MARKOV INEQUALITY FOR POLYNOMIALS OF DEGREE n WITH m DISTINCT ZEROS

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Abstract. Let $\mathcal{P}_n^m$ be the collection of all polynomials of degree at most n with real coefficients that have at most m distinct complex zeros. We prove that

$$\max_{x\in[0,1]}|P'(x)| \leq 32\cdot 8^m n \max_{x\in[0,1]}|P(x)|$$

for every $P \in \mathcal{P}_n^m$. This is far away from what we expect. We conjecture that the Markov factor $32\cdot 8^m n$ above may be replaced by cmn with an absolute constant $c > 0$. We are not able to prove this conjecture at the moment. However, we think that our result above gives the best known Markov-type inequality for $\mathcal{P}_n^m$ on a finite interval when $m \leq c\log n$.

1. Introduction, Notation, New Result

Markov's inequality asserts that

$$\max_{x\in[0,1]}|P'(x)| \leq 2n^2 \max_{x\in[0,1]}|P(x)|$$

for all polynomials of degree at most n with real coefficients. There is a huge literature about Markov-type inequalities for constrained polynomials. In particular, several essentially sharp improvements are known for various classes of polynomials with restricted zeros. Here we just refer to [1], and the references therein.

Let $\mathcal{P}_n^m$ be the collection of all polynomials of degree at most n with real coefficients that have at most m distinct complex zeros. We prove the following.

Theorem. *We have*

$$\max_{x\in[0,1]}|P'(x)| \leq 32\cdot 8^m n \max_{x\in[0,1]}|P(x)|$$

for every $P \in \mathcal{P}_n^m$.

This is far away from what we expect. We conjecture that the Markov factor $32\cdot 8^m n$ above may be replaced by cmn with an absolute constant $c > 0$. We are not able to prove this conjecture at the moment. However, we think that our result above gives the best known Markov-type inequality for $\mathcal{P}_n^m$ on a finite interval when $m \leq c\log n$.

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2. Proof

It is easy to see by Rouche's Theorem that $\mathcal{P}_n^m$ is closed in the maximum norm on $[0,1]$, and hence in any norm. Therefore it is easy to argue that there is a $P^* \in \mathcal{P}_n^m$ with minimal L_1 norm on $[0,1]$ such that

$$\frac{|P^{*\prime}(0)|}{\max\limits_{x\in[0,1]} |P^*(x)|} = \sup_{P\in\mathcal{P}_n^m} \frac{|P'(0)|}{\max\limits_{x\in[0,1]} |P(x)|}\,.$$

Lemma 1. *There is a polynomial $T \in \mathcal{P}_n^{m+1}$ of the form*

$$T(x) = Q(x)(x-a)\,,$$

where $Q \in \mathcal{P}_{n-1}^m$ has all its zeros in $[0,1]$, $a \in \mathbb{R}$, and

$$\frac{|P^{*\prime}(0)|}{\max\limits_{x\in[0,1]} |P^*(x)|} \leq \frac{|T'(0)|}{\max\limits_{x\in[0,1]} |T(x)|}\,.$$

Proof. Assume that $z_0 \in \mathbb{C}\setminus\mathbb{R}$ is a zero of P^* with multiplicity k. Then

$$P_\varepsilon^*(x) := P^*(x)\left(1 - \varepsilon\frac{x^2}{(x-z_0)(x-\overline{z}_0)}\right)^k$$

with a sufficiently small $\varepsilon > 0$ is in $\mathcal{P}_n^m$ and it contradicts the defining properties of P^*. So each of the zeros of P^* is real. Now let $P^* = RS$ where all the zeros of R are in $[0,1]$, while $S(0) > 0$ and all the zeros of S are in $\mathbb{R}\setminus[0,1]$. We may assume that S is not identically constant, otherwise $T := P^* \in \mathcal{P}_n^{m+1}$ with $Q \in \mathcal{P}_{n-1}^m$ defined by

$$Q(x) := \frac{P^*(x)}{x-a}$$

is a suitable choice, where $x-a$ is any linear factor of P^*. It is easy to see that S can be written as

$$S(x) := \sum_{j=0}^d A_j x^j (1-x)^{d-j}\,, \qquad A_j \geq 0\,,\ \ j = 0,1,\dots,d\,,$$

where $d \geq 1$ is the degree of S. Now let

$$T(x) = R(x)\sum_{j=0}^1 A_j x^j (1-x)^{d-j}\,.$$

Then T is of the form

$$T(x) = Q(x)(x-a)\,,$$

where $Q \in \mathcal{P}_{n-1}^m$ has all its zeros in $[0,1]$, $a \in \mathbb{R}$, and

$$\frac{|P^{*\prime}(0)|}{\max\limits_{x\in[0,1]} |P^*(x)|} \leq \frac{|T'(0)|}{\max\limits_{x\in[0,1]} |T(x)|}\,,$$

and the proof is finished. $\square$

For the sake of brevity let

$$n \leq M(n,m) := \sup_P \frac{|P'(0)|}{\max\limits_{x\in[0,1]} |P(x)|}\,,$$

where the supremum is taken for all $P \in \mathcal{P}_n^m$ having all their zeros in $[0,1]$.

Lemma 2. *Let P^* and $T(x) = Q(x)(x-a)$ be as in Lemma 1. Suppose $a < 0$ or $a > 2$. Then*

$$\max_{x\in[0,1]} |Q(x)| \leq 4M(n,m) \max_{x\in[0,1]} |T(x)|\,.$$

Proof. Let $b \in [0,1]$ be a point for which

$$|Q(b)| = \max_{x\in[0,1]} |Q(x)|\,.$$

Case 1: $b \in [1/2, 1]$. In this case

$$\max_{x\in[0,1]} |Q(x)| = |Q(b)| = \frac{|T(b)|}{|b-a|} \leq 2|T(b)| \leq 2 \max_{x\in[0,1]} |T(x)|\,.$$

Case 2: $b \in [0, 1/2]$. In this case $Q = UV$, where $U \in \mathcal{P}_n^m$ has all its zeros in $[b,1]$, and $V \in \mathcal{P}_n^m$ has all its zeros in $\mathbb{R} \setminus [b,1]$. It is easy to see that V can be written as

$$V(x) := \sum_{j=0}^{d} B_j (x-b)^j (1-x)^{d-j}\,, \qquad B_j \geq 0\,, \qquad j = 0,1,\ldots,d\,,$$

where d is the degree of V. Now let

$$W(x) = U(x) B_0 (1-x)^d\,.$$

Then

$$|W(b)| = |(UV)(b)| = |Q(b)| = \max_{x\in[b,1]} |Q(x)| \tag{1}$$

and

$$|W(x)| \leq |Q(x)|\,, \qquad x \in [b,1]\,. \tag{2}$$

Also $W \in \mathcal{P}_n^m$ has all its zeros in $[b,1]$. Let $\eta > b$ be the smallest point for which

$$|W(\eta)| = \frac{1}{2} \max_{x\in[b,1]} |W(x)|\,.$$

Then $|W'(x)|$ is decreasing on $[b,\eta]$, and it follows by a linear transformation that

$$|W'(b)| \le \frac{M(n,m)}{1-b} \max_{x\in[b,1]} |W(x)|\,. \tag{3}$$

Combining the above by the Mean Value Theorem, we obtain

$$\begin{aligned}\frac{1}{2} \max_{x\in[b,1]} |W(x)| &= |W(b) - W(\eta)| = (\eta - b)|W'(\xi)| \\ &\le (\eta - b)||W'(b)| \le \frac{\eta - b}{1-b} M(n,m) \max_{x\in[b,1]} |W(x)|\,,\end{aligned}$$

whence

$$\eta - b \ge \frac{1-b}{2M(n,m)}\,.$$

This, together with (1), (2), (3), yields

$$\begin{aligned}\max_{x\in[0,1]} |Q(x)| &\le 2|Q(\eta)| = \frac{2|T(\eta)|}{|\eta - a|} = \frac{2|T(\eta)}{|\eta - b|}\,\frac{|\eta - b|}{|\eta - a|} \\ &\le 2|T(\eta)|\,\frac{2M(n,m)}{1-b}\,\frac{1-b}{|1-a|} \le 4M(n,m) \max_{x\in[0,1]} |T(x)|\,,\end{aligned}$$

and the proof is finished. □

Lemma 3. *Let P^* be as in Lemma 1. Then there exists a polynomial $U \in \mathcal{P}_n^{m+1}$ having all its zeros in $[0,1]$ such that*

$$\frac{|U'(0)|}{\max_{x\in[0,1]} |U(x)|} \ge \frac{1}{7}\,\frac{|P^{*\prime}(0)|}{\max_{x\in[0,1]} |P^*(x)|}\,.$$

Proof. Let $T(x) = Q(x)(x-a)$ as in Lemma 1. We distinguish three cases.
Case 1: $a \in [0,1]$. In this case $U(x) = T(x)$ is a suitable choice.
Case 2: $a \in [1,2]$. In this case $U(x) = T(ax)$ is a suitable choice.
Case 3: $a < 0$ or $a > 2$. Then we have

$$T'(0) = -aQ'(0) + Q(0)\,.$$

Combining this with Lemma 2 we obtain

$$\frac{|P^{*\prime}(0)|}{\max\limits_{x\in[0,1]}|P^*(x)|}\leq\frac{|T'(0)|}{\max\limits_{x\in[0,1]}|T(x)|}\leq\frac{|aQ'(0)|}{\max\limits_{x\in[0,1]}|Q(x)(x-a)|}+\frac{|Q(0)|}{\max\limits_{x\in[0,1]}|Q(x)(x-a)|}$$
$$\leq\frac{|aQ'(0)|}{\left|\frac{a}{2}\right|\max\limits_{x\in[0,1]}|Q(x)|}+\frac{|Q(0)|}{(4M(n,m))^{-1}\max\limits_{x\in[0,1]}|Q(x)|}$$
$$\leq 2M(n-1,m)+4M(n,m+1)\leq 6M(n,m)\,.$$

This means that there is a polynomial $U\in\mathcal{P}_n^{m+1}$ having all its zeros in $[0,1]$ such that

$$\frac{|U'(0)|}{\max\limits_{x\in[0,1]}|U(x)|}\geq(1/7)\frac{|P^{*\prime}(0)|}{\max\limits_{x\in[0,1]}|P^*(x)|}\,.\quad\square$$

We introduce

$$n\leq M^*(n,m):=\sup_P\frac{|P'(0)|}{\max\limits_{x\in[0,1]}|P(x)|}\,,$$

where the supremum is taken for all $P\in\mathcal{P}_n^m$ having all their zeros in $[0,1]$ for which

$$|P(0)|=\max_{x\in[0,1]}|P(x)|\,.$$

Lemma 4. *We have $M(n,m+1)=M^*(n,m+1)$.*

Proof. Since $M(n,m+1)\geq M^*(n,m+1)$ is trivial, we need to see only $M(n,m+1)\leq M^*(n,m+1)$. To this end take a $P\in\mathcal{P}_n^{m+1}$ and choose $\alpha\in(-\infty,0]$ so that

$$|P(\alpha)|=\max_{x\in[0,1]}|P(x)|\,.$$

Now let

$$U(x):=P((1-\alpha)x+\alpha)\,.$$

Then $U\in\mathcal{P}_n^{m+1}$ has all its zeros in $[0,1]$ and

$$|U(0)|=|P(\alpha)|=\max_{x\in[0,1]}|P(x)|=\max_{x\in[\alpha,1]}|P(x)|=\max_{x\in[0,1]}|U(x)|\,,$$

while, since $|P'(x)|$ is decreasing on $(-\infty,0]$, we have

$$|U'(0)|=(1-\alpha)|P'(\alpha)|\geq(1-\alpha)|P'(0)|\geq|P'(0)|\,.$$

Therefore

$$\frac{|P'(0)|}{\max\limits_{x\in[0,1]}|P(x)|}\leq\frac{|U'(0)|}{\max\limits_{x\in[0,1]}|U(x)|}\,.\quad\square$$

From Lemmas 3 and 4 we can draw the following conclusion.

Lemma 5. *We have*

$$\sup_{P\in\mathcal{P}_n^m} \frac{|P'(0)|}{\max_{x\in[0,1]}|P(x)|} \le 7M^*(n,m+1)\,.$$

Lemma 6. *We have* $M^*(n,m) \le \frac{2}{7}8^m n\,.$

Proof. Suppose that $P\in\mathcal{P}_n^m$ has all its zeros in $[0,1]$, and

$$|P(0)| = \max_{x\in[0,1]}|P(x)|\,.$$

Let $F(x) := |P(x)|^{1/d}$, where $d(\le n)$ is the degree of P. Then

$$|F(0)| = \max_{x\in[0,1]}|F(x)|\,. \tag{4}$$

Let

$$F(x) = \prod_{i=1}^m |x-x_i|^{\alpha_i}\,,$$

where

$$0<x_1<\ldots<x_m<1\,,\quad 0<\alpha_i,\quad i=1,2,\ldots,m\,,\quad \sum_{i=1}^m \alpha_i = 1\,.$$

We show that

$$\frac{\alpha_i}{x_i} \le 2\cdot 8^{m-i} \tag{5}$$

for $i=1,2,\ldots,m$. To see this let

$$\begin{aligned} A_1 &:= \{1,2,\ldots,i_1\}\,,\\ A_2 &:= \{i_1+1,i_1+2,\ldots,i_2\}\,,\\ &\vdots\\ A_\mu &:= \{i_{\mu-1}+1,i_{\mu-1}+2,\ldots,i_\mu := m\}\,, \end{aligned}$$

be the sets of indices for which

$$\frac{x_{i+1}}{x_i}\le 8 \quad \text{whenever } i \text{ and } i+1 \text{ are in the same set}\,,$$

$$\frac{x_{i+1}}{x_i} > 8 \quad \text{whenever } i \text{ and } i+1 \text{ are in two distinct sets}\,.$$

Now (5) is clear for any $i\in A_\mu$, since (4) implies that

$$\frac{\alpha_i}{x_i} \le \frac{1}{x_i} \le \frac{8^{m-i}}{x_m} \le 2\cdot 8^{m-i}\,.$$

We continue by induction. Assume that (5) holds for any $i \in A_\nu \cup A_{\nu+1} \cup \ldots \cup A_\mu$. We prove that it holds for any $j \in A_{\nu-1}$. Since

$$\prod_{i=1}^{m} |x - x_i|^{\alpha_i} \le F(0) = \prod_{i=1}^{m} |x_i|^{\alpha_i}\,, \qquad x \in [0,1]\,,$$

we have

$$\sum_{i=1}^{m} \alpha_i \log \left| \frac{x}{x_i} - 1 \right| \le 0\,, \qquad x \in [0,1]\,.$$

Let $j \in A_{\nu-1}$ arbitrary and $x^* := 4x_{i_{\nu-1}}$. For $k \in A_\nu \cup A_{\nu+1} \cup \ldots \cup A_\mu$ we have $x^*/x_k \le 1/2$, so

$$\log\left(1 - \frac{x^*}{x_k}\right) \ge -2(\log 2) \cdot \frac{x^*}{x_k}\,.$$

Thus

$$(\log 3) \sum_{i=1}^{i_{\nu-1}} \alpha_i \le 2(\log 2) \cdot x^* \sum_{i=i_{\nu-1}+1}^{m} \frac{\alpha_i}{x_i}\,,$$

$$\frac{\alpha_j}{x_j} \le \frac{2(\log 2)}{\log 3} \frac{x^*}{x_j} \sum_{i=i_{\nu-1}+1}^{m} \frac{\alpha_i}{x_i} \le \frac{2(\log 2)}{\log 3} 4 \cdot 8^{i_{\nu-1}-j} \left(2 + 2\cdot 8 + \cdots + 2 \cdot 8^{m-i_{\nu-1}-1}\right)\,,$$

from which

$$\frac{\alpha_j}{x_j} \le 2 \cdot 8^{m-j}$$

follows immediately. The proof of (5) is complete now for all $i = 1, 2, \ldots, m$. The lemma follows now from (5):

$$\frac{|P'(0)|}{|P(0)|} = d \frac{|F'(0)|}{|F(0)|} \le d\, \frac{2}{7}\, 8^m\,. \quad \square$$

Now it follows from Lemmas 5 and 6 that

Corollary 7. *We have*

$$|P'(0)| \le 2 \cdot 8^{m+1} n \max_{x \in [0,1]} |P(x)|\,.$$

for every $P \in \mathcal{P}_n^m$.

Proof of the Theorem. We need to prove that

$$|P'(y)| \le 4 \cdot 8^{m+1} n \max_{x \in [0,1]} |P(x)|\,.$$

for every $P \in \mathcal{P}_n^m$ and $y \in [0,1]$. However, it follows from Corollary 7 by a simple linear transformation that

$$|P'(y)| \le 2 \cdot 2 \cdot 8^{m+1} n \max_{x \in [y,1]} |P(x)| \le 4 \cdot 8^{m+1} n \max_{x \in [0,1]} |P(x)|\,, \qquad y \in [0, 1/2]\,,$$

and

$$|P'(y)| \le 2 \cdot 2 \cdot 8^{m+1} n \max_{x \in [0,y]} |P(x)| \le 4 \cdot 8^{m+1} n \max_{x \in [0,1]} |P(x)|\,, \qquad y \in [1/2, 1]\,.$$

This finishes the proof. $\square$

References

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