product of some matrices is independent of the order of the factors. So what he ended up with cannot be viewed as a correctly proved result. Building on what is correct in [T-97] Stephen Choi made some computations leading to the above correct form of Taghavi's upper bound on the autocorrelation coefficients of the Rudin-Shapiro polynomials. The correction based on Choi's computations will be the subject of a forthcoming note [AC-17] perhaps even in a more optimized form.

Our next lemma is due to Littlewood, see [Theorem 1 in L-66a].

Lemma 3.5. *If $S \in \mathcal{T}_n$ of the form*

$$S(t) = \sum_{m=0}^{n} b_m \cos(mt + \alpha_m)\,, \qquad b_m, \alpha_m \in \mathbb{R}\,, \tag{3.1}$$

satisfies

$$M_1(S) \ge c\mu\,, \qquad \mu := M_2(S)\,,$$

where $c > 0$ is a constant, $b_0 = 0$,

$$s_{\lfloor n/h \rfloor} = \sum_{m=1}^{\lfloor n/h \rfloor} \frac{b_m^2}{\mu^2} \le 2^{-9}c^6$$

for some constant $h > 0$, and $v \in \mathbb{R}$ satisfies

$$|v| \le V = 2^{-5}c^3\,,$$

then

$$\mathcal{N}(S,v) > c_{12}h^{-1}c^5 n\,,$$

where $\mathcal{N}(S,v)$ denotes the number of real zeros of $S - v\mu$ in $(-\pi,\pi)$, and $c_{12} > 0$ is an absolute constant.

To prove Lemma 3.7 we need the lemma below stated as Lemma 4.3 in [E-98].

Lemma 3.6. *Let $0 < s \le \lambda \le 1$. We have*

$$\|f\|_{[-1,1+s]} \le \exp\left(8n\lambda^{-1/2}s\right) \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}_n^c$ having no zeros in the disk $D(1-\lambda,\lambda)$.

Now we are ready to prove the following.

Lemma 3.7. *Let $t_0 \in K$ and $1/n \le r \le 1$. We have*

$$|S(t_0+i\tau)| \le e\|S\|_K$$

for every $S \in \mathcal{T}_n$ having no zeros in the disk $D(t_0,r)$, and for every $\tau \in [-\rho,\rho]$ with $\rho := \frac{1}{5}\left(\frac{r}{n}\right)^{1/2}$.

Proof. It is sufficient to prove the lemma for $t_0 = 0$, since for $t_0 \ne 0$ we can study the polynomial $\widetilde{S} \in \mathcal{T}_n$ defined by $\widetilde{S}(\zeta) := S(\zeta - t_0)$ having no zeros in the disk $D(0,r)$. Associated with $S \in \mathcal{T}_n$ we define $U \in \mathcal{T}_{2n}$ by $U(\zeta) := S(\zeta)S(-\zeta)$. Observe that U is an even trigonometric polynomial of degree at most $2n$, hence we can define $f \in \mathcal{P}_{2n}^c$ (in fact, with real coefficients) by

$$f(\cos\zeta) := U(\zeta), \qquad \zeta \in \mathbb{C}\,.$$

Assume that S, and hence U, has no zeros in the disk $D(0,r)$. We show that f has no zeros in the disk $D(1,2\lambda)$, where $2\lambda := r^2/4$. Indeed, as S, and hence U, has no zeros in the disk $D(0,r)$, f has no zeros in the region $H := \{z = \cos\zeta : \zeta \in D(0,r)\}$ bounded by the curve $\Gamma := \partial H := \{z = \cos\zeta \in \mathbb{C} : |\zeta| = r\}$. As $\cos 0 = 1$, Γ goes around 1 at least once by the Argument Principle. Observe that if $z = \cos\zeta \in \Gamma$, then $|\zeta| = r \le 1$ implies that

$$|1-\cos\zeta| = \left|\sum_{k=1}^{\infty} \frac{\zeta^{2k}}{(2k)!}\right| \ge |\zeta|^2\left(\frac{1}{2} - \sum_{k=1}^{\infty}\frac{|\zeta|^{2k}}{(2k+2)!}\right) \ge \frac{|\zeta|^2}{4} = \frac{r^2}{4}\,,$$

and hence H contains the disk $D(1,2\lambda) = D(1,r^2/4)$. In conclusion, f has no zeros in the disk $D(1,2\lambda)$ as we claimed.

Using Lemma 3.6 with $\lambda := \dfrac{r^2}{8}$ and $s := u-1$, we have

(3.1)
$$\begin{aligned}|f(u)| &\le \exp(8(2n)\lambda^{-1/2}(u-1))\|f\|_{[-1,1]}\\ &\le \exp(48nr^{-1}(u-1))\|f\|_{[-1,1]}\,, \qquad 0 < s := u-1 \le \lambda = \frac{r^2}{8} \le 1\,.\end{aligned}$$

Now let

$$\rho := \frac{1}{5}\left(\frac{r}{n}\right)^{1/2}, \qquad \tau \in [-\rho,\rho]\,, \qquad u := \cosh\tau\,.$$

Then $1/n \le r \le 1$ and $\lambda := \dfrac{r^2}{8}$ imply that

$$u - 1 = \cosh\tau - 1 \le \tau^2 \le \rho^2 = \frac{r}{25n} < \frac{r^2}{8} = \lambda\,.$$

Using (3.1) we have

$$\begin{aligned}|S(i\tau)|^2 = |U(i\tau)| = |f(\cos(i\tau))| = |f(\cosh\tau)| &\le \exp(48nr^{-1}(\cosh\tau - 1))\|f\|_{[-1,1]}\\ &\le \exp(48nr^{-1}\rho^2)\|f\|_{[-1,1]} \le \exp\left(48nr^{-1}\frac{r}{25n}\right)\|f\|_{[-1,1]}\\ &\le e^2\|S\|_K^2\end{aligned}$$

for every $\tau \in [-\rho, \rho]$ with $\rho := \frac{1}{5}\left(\frac{r}{n}\right)^{1/2}$, $1/n \le r \le 1$. $\square$

Lemma 3.7 implies the following.

Lemma 3.8. *Let $t_0 \in K$ and $2/n < r \le 2$. We have*

$$|S(\zeta)| \le e\|S\|_K$$

for every $S \in \mathcal{T}_n$ having no zeros in the disk $D(t_0, r)$, and for every ζ in the square

$$\{\zeta = t + i\tau : t \in [t_0 - \rho, t_0 + \rho],\ \ \tau \in [-\rho, \rho]\}$$

with $\rho := \frac{1}{5}\left(\frac{r}{2n}\right)^{1/2}$.

Proof. Observe that if $S \in \mathcal{T}_n$ has no zeros in the disk $D(t_0, r)$, then it has no zeros in the disks $D(t, r/2)$ whenever $t \in [t_0 - r/2, t_0 + r/2]$. Observe that $2/n \le r \le 2$ implies $1/n \le r/2 \le 1$. Using Lemma 3.7 we obtain that

$$|S(\zeta)| \le e\|S\|_K$$

for every $S \in \mathcal{T}_n$ having no zeros in the disk $D(t_0, r)$, and for every

$$\zeta \in \{\zeta = t + i\tau : x \in [t_0 - r/2, t_0 + r/2],\ \ \tau \in [-\rho, \rho]\}$$

with $\rho := \frac{1}{5}\left(\frac{r}{2n}\right)^{1/2}$. As $2/n \le r \le 2$ implies $0 < \rho < r/2$, the lemma follows. $\square$

Our next lemma is a key to prove Theorem 2.7. It is an extension of Theorem 1 in [E-02] establishing the right Bernstein inequality for trigonometric polynomials $S \in \mathcal{T}_n$ not vanishing in the strip

$$\{z \in \mathbb{C} : |\mathrm{Im}(z)| < r\}\,, \qquad 0 < r \le 1\,.$$

Lemma 3.9. *Let $t_0 \in K$ and $0 < r \le 2$. We have*

$$|S'(t_0)| \le 5e\left(\frac{2n}{r}\right)^{1/2} \|S\|_K$$

for every $S \in \mathcal{T}_n$ having no zeros in the disk $D(t_0, r)$.

Proof. If $2/n \le r \le 2$ then using Cauchy's integral formula and Lemma 3.8, we obtain

$$\begin{aligned}|S'(t_0)| &\le \left|\frac{1}{2\pi}\int_{|\zeta - t_0| = \rho} \frac{S(\zeta)}{(\zeta - t_0)^2}\, d\zeta\right| \le e\rho^{-1}\|S\|_K \\ &\le 5e\left(\frac{2n}{r}\right)^{1/2} \|S\|_K\end{aligned}$$

for every $S \in \mathcal{T}_n$ having no zeros in the disk $D(t_0, r)$. If $r < 2/n$ then the classical Bernstein inequality valid for all $S \in \mathcal{T}_n$ gives the lemma. $\square$

4. Proofs of the Theorems

Proof of Theorem 2.2. Let $S \in \mathcal{T}_n$ be of the form $S(t) = |f(e^{it})|^2$, where $f \in \mathcal{P}_n^c$. We define $U \in \mathcal{T}_n$ and $V \in \mathcal{T}_n$ by

$$U(t) := \mathrm{Re}(f(e^{it})) \qquad \text{and} \qquad V(t) := \mathrm{Im}(f(e^{it}))\,, \qquad t \in K\,.$$

Then

$$S(t) = |f(e^{it})|^2 = U(t)^2 + V(t)^2\,, \qquad t \in K\,. \tag{4.1}$$

Suppose $S \in \mathcal{T}_n$ defined by $S(t) = |f(e^{it})|^2$ has at least u zeros in K, and let $\alpha \in (0,1)$. Then the set

$$\{t \in K : |S(t)| \le \alpha \|S\|_K\}$$

can be written as the union of pairwise disjoint intervals I_j, $j = 1, 2, \ldots, m$. Each of the intervals I_j contains a point $y_j \in I_j$ such that

$$|S(y_j)| = \alpha \|S\|_K\,.$$

Hence, (4.1) implies that for each $j = 1, 2, \ldots, m$, we have either

$$|U(y_j)| \ge \sqrt{\alpha/2}\,\|f\|_K \ge \sqrt{\alpha/2}\,\|U\|_K \tag{4.2}$$

or

$$|V(y_j)| \ge \sqrt{\alpha/2}\,\|f\|_K \ge \sqrt{\alpha/2}\,\|V\|_K\,. \tag{4.3}$$

Also, each zero of S lying in K is contained in one of the intervals I_j. Let μ_j denote the number of zeros of S lying in I_j. Since $S \in \mathcal{T}_n$ has at least u zeros in K, so do $U \in \mathcal{T}_n$ and $V \in \mathcal{T}_n$, and we have $\sum_{j=1}^m \mu_j \ge u$. Note that Lemma 3.1 applied to $U \in \mathcal{T}_n$ yields that

$$\mu_j \le en|I_j| \left(\sqrt{\alpha/2}\,\|U\|_K\right)^{-1} \|U\|_K = \frac{e\sqrt{2}\,n}{\sqrt{\alpha}}\,|I_j|$$

for each $j = 1, 2, \ldots, m$ for which (4.2) holds. Also, Lemma 3.1 applied to $V \in \mathcal{T}_n$ yields that

$$\mu_j \le en|I_j| \left(\sqrt{\alpha/2}\,\|V\|_K\right)^{-1} \|V\|_K = \frac{e\sqrt{2}\,n}{\sqrt{\alpha}}\,|I_j|$$

for each $j = 1, 2, \ldots, m$ for which (4.3) holds. Hence

$$\mu_j \le \frac{e\sqrt{2}\,n}{\sqrt{\alpha}}|I_j|\,, \qquad j = 1, 2, \ldots, m\,.$$

Therefore

$$u \le \sum_{j=1}^m \mu_j \le \frac{e\sqrt{2}\,n}{\sqrt{\alpha}} \sum_{j=1}^m |I_j| = \frac{e\sqrt{2}\,n}{\sqrt{\alpha}}\, m(\{t \in K : |S(t)| \le \alpha |S\|_K\})\,,$$

and the theorem follows. $\square$

Proof of Theorem 2.1. We show that the P_k has $o(n)$ zeros on the unit circle, where $n-1 = 2^k - 1$ is the degree of P_k. The proof of the fact that Q_k has $o(n)$ zeros on the unit circle is analogous. Suppose to the contrary that there are $\varepsilon > 0$ and an increasing sequence $(k_j)_{j=1}^{\infty}$ of positive integers such that the Rudin-Shapiro polynomials P_{k_j} have at least εn_j zeros on the unit circle, where $n_j := 2^{k_j}$ for each $j = 1, 2, \ldots$. Then P_{k_j} has at least one zero on the unit circle and hence (1.1) and (1.2) imply that

$$\|P_{k_j}(e^{it})\|_K^2 = 2^{k_j+1}\,. \tag{4.4}$$

Then Theorem 2.2 implies that

$$m\big(\{t \in K : |P_{k_j}(e^{it})|^2 \leq \alpha \|P_{k_j}\|_K^2\}\big) \geq \frac{\sqrt{\alpha}}{e\sqrt{2}}\,\frac{\varepsilon n_j}{n_j} = \frac{\varepsilon\sqrt{\alpha}}{e\sqrt{2}}$$

for every $\alpha \in (0,1)$ and $j = 1, 2, \ldots$. Hence,

$$\liminf_{j\to\infty} m\big(\{t \in K : |P_{k_j}(e^{it})|^2 \leq \alpha \|P_{k_j}(e^{it})\|_K^2\}\big) \geq \frac{\varepsilon\sqrt{\alpha}}{e\sqrt{2}} \tag{4.5}$$

for every $\alpha \in (0,1)$. On the other hand, Conjecture 1.1 proved in [R-16] combined with (4.4) imply that

$$\lim_{j\to\infty} m\big(\{t \in K : |P_{k_j}(e^{it})|^2 \leq \alpha \|P_{k_j}(e^{it})\|_K^2\}\big) = 2\pi\alpha \tag{4.6}$$

for every $\alpha \in (0,1)$. Combining (4.5) and (4.6) we obtain

$$\frac{\varepsilon\sqrt{\alpha}}{e\sqrt{2}} \leq 2\pi\alpha\,,$$

that is, $\varepsilon/e \leq 2\pi\sqrt{2\alpha}$ for every $\alpha \in (0,1)$, a contradiction. $\square$

Proof of Theorem 2.3. We prove that there is an absolute constant $c_1 > 0$ such that $\mathrm{Re}(P_k)$ has at least $c_1 n$ zeros on the unit circle; the fact that each of the functions $\mathrm{Re}(Q_k)$, $\mathrm{Im}(P_k)$, and $\mathrm{Im}(Q_k)$ has at least $c_1 n$ zeros on the unit circle can be proved similarly. Let, as before, $K := \mathbb{R} \pmod{2\pi}$. Let

$$\mathcal{A}_n := \left\{ f : f(t) = \sum_{j=1}^{n} \cos(jt + \alpha_j)\,, \quad \alpha_j \in \mathbb{R} \right\}.$$

Let $S \in \mathcal{A}_{n-1}$ with $n := 2^k$ be defined by

$$S(t) := \mathrm{Re}(P_k(e^{it})) - 1\,.$$

We have

$$\mu = M_2(S) := \left(\frac{1}{2\pi}\int_K |S(t)|^2\,dt\right)^{1/2} = \left(\frac{n-1}{2}\right)^{1/2}.$$

Let $\mathcal{N}(S,v)$ be the number of real roots of $S - v\mu$ in $[-\pi,\pi)$. Observe that (1.1) implies that $|S(t)| \leq (2n)^{1/2} + 1 \leq 2(n-1)^{1/2}$ for all $t \in K$ and $n = 2^k - 1 \geq 3$, and hence

$$\begin{aligned} M_1(S) &= \frac{1}{2\pi}\int_0^{2\pi} |S(t)|\,dt \geq \frac{1}{2\pi}\frac{1}{2(n-1)^{1/2}}\int_0^{2\pi} |S(t)|^2\,dt \\ &= \frac{1}{2(n-1)^{1/2}}\frac{n-1}{2} = \frac{(n-1)^{1/2}}{4} = \frac{\mu}{2\sqrt{2}} =: c\mu \end{aligned}$$

for all $n = 2^k - 1 \geq 3$. Thus, applying Lemma 3.5 with $h := 2^{10}c^{-6}$ we can deduce that there is an absolute constant $c_{12} > 0$ such that

$$S(t) + 1 = \mathrm{Re}(P_k(e^{it}))$$

has at least $c_{12}2^{-10}c^{11} = c_{12}2^{-10}2^{-33/2}(n-1) = c_{12}2^{-53/2}(n-1)$ zeros in $[-\pi,\pi)$ whenever

$$2^{-5}c^3\mu = 2^{-19/2}\sqrt{(n-1)/2} \geq 1\,.$$

This finishes the proof when $n = 2^k - 1$ is sufficiently large.

Observe that the product of all the zeros of P_k is ± 1, so if $k \geq 1$ then P_k always has at least one zero in the closed unit disk. Hence, if $k \geq 1$, then it follows from the Argument Principle that $\mathrm{Re}(P_k(e^{it}))$ has at least two zeros in $[-\pi,\pi)$. $\square$

Proof of Theorem 2.4. The proof is a combination of Lemmas 3.1, 3.2, and 3.3. Recalling (1.2) we can observe that without loss of generality we may assume that $\eta \in (0,1]$, that is, it is sufficient to prove only the first statement of the theorem. As the trigonometric polynomial $R_k(t) - \eta n$ of degree $n-1$ has at most $2(n-1)$ zeros in K, without loss of generality we may assume also that $\eta < \gamma/2$, where $\gamma := \sin^2(\pi/8)$ as before. In the light of Lemma 3.3 it is sufficient to prove that there is an absolute constant $c > 0$ such that the equation $R_k(t) = \eta n$ has at most c solutions in the interval $[\tau_{j-1},\tau_j]$ for every $j \in \{2,3,\ldots,m+1\}$ for which

$$\min_{t\in[\tau_{j-1},\tau_j]} R_k(t) \leq \eta n\,.$$

However, this follows from Lemmas 3.1 combined with Lemma 3.2. $\square$

Proof of Theorem 2.5. Recalling (1.1), without loss of generality we may assume that $\eta \in (0,2\gamma)$. Let

$$I_j := \left[\frac{(2j-2)\pi}{n}, \frac{2j\pi}{n}\right), \qquad j = 1,2,\ldots,n\,.$$

By Saffari's Conjecture 1.1 proved by Rodgers [R-16] we have

$$m(\{t \in K : R_k(t) \leq \eta n\}) > \pi(1-\varepsilon)\eta$$

for every $\eta \in (0,1)$, $\varepsilon > 0$, and sufficiently large $k \geq k_{\eta,\varepsilon}$. Hence, with the notation

$$A_\eta := \{t \in K : R_k(t) \leq \eta n\} ,$$

there are at least $(1-\varepsilon)\eta n/2$ distinct values of $j \in \{1,2,\dots,n\}$ such that $A_\eta \cap I_j \neq \emptyset$ for every $\eta \in (0,1)$ and sufficiently large $k \geq k_{\eta,\varepsilon}$. On the other hand, by Lemma 3.2, for each $j \in \{1,2,\dots,n\}$ there is a $t_j \in I_j$ such that $R_k(t_j) \geq 2\gamma n$. Hence by the Intermediate Value Theorem there are at least $(1-\varepsilon)\eta n/2$ distinct values of $j \in \{1,2,\dots,n\}$ for which there is a $\tau_j \in I_j$ such that $R_k(\tau_j) = \eta n$ for every $\eta \in (0, 2\gamma)$, $\varepsilon > 0$, and sufficiently large $k \geq k_{\eta,\varepsilon}$. $\square$

Proof of Theorem 2.6. Let $S_n \in \mathcal{T}_{n-1}$ be defined by

$$S_n(t) := R_k(t) - n = |P_k(e^{it})|^2 - n = \sum_{j=-n+1}^{n-1} a_j z^j - n \, .$$

We show that $S := S_n$ satisfies the assumptions of Lemma 3.5 with $c = 1/2$ and $h := n^{0.64}$ if $n = 2^k$ is sufficiently large. Clearly, S_n is of the form (3.1) with $b_0 = 0$, $b_m = 2a_m$, and $\gamma_m = 0$ for $m = 1,2,\dots,n-1$. As it is already mentioned in Section 1, Littlewood [L-68] evaluated $M_4(P_k)$ and found that $M_4(P_k) \sim (4^{k+1}/3)^{1/4} = (4n^2/3)^{1/4}$. Hence $\mu := M_2(S_n) \sim (1/3)^{1/2} n$. Also, it follows from (1.1) that $M_\infty(S_n) \leq n$, hence

$$(1/3)n^2 \sim (M_2(S_n))^2 \leq M_1(S_n) M_\infty(f) \leq n M_1(S_n)$$

implies that $M_1(S_n) \geq c\mu$ with $c := 1/2$ if $n = 2^k$ is sufficiently large (in fact, any number $0 < c < 3^{-1/2}$ can be chosen). Now Lemma 3.4, $b_0 = 0$, $b_m = 2a_m \in \{-2, 2\}$, $m = 1,2,\dots,n-1$, and $h := n^{0.64}$ imply that

$$\begin{aligned} s_{\lfloor (n-1)/h \rfloor} = & \sum_{m=1}^{\lfloor (n-1)/h \rfloor} \frac{b_m^2}{\mu^2} \leq \frac{n-1}{h} \frac{(2c_{11} n^{0.8190})^2}{\mu^2} \leq \frac{n}{n^{0.64}} \frac{4c_{11}^2 n^{1.6380}}{(1/4)n^2} \leq 16 c_{11}^2 n^{-0.0020} \\ & \leq 2^{-9} c^6 \end{aligned}$$

if $n = 2^k$ is sufficiently large. So S_n satisfies the assumptions of Lemma 3.5 with $c = 1/2$ and $h := n^{0.64}$ if $n = 2^k$ is sufficiently large, indeed. Thus Lemma 3.5 implies that

$$\mathcal{N}(S_n, v) > c_{12} h^{-1} c^5 n = c_{12} c^5 n^{0.36}$$

whenever v is real with $|v| \leq 2^{-5} c^3 = 2^{-8}$ and $n = 2^k$ is sufficiently large. $\square$

Proof of Theorem 2.8. Suppose P_k does not have a zero in the disk

$$\left\{ z \in \mathbb{C} : |z - e^{it_0}| < \frac{c_8}{n} \right\} .$$

Observe that

$$|e^{i\zeta} - e^{it_0}| = |e^{it_0}(1 - e^{i(\zeta - t_0)})| = |\zeta - t_0| \left| \sum_{j=1}^{\infty} \frac{(i(\zeta - t_0))^{j-1}}{j!} \right|$$
$$\leq 2|\zeta - t_0|\,, \qquad |\zeta - t_0| \leq 1\,,$$

implies that $R_k \in \mathcal{T}_n$ defined by $R_k(\zeta) = P_k(e^{i\zeta})P_k(e^{-i\zeta})$ does not have a zero in

$$\left\{ \zeta \in \mathbb{C} : |\zeta - t_0| < \frac{c_8/2}{n} \right\}.$$

It follows from Lemma 3.6 and $\|R_k\|_K \leq 2n$ that

$$|R_k'(t_0)| \leq 5e\sqrt{\frac{2n}{(c_8/2)/n}}\,\|R_k\|_K \leq \frac{20e}{\sqrt{c_8}}\,n^2 < c_9 n^2$$

whenever

$$0 < c_8 < \frac{c_9^2}{400e^2}\,.$$

Hence, if we chose $c_8 > 0$ as above, P_k must have a zero in the disk

$$\left\{ \zeta \in \mathbb{C} : |\zeta - e^{it_0}| < \frac{c_8}{n} \right\}$$

whenever $R_k'(t_0) \geq c_9 n^2$. □

Proof of Theorem 2.7.

$$I_j := \left[\frac{(2j-2)\pi}{n}, \frac{2j\pi}{n} \right)\,, \qquad j = 1, 2, \ldots, n\,.$$

Let $\gamma := \sin^2(\pi/8)$ as before. By Saffari's Conjecture 1.1 proved by Rodgers [R-16] we have

$$m(\{t \in K : R_k(t) \leq \gamma n\}) > 2\pi(\gamma/4)$$

for every sufficiently large n. Hence, with the notation

$$A := \{t \in K : R_k(t) \leq \gamma n\}\,,$$

there are at least $n\gamma/4$ distinct values of $j \in \{1, 2, \ldots, n\}$ such that $A \cap I_j \neq \emptyset$ for every sufficiently large n. On the other hand, by Lemma 3.2, for each $j \in \{1, 2, \ldots, n\}$ there is a $t_j \in I_j$ such that $R_k(t_j) \geq 2\gamma n$. Hence by the Mean Value Theorem there are at least $n\gamma/4$ distinct values of $j \in \{1, 2, \ldots, n\}$ for which there is a $\tau_j \in I_j$ such that

$$R_k'(\tau_j) \geq \gamma n(2\pi/n)^{-1} \geq \frac{\gamma}{2\pi} n^2$$

for every sufficiently large n. Hence, by Theorem 2.8, there are at least $n\gamma/4$ distinct values of $j \in \{1, 2, \ldots, n\}$ such that the open disk D_j centered at $e^{i\tau_j}$ of radius $c_8 n^{-1}$ has at least one zero of P_k, where the absolute constant $c_8 > 0$ is chosen to $c_9 := \gamma/(2\pi)$ as in the proof of Theorem 2.8, that is,

$$0 < c_8 < \frac{\gamma^2}{1600\pi^2 e^2}\,.$$

□

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