THE "FULL MÜNTZ THEOREM" IN $L_p[0,1]$ FOR $0 < p < \infty$

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Abstract. Denote by $\text{span}\{f_1, f_2, \dots\}$ the collection of all finite linear combinations of the functions $f_1, f_2, \dots$ over $\mathbb{R}$. The principal result of the paper is the following.

Theorem (Full Müntz Theorem in $L_p(A)$ for $p \in (0,\infty)$ and for compact sets $A \subset [0,1]$ with positive lower density at 0). *Let $A \subset [0,1]$ be a compact set with positive lower density at 0. Let $p \in (0,\infty)$. Suppose $(\lambda_j)_{j=1}^\infty$ is a sequence of distinct real numbers greater than $-(1/p)$. Then $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $L_p(A)$ if and only if*

$$\sum_{j=1}^\infty \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty\,.$$

Moreover, if

$$\sum_{j=1}^\infty \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} < \infty\,,$$

then every function from the $L_p(A)$ closure of $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ can be represented as an analytic function on $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}$ restricted to $A \cap (0, r_A)$, where

$$r_A := \sup\{y \in \mathbb{R} : m(A \cap [y,\infty)) > 0\}$$

($m(\cdot)$ denotes the one-dimensional Lebesgue measure).

This improves and extends earlier results of Müntz, Szász, Clarkson, Erdős, P. Borwein, Erdélyi, and Operstein. Related issues about the denseness of $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ are also considered.

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1. Introduction and Notation

Müntz's beautiful classical theorem characterizes sequences $(\lambda_j)_{j=0}^\infty$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

for which the Müntz space $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C[0,1]$. Here, and in what follows, $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ denotes the collection of finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \dots$ with real coefficients, and $C[a,b]$ is the space of all real-valued continuous functions on $[a,b] \subset \mathbb{R}$ equipped with the uniform norm. Müntz's Theorem [Bo-Er3, De-Lo, Go, Mü, Szá] states the following.

Theorem 1.1 (Müntz). *Suppose* $(\lambda_j)_{j=0}^\infty$ *is a sequence with*

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots .$$

Then $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ *is dense in* $C[0,1]$ *if and only if* $\sum_{j=1}^\infty 1/\lambda_j = \infty$.

The original Müntz Theorem proved by Müntz [Mü] in 1914, by Szász [Szá] in 1916, and anticipated by Bernstein [Be] was only for sequences of exponents tending to infinity. The point 0 is special in the study of Müntz spaces. Even replacing $[0,1]$ by an interval $[a,b] \subset [0,\infty)$ in Müntz's Theorem is a non-trivial issue. This is, in large measure, due to Clarkson and Erdős [Cl-Er] and Schwartz [Sch] whose works include the result that if $\sum_{j=1}^\infty 1/\lambda_j < \infty$ then every function belonging to the uniform closure of

$$\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$$

on $[a,b]$ can be extended analytically throughout the region $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < b\}$.

There are many variations and generalizations of Müntz's Theorem [An, Be, Boa, Bo1, Bo2, Bo-Er1-Bo-Er7, B-E-Z, Ch, Cl-Er, De-Lo, Go, Lu-Ko, Op, Sch, So]. There are also still many open problems. In [Bo-Er6] it is shown that the interval $[0,1]$ in Müntz's Theorem can be replaced by an arbitrary compact set $A \subset [0,\infty)$ of positive Lebesgue measure. That is, if $A \subset [0,\infty)$ is a compact set of positive Lebesgue measure, then $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C(A)$ if and only if $\sum_{j=1}^\infty 1/\lambda_j = \infty$. Here $C(A)$ denotes the space of all real-valued continous functions on A equipped with the uniform norm. If A contains an interval then this follows from the already mentioned results of Clarkson, Erdős, and Schwartz. However, their results and methods cannot handle the case when, for example, $A \subset [0,1]$ is a a Cantor type set of positive measure.

In the case that $\sum_{j=1}^\infty 1/\lambda_j < \infty$, analyticity properties of the functions belonging to the uniform closure of $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ on A are also established in [Bo-Er6].

From Theorem 1.1 we can easily obtain the following $L_p[0,1]$ version of the Müntz Theorem.

Theorem 1.2 (Müntz). *Let* $p \in (0,\infty)$. *Suppose* $(\lambda_j)_{j=0}^\infty$ *is a sequence with*

$$0 \le \lambda_0 < \lambda_1 < \lambda_2 < \cdots .$$

Then $\operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ *is dense in* $L_p[0,1]$ *if and only if* $\sum_{j=1}^\infty 1/\lambda_j = \infty$.

The main result of this paper is the following.

Theorem 1.3 (Full Müntz Theorem in $L_p(A)$ for $p \in (0,\infty)$ and for compact sets $A \subset [0,1]$ with positive lower density at 0). *Let $A \subset [0,1]$ be a compact set with positive lower density at 0. Let $p \in (0,\infty)$. Suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers greater than $-(1/p)$. Then $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $L_p(A)$ if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty\,.$$

Moreover, if

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} < \infty\,,$$

then every function from the $L_p(A)$ closure of $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ can be represented as an analytic function on $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}$ restricted to $A \cap (0, r_A)$, where

$$r_A := \sup\{y \in \mathbb{R} : m(A \cap [y,\infty)) > 0\}$$

($m(\cdot)$ denotes the one-dimensional Lebesgue measure).

This corrects, improves, and extends earlier results of Müntz [Mü], Szász [Szá], Clarkson and Erdős [Cl-Er], P. Borwein and Erdélyi [Bo-Er3, Bo-Er4], and Operstein [Op]. Related issues about the denseness of $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ are also considered.

The notations

$$\|f\|_A := \sup_{x \in A} |f(x)|\,,$$

$$\|f\|_{L_{p,w}(A)} := \left(\int_A |f(x)|^p w(x)\,dx\right)^{1/p}\,,$$

$$\|f\|_{L_{\infty,w}(A)} := \inf\{\alpha \in \mathbb{R} :\ |f(x)|w(x) \le \alpha \text{ a.e. on } A\}\,,$$

$$\|f\|_{L_p(A)} := \left(\int_A |f(x)|^p\,dx\right)^{1/p}$$

$$\|f\|_{L_\infty(A)} := \inf\{\alpha \in \mathbb{R} :\ |f(x)| \le \alpha \text{ a.e. on } A\}\,,$$

are used throughout this paper for real-valued measurable functions f defined on a measurable set $A \subset \mathbb{R}$ with positive Lebesgue measure, for nonnegative measurable weight functions w defined on A, and for $p \in (0,\infty)$. The space of all real-valued continuous functions on a set $A \subset \mathbb{R}$ equipped with the uniform norm is denoted by $C(A)$. For $0 < p \le \infty$ the space $L_{p,w}(A)$ is defined as the collection of equivalence classes of real-valued measurable functions for which $\|f\|_{L_{p,w}(A)} < \infty$. The equivalence classes are defined by the equivalence relation $f \sim g$ if $fw = gw$ almost everywhere on A. When $A := [a,b]$ is a finite closed interval, we use the notation $L_{p,w}[a,b] := L_{p,w}(A)$. When $w = 1$, we use the notation $L_p[a,b] := L_{p,w}[a,b]$. It is always our understanding that the space $L_{p,w}(A)$ is equipped with the $L_{p,w}(A)$ norm. Denote by $\operatorname{span}\{f_1, f_2, \dots\}$ the collection of all finite linear combinations of the functions $f_1, f_2, \dots$ over $\mathbb{R}$.

The lower density of a measurable set $A \subset [0,\infty)$ at 0 is defined by

$$d(A) := \liminf_{y \to 0+} \frac{m(A \cap [0,y])}{y}\,.$$

2. Auxiliary Results

In [Bo-Er3, Section 4.2], [Op], and partially in [Bo-Er4] the following two theorems are proved.

Theorem 2.1 (Full Müntz Theorem in $C[0,1]$). *Suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct positive real numbers. Then* $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *is dense in $C[0,1]$ if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} = \infty\,.$$

Moreover, if

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} < \infty\,,$$

then every function from the $C[0,1]$ closure of $\operatorname{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *is infinitely many times differentiable on $(0,1)$.*

Theorem 2.2 (Full Müntz Theorem in $L_p[0,1]$ for $p \in [1,\infty)$). *Suppose $p \in [1,\infty)$. Suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers greater than $-(1/p)$. Then* $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *is dense in $L_p[0,1]$ if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty\,.$$

Moreover, if

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} < \infty\,,$$

then every function from the $L_p[0,1]$ closure of $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *is infinitely many times differentiable on $(0,1)$.*

Unfortunately each of the works mentioned above has some shortcomings in proving the sufficiency part of Theorem 2.2. Hence in Section 4 we present the correct arguments to prove the sufficiency part of Theorem 2.2. This part is based on discussions with Peter Borwein.

Theorems 2.3 and 2.4 are restatements of some earlier results giving sufficient conditions for the non-denseness of $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ in $L_p[0,1]$ when $0 < p < \infty$ and $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct nonnegative numbers. See Theorems 6.1 and 5.6 in [Bo-Er6].

Theorem 2.3. *Suppose $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct nonnegative numbers satisfying $\sum_{j=1}^{\infty} 1/\lambda_j < \infty$. Suppose that $A \subset [0,\infty)$ is a set of positive Lebesgue measure, w is a nonnegative-valued, integrable weight function on A with $\int_A w > 0$, and $p \in (0,\infty)$. Then* $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *is not dense in $L_{p,w}(A)$.*

Moreover, if the gap condition

$$\inf\{\lambda_{j+1} - \lambda_j : \ j = 1, 2, \dots\} > 0 \tag{2.1}$$

holds, then every function $f \in L_{p,w}(A)$ *belonging to the* $L_{p,w}(A)$ *closure of* $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *can be represented as*

$$f(x) = \sum_{j=1}^{\infty} a_j x^{\lambda_j}\,, \qquad x \in A \cap [0, r_w)\,,$$

where

$$r_w := \sup\left\{x \in [0,\infty) : \int_{A\cap(x,\infty)} w(t)\,dt > 0\right\}.$$

If the gap condition (2.1) does not hold, then every function $f \in L_{p,w}(A)$ *belonging to the* $L_{p,w}(A)$ *closure of* $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *can still be represented as an analytic function on*

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_w\}$$

restricted to $A \cap (0, r_w)$.

Theorem 2.4. *Suppose* $\sum_{j=1}^{\infty} 1/\lambda_j < \infty$. *Let* $s > 0$ *and* $p \in (0,\infty)$. *Then there exists a constant* c *depending only on* $\Lambda := (\lambda_j)_{j=1}^{\infty}$, s , *and* p *(and not on* ϱ , A , *or the "length" of* f *) so that*

$$\|f\|_{[0,\varrho]} \leq c\,\|f\|_{L_p(A)}$$

for every $f \in \text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *and for every set* $A \subset [\varrho, 1]$ *of Lebesgue measure at least* s .

Now we offer a sufficient condition for a sequence $(\lambda_j)_{j=1}^{\infty}$ of distinct real numbers greater than $-(1/p)$ converging to $-(1/p)$, to guarantee the nondenseness of $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ in $L_p[0,1]$, where $p \in (0,\infty)$.

Theorem 2.5. *Let* $p \in (0,\infty)$. *Suppose that* $(\lambda_j)_{j=1}^{\infty}$ *is a sequence of distinct real numbers greater than* $-(1/p)$ *satisfying*

$$\sum_{j=1}^{\infty} (\lambda_j + (1/p)) =: \eta < \infty\,.$$

Then $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *is not dense in* $L_p[0,1]$. *Moreover, every function in the* $L_p[0,1]$ *closure of* $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *can be represented as an analytic function on* $\mathbb{C} \setminus (-\infty, 0]$ *restricted to* $(0,1)$.

Proof. The theorem is a consequence of D. J. Newman's Markov-type inequality [Bo-Er3, Theorem 6.1.1 on page 276] (see also [Ne]) and a Nikolskii-type inequality [Bo-Er3, page 281] (see also [Bo-Er5]). We state these as Theorems 2.6 and 2.7. Indeed, it follows from Theorem 2.7 that

$$\|x^{1/p}Q(x)\|_{L_\infty[0,1]} \leq (18 \cdot 2^p \eta)^{1/p}\,\|Q\|_{L_p[0,1]} \tag{2.2}$$

for every $Q \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$. Now repeated applications of Theorem 2.6 with the substitution $x = e^{-t}$ imply that

$$\|(e^{-t/p}Q(e^{-t}))^{(m)}\|_{L_\infty[0,\infty)} \le (9\eta)^m \|e^{-t/p}Q(e^{-t})\|_{L_\infty[0,\infty)}\,, \qquad m = 1, 2, \dots\,,$$

in particular

$$|(e^{-t/p}Q(e^{-t}))^{(m)}(0)| \le (9\eta)^m \|e^{-t/p}Q(e^{-t})\|_{L_\infty[0,\infty)}\,, \qquad m = 1, 2, \dots\,,$$

for every $Q \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$. By using the Taylor series expansion of $e^{-t/p}Q(e^{-t})$ around 0, we obtain that

$$|z^{1/p}Q(z)| \le c_1(K,\eta)\|x^{1/p}Q(x)\|_{L_\infty[0,1]}\,, \qquad z \in K\,, \tag{2.3}$$

for every $Q \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ and for every compact $K \subset \mathbb{C} \setminus \{0\}$, where

$$c_1(K,\eta) := \sum_{m=0}^{\infty} \frac{(9\eta)^m \Big(\max_{z\in K} |\log z|\Big)^m}{m!} = \exp\Big(9\eta \max_{z\in K} |\log z|\Big)$$

is a constant depending only on K and η. Now combining (2.2) and (2.3) gives

$$|Q(z)| \le c_2(K,p,\eta)\|x^{1/p}Q(x)\|_{L_p[0,1]}\,, \qquad z \in K\,, \tag{2.4}$$

for every $Q \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ and for every compact $K \subset \mathbb{C} \setminus \{0\}$, where

$$c_2(K,p,\eta) := c_1(K,\eta) \max_{z\in K} |\log z|^{-(1/p)} = \exp\Big(9\eta \max_{z\in K} |\log z|\Big) \max_{z\in K} |\log z|^{-(1/p)}$$

is a constant depending only on K, p, and η. Now (2.4) shows that if

$$Q_n \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

converges in $L_p[0,1]$, then it converges uniformly on every compact $K \subset \mathbb{C} \setminus \{0\}$, and the theorem is proved. $\square$

Theorem 2.6 (Markov-Type Inequality for Müntz Polynomials). *Suppose that $\gamma_1, \gamma_2, \dots, \gamma_n$ are distinct nonnegative numbers. Then*

$$\|xQ'(x)\|_{[0,1]} \le 9 \left(\sum_{j=1}^{n} \gamma_j\right) \|Q\|_{[0,1]}$$

for every $Q \in \operatorname{span}\{x^{\gamma_1}, x^{\gamma_2}, \dots, x^{\gamma_n}\}$.

Theorem 2.7 (Nikolskii-Type Inequality for Müntz Polynomials). *Let $p \in (0,\infty)$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct real numbers greater than $-(1/p)$. Then*

$$\|x^{1/p}Q(x)\|_{L_\infty[0,1]} \leq \left(18 \cdot 2^p \sum_{j=1}^{n} (\lambda_j + (1/p))\right)^{1/p} \|Q\|_{L_p[0,1]}$$

for every $Q \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_n}\}$.

Our next tool is an extension of the above Nikolskii-type inequality.

Lemma 2.8 (Another Nikolskii-Type Inequality for Müntz Polynomials). *Let $p \in (0,\infty)$. Let $B \subset [0,b]$ be a measurable set satisfying $m(B \cap [0,\beta]) \geq \delta\beta$ for every $\beta \in [0,b]$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct real numbers greater than $-(1/p)$. Suppose that*

$$\sum_{j=1}^{n} (\lambda_j + (1/p)) =: \eta \leq \delta b/36\,,$$

where $\delta \in (0,1]$. Then

$$\|x^{1/p}Q(x)\|_{L_\infty[0,b]} \leq ((2/\delta)b \cdot 2^p)^{1/p}\, \|Q\|_{L_p(B)}\,,$$

and hence

$$\max_{z \in K} |Q(z)| \leq c(K,p,\eta,b,\delta)\|Q\|_{L_p(B)}$$

for every $Q \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_n}\}$ and for every compact $K \subset \mathbb{C} \setminus \{0\}$, where the constant $c(K,p,\eta,b,\delta)$ depends only on K, p, η, b, and δ.

Proof of Lemma 2.8. By using a linear scaling if necessary, without loss of generality we may assume that $b = 1$. Let $Q \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_n}\}$, and pick a point $y \in (0,1]$ for which

$$|y^{1/p}Q(y)| = \max_{t \in [0,1]} |t^{1/p}Q(t)|\,.$$

Then using the Mean Value Theorem and applying Theorem 2.6 (Markov-Type Inequality for Müntz Polynomials) to

$$x^{1/p}Q(x) \in \operatorname{span}\{x^{\lambda_1+(1/p)}, x^{\lambda_2+(1/p)}, \dots, x^{\lambda_n+(1/p)}\}\,,$$

we obtain for $x \in [(\delta/2)y, y]$ that

$$\begin{aligned}
\left(\max_{t\in[0,1]} |t^{1/p}Q(t)|\right) - |x^{1/p}Q(x)| &\leq |y^{1/p}Q(y)| - |x^{1/p}Q(x)| \\
&\leq |y^{1/p}Q(y) - x^{1/p}Q(x)| \leq (y-x) \max_{t\in[x,y]} |(t^{1/p}Q(t))'| \\
&\leq y\frac{1}{x} \max_{t\in[x,y]} |t(t^{1/p}Q(t))'| \leq \frac{2}{\delta}\, x\, \frac{9\eta}{x} \max_{t\in[0,1]} |t^{1/p}Q(t)| \\
&\leq \frac{18\eta}{\delta} \max_{t\in[0,1]} |t^{1/p}Q(t)| \leq \frac{1}{2} \max_{t\in[0,1]} |t^{1/p}Q(t)|\,.
\end{aligned}$$

Hence, for $x \in [(\delta/2)y, y]$ we have

$$|x^{1/p}Q(x)| \geq \frac{1}{2} \max_{t \in [0,1]} |t^{1/p}Q(t)|\,.$$

Using the assumption on the set B, we conclude that

$$m(B \cap [(\delta/2)y, y]) \geq \delta y - (\delta/2)y = (\delta/2)y\,,$$

and hence

$$\begin{aligned} \|Q\|^p_{L_p(B)} &= \int_B |Q(t)|^p\,dt \geq \int_{B \cap [(\delta/2)y,y]} |Q(t)|^p\,dt \\ &\geq (\delta/2)y2^{-p}\left(y^{-(1/p)}\right)^p \left(\max_{t \in [0,1]} |t^{1/p}Q(t)|\right)^p \\ &= (\delta/2)2^{-p}\left(\max_{t \in [0,1]} |t^{1/p}Q(t)|\right)^p\,. \end{aligned}$$

This finishes the proof of the first inequality of the lemma when $b = 1$. As we have already remarked the case of an arbitrary $b > 0$ follows by a linear scaling. The second inequality of the lemma follows from the first one and from (2.3) applied to $\widetilde{Q}(x) = Q(bx)$, where $\widetilde{Q} \in \text{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$. $\square$

Corollary 2.9. *Let $p \in (0,\infty)$ and $\delta \in (0,1]$. Let $B \subset [0,b]$ be a measurable set satisfying $m(B \cap [0,\beta]) \geq \delta\beta$ for every $\beta \in [0,b]$. Let $(\lambda_j)_{j=1}^{\infty}$ be a sequence of distinct real numbers greater than $-(1/p)$ satisfying*

$$\sum_{j=1}^{\infty} (\lambda_j + (1/p)) =: \eta \leq \delta b/36\,.$$

Then $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *is not dense in $L_p(B)$. Moreover, every function from the $L_p(B)$ closure of* $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ *can be represented as an analytic function on $\mathbb{C} \setminus (-\infty, 0]$ restricted to $B \setminus \{0\}$.*

Proof of Corollary 2.9. The corollary is a consequence of D. J. Newman's Markov-type inequality formulated in Theorem 2.6, and our Nikolskii-type inequality given by Lemma 2.8. Indeed, it follows from Lemma 2.8 and Theorem 2.6 by the substition $z = e^{-t}$ and by the Taylor expansion of $e^{-t/p}Q(e^{-t})$ around 0 that

$$|z^{1/p}Q(z)| \leq c(K,p,b,\delta)\|Q\|_{L_p(B)}$$

whenever $p \in (0,\infty)$, $B \subset [0,b]$ is a measurable set satisfying $m(B \cap [0,\beta]) \geq \delta\beta$ for every $\beta \in [0,b]$, $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers greater than $-(1/p)$ satisfying

$$\sum_{j=1}^{n} (\lambda_j + (1/p)) = \eta \leq \delta b/36\,,$$

$\delta \in (0,1]$, $Q \in \text{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$, $K \subset \mathbb{C}$ is bounded, and $z \in K$, where $c(K,p,b,\delta)$ is a constant depending only on K, p, b, and δ. $\square$

Corollary 2.10. *Let $p \in (0, \infty)$. Let $A \subset [0,1]$ be a measurable set with lower density $\delta > 0$ at 0. Let $(\lambda_j)_{j=1}^{\infty}$ be a sequence of distinct real numbers greater than $-(1/p)$ satisfying*

$$\sum_{j=1}^{\infty} (\lambda_j + (1/p)) < \infty\,.$$

Then $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *is not dense in $L_p(A)$. Moreover, every function from the $L_p(A)$ closure of* $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ *can be represented as an analytic function on $\mathbb{C} \setminus (-\infty, 0]$ restricted to $A \setminus \{0\}$.*

Proof of Corollary 2.10. The corollary follows easily from Corollary 2.9. To see this, choose $b \in (0,1]$ such that with $B := A \cap [0,b]$ we have $m(B \cap [0,\beta]) \geq \delta\beta$ for every $\beta \in [0,b]$. Then choose $N \in \mathbb{N}$ such that

$$\sum_{j=N+1}^{\infty} (\lambda_j + (1/p)) =: \eta \leq \delta b/36\,.$$

Let U be the $L_p(A)$ closure of

$$\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}\,.$$

Let V be the $L_p(A)$ closure of

$$\operatorname{span}\{x^{\lambda_{N+1}}, x^{\lambda_{N+2}}, \dots\}\,.$$

Since the space

$$W := \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_N}\}$$

is finite dimensional, we have $U \subset V + W$. Therefore, by Corollary 2.9 every function from the $L_p(A)$ closure of $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ can be represented as an analytic function on $\mathbb{C} \setminus (-\infty, 0]$ restricted to $A \setminus \{0\}$. $\square$

Finally in this section we restate a Nikolskii-type inequality that is proved in [Bo-Er3, pages 216–217] for $1 \leq p < \infty$.

Theorem 2.11. *Let $p \in [1, \infty)$. Suppose that $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers greater than $-(1/p)$ satisfying*

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} < \infty\,.$$

Then for every $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ depending only on ε so that

$$|Q(x)| \leq c_\varepsilon x^{-(1/p)} \|Q\|_{L_p[0,1]}$$

for every $Q \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ and for every $x \in [0, 1-\varepsilon)$.

We suspect that the above theorem may extend to all $0 < p < \infty$ and would offer a natural approach to prove one half of the "Full Müntz Theorem in $L_p[0,1]$" when $0 < p < 1$. However, we are unable to prove this extension. Nevertheless we can still prove the "Full Müntz Theorem in $L_p[0,1]$" for all $0 < p < \infty$ with the help of Theorems 2.3 – 2.8 and Theorem 3.5. This "Full Müntz Theorem in $L_p[0,1]$" for all $0 < p < \infty$ is formulated by Theorem 3.6.

3. New Results

The new results of the paper include the resolution of the conjecture that the "Full Müntz Theorem in $L_p[0,1]$" remains valid when $0 < p < 1$. Theorems 3.1 and 3.2 offer the right sufficient conditions for the denseness of $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ in $L_p[0,1]$ when $0 < p < 1$. The "easy case" when $\Lambda := (\lambda_j)_{j=1}^\infty$ is a sequence of distinct real numbers greater than $-(1/p)$ tending to $-(1/p)+\alpha$, where $\alpha > 0$, is handled by Theorem 3.1.

Theorem 3.1. *Let $p \in (0,\infty)$. Suppose $\Lambda := (\lambda_j)_{j=1}^\infty$ is a sequence of distinct real numbers greater than $-(1/p)$ tending to $-(1/p)+\alpha$, where $\alpha > 0$. Then $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $L_p[0,1]$.*

In the much more interesting case, when $\Lambda := (\lambda_j)_{j=1}^\infty$ is a sequence of distinct real numbers greater than $-(1/p)$ tending to $-(1/p)$, our next theorem offers a sufficient condition for the denseness of $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ in $L_p[0,1]$, $p \in (0,1]$.

Theorem 3.2. *Let $p \in (0,\infty)$. Let $\Lambda := (\lambda_j)_{j=1}^\infty$ be a sequence of distinct real numbers greater than $-(1/p)$ tending to $-(1/p)$. Suppose that*

$$\sum_{j=1}^{\infty} (\lambda_j + (1/p)) = \infty .$$

Then $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $L_p[0,1]$.

Our next theorem establishes a sufficient condition for the non-denseness of $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ in $L_p(A)$ where $0 < p < \infty$ and $A \subset [0,1]$ is a compact set with positive lower density at 0. It extends one direction of the "Full Müntz Theorem" in $L_p[0,1]$ proved earlier for $p \in [1,\infty)$, see Theorem 2.2. Moreover, the statement about the $L_p(A)$ closure of $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ in the non-dense case is new even for $A = [0,1]$ and $1 \le p < \infty$.

Theorem 3.3. *Let $A \subset [0,1]$ be a compact set with positive lower density at 0. Let $p \in (0,\infty)$. Suppose $(\lambda_j)_{j=1}^\infty$ is a sequence of distinct real numbers greater than $-(1/p)$. Suppose*

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} < \infty .$$

Then every function from the $L_p(A)$ closure of $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ can be represented as an analytic function on $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}$ restricted to $A \cap (0, r_A)$, where

$$r_A := \sup\{y \in \mathbb{R} : m(A \cap [y,\infty)) > 0\}$$

($m(\cdot)$ denotes the one-dimensional Lebesgue measure).

The key to the proof of Theorem 3.3 is a combination of Theorems 2.3 – 2.7 with the following functional analytic theorem.

Theorem 3.4. *Let $p \in (0,\infty)$. Assume that W and V are closed linear subspaces of $L_p[0,1]$ such that*

$$\|f\|_{L_\infty[0,1/2]} \leq C_1 \|f\|_{L_p[0,1]}$$

for every $f \in W$, and

$$\|f\|_{L_\infty[1/2,1]} \leq C_2 \|f\|_{L_p[0,1]}$$

for every $f \in V$, where C_1 and C_2 are positive constants depending only on W and V, respectively. Then $W+V$ is closed in $L_p[0,1]$.

A straightforward modification of the proof of the above theorem yields

Theorem 3.5. *Let $p \in (0,\infty)$. Let $A_1, A_2 \subset \mathbb{R}$ be sets of finite positive measure with $A_1 \cap A_2 = \emptyset$. Assume that W and V are closed linear subspaces of $L_p(A_1 \cup A_2)$ such that*

$$\|f\|_{L_\infty(A_1)} \leq C_1 \|f\|_{L_p(A_1 \cup A_2)}$$

for every $f \in W$, and

$$\|f\|_{L_\infty(A_2)} \leq C_2 \|f\|_{L_p(A_1 \cup A_2)}$$

for every $f \in V$, where C_1 and C_2 are positive constants depending only on W and V, respectively. Then $W+V$ is closed in $L_p(A_1 \cup A_2)$.

Theorem 3.6 (Full Müntz Theorem in $L_p(A)$ for $p \in (0,\infty)$ and for compact sets $A \subset [0,1]$ with positive lower density at 0.). *Let $A \subset [0,1]$ be a compact set with positive lower density at 0. Let $p \in (0,\infty)$. Suppose $(\lambda_j)_{j=1}^\infty$ is a sequence of distinct real numbers greater than $-(1/p)$. Then $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $L_p(A)$ if and only if*

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty .$$

Moreover, if

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} < \infty ,$$

then every function from the $L_p(A)$ closure of $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ can be represented as an analytic function on $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A]\}$ restricted to $A \cap (0, r_A)$, where

$$r_A := \sup\{y \in \mathbb{R} : m(A \cap [y,\infty)) > 0\}$$

($m(\cdot)$ denotes the one-dimensional Lebesgue measure).

It may be interesting to compare Theorem 3.6 with Theorems 3.A and 3.B below proved in [Bo-Er7]. Let

$$\|f\|_{L_{p,w}(A)} := \left(\int_A |f(x)|^p w(x)\, dx \right)^{1/p} .$$

The space $L_{p,w}(A)$ is the collection of all real-valued measurable functions on A for which $\|f\|_{L_{p,w}(A)} < \infty$.

Theorem 3.A (Full Müntz Theorem in $L_p(A)$ for $p \in (0,\infty)$ when $A \subset [0,1]$ is compact and $\inf A > 0$). *Suppose $(\lambda_j)_{j=-\infty}^{\infty}$ is a sequence of distinct real numbers satisfying*

$$\sum_{\substack{j=-\infty \\ \lambda_j \neq 0}}^{\infty} \frac{1}{|\lambda_j|} < \infty$$

with $\lambda_j < 0$ for $j < 0$ and $\lambda_j \geq 0$ for $j \geq 0$. Suppose $A \subset [0,\infty)$ is a set of positive Lebesgue measure with $\inf A > 0$, w is a nonnegative-valued, integrable weight function on A with $\int_A w > 0$, and $p \in (0,\infty)$. Then

$$\operatorname{span}\{x^{\lambda_j} : j \in \mathbb{Z}\}$$

is not dense in $L_{p,w}(A)$.

Suppose the gap condition

$$\inf\{\lambda_j - \lambda_{j-1} : j \in \mathbb{Z}\} > 0$$

holds. Then every function $f \in L_{p,w}(A)$ belonging to the $L_{p,w}(A)$ closure of

$$\operatorname{span}\{x^{\lambda_j} : j \in \mathbb{Z}\}$$

can be represented as

$$f(x) = \sum_{j=-\infty}^{\infty} a_j x^{\lambda_j}, \qquad x \in A \cap (a_w, b_w),$$

where

$$a_w := \inf\left\{ y \in [0,\infty) : \int_{A\cap(0,y)} w(x)\,dx > 0 \right\}$$

and

$$b_w := \sup\left\{ y \in [0,\infty) : \int_{A\cap(y,\infty)} w(x)\,dx > 0 \right\}.$$

If the above gap condition does not hold, then every function $f \in L_{p,w}(A)$ belonging to the $L_{p,w}(A)$ closure of

$$\operatorname{span}\{x^{\lambda_j} : j \in \mathbb{Z}\}$$

can still be represented as an analytic function on

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : a_w < |z| < b_w\}$$

restricted to $A \cap (a_w, b_w)$.

Theorem 3.B (Full Müntz Theorem in $L_p(A)$ for $p \in (0,\infty)$ when $A \subset [0,1]$ is compact and $\inf A > 0$, Part 2). *Suppose $(\lambda_j)_{j=-\infty}^{\infty}$ is a sequence of distinct real numbers. Suppose $A \subset (0,\infty)$ is a bounded set of positive Lebesgue measure, $\inf A > 0$, w is a nonnegative-valued integrable weight function on A with $\int_A w > 0$, and $p \in (0,\infty)$. Then*

$$\operatorname{span}\{x^{\lambda_j} : j \in \mathbb{Z}\}$$

is dense in $L_{p,w}(A)$ if and only if

$$\sum_{\substack{j=-\infty \\ \lambda_j \neq 0}}^{\infty} \frac{1}{|\lambda_j|} < \infty \, .$$

Finally our next theorem offers an upper bound for the $L_p[0,1]$ distance from x^m to

$$\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_n}\}\,,$$

when $\Lambda := (\lambda_j)_{j=1}^{\infty}$ is a sequence of real numbers tending to $-(1/p)$ and $m = -(1/p) + \alpha$ for some $\alpha > 0$.

Theorem 3.7. *Let $p > 0$. Let $\Lambda := (\lambda_j)_{j=1}^{\infty}$ be a strictly decreasing sequence of real numbers tending to $-(1/p)$. Let $m = -(1/p) + \alpha$ for some $\alpha > 0$. Then there are*

$$R_n \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_n}\}\,.$$

such that

$$\int_0^1 |x^m - R_n(x)|^p \, dx$$

$$\leq \frac{c(\Lambda,\alpha)^p}{p \min_{1 \leq j \leq n}(\lambda_j + (1/p))} \exp\left(-p\left(\frac{1}{2\alpha} - \frac{1}{2}\right)\sum_{j=1}^{n}(\lambda_j + (1/p))\right)$$

whenever $\min_{1\leq j\leq n}(\lambda_j + (1/p)) \leq \alpha$, where $c(\Lambda,\alpha)$ is a constant depending only on Λ and α.

4. Proof of Theorems 3.1, 3.2, 3,7, and the sufficiency part of Theorem 2.2

To prove the sufficiency part of Theorem 2.2 we need the following; see [1, page 191].

Blaschke's Theorem. *Suppose $(\beta_j)_{j=1}^{\infty}$ is a sequence in $D := \{z \in \mathbb{C} : |z| < 1\}$ satisfying*

$$\sum_{j=1}^{\infty}(1 - |\beta_j|) = \infty \, .$$

Denote the multiplicity of β_k in $(\beta_j)_{j=1}^{\infty}$ by m_k. Assume that f is a bounded analytic function on D having a zero at each β_j with multiplicity m_j. Then $f = 0$ on D.

The proof below is based on the Riesz Representation Theorem for continuous linear functionals on $L_p[0,1]$, valid for $p \in [1,\infty)$, so the assumption $p \in [1,\infty)$ in Theorem 2.2 is essential for our arguments.

Proof of the sufficiency part of Theorem 2.2. Choosing a subsequence if necessary, without loss of generality we may assume that one of the following three cases occurs.
Case 1: $\lambda_j \geq 1$ for each $j = 1,2,\dots$ with $\sum_{j=1}^{\infty}(1/\lambda_j) = \infty$.
Case 2: $(\lambda_j)_{j=1}^{\infty}$ is a sequence of distinct real numbers tending to $-(1/p)+\alpha$, where $\alpha > 0$.
Case 3: $-(1/p) < \lambda_j \leq 0$ for each $j = 1,2,\dots$ with $\sum_{j=1}^{\infty}(\lambda_j + (1/p)) = \infty$ and $\lim_{j\to\infty}\lambda_j = -(1/p)$.

In Case 1, Theorem 2.1 (Full Müntz Theorem in $C[0,1]$) yields that $\text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $C[0,1]$. From this we can easily deduce that $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $L_p[0,1]$.

In Case 2, Theorem 3.1 implies that $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ is dense in $L_p[0,1]$.

In Case 3, we argue as follows. By the Hahn-Banach Theorem and the Riesz Representation Theorem for continuous linear functionals on $L_p[0,1]$ we know that

$$\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$$

is not dense in $L_p[0,1]$ if and only if there exists a $0 \neq h \in L_q[0,1]$ satisfying

$$\int_0^1 t^{\lambda_j} h(t)\,dt = 0\,, \qquad j = 1,2,\dots\,, \tag{4.1}$$

where q is the conjugate exponent of p defined by $p^{-1} + q^{-1} = 1$. Suppose there exists a $0 \neq h \in L_q[0,1]$ such that (4.1) holds. Let

$$f(z) := \int_0^1 t^z h(t)\,dt\,, \qquad \text{Re}(z) > -(1/p)\,.$$

We can easily show by using Hölder's inequality that

$$g(z) := (z+1)^2 f(z+1-(1/p))$$

is a bounded analytic function on the open unit disk, that satisfies

$$g(\lambda_j + (1/p) - 1) = 0\,.$$

Now

$$\sum_{j=1}^{\infty}(1 - |\lambda_j + (1/p) - 1|) = \sum_{j=1}^{\infty}(1 - (1 - \lambda_j - (1/p)) = \sum_{j=1}^{\infty}(\lambda_j + (1/p)) = \infty\,.$$

Hence Blaschke's Theorem with $\beta_j := \lambda_j + (1/p) - 1$, $j = 1,2,\dots$, yields that $g = 0$ on the open unit disk. Therefore $f = 0$ on the open disk with diameter $[-(1/p), 2-(1/p)]$.

Now observe that f is an analytic function on the half plane $\{z \in \mathbb{C} : \operatorname{Re}(z) > -(1/p)\}$, hence $f(z) = 0$ whenever $\operatorname{Re}(z) > -(1/p)$, so

$$f(n) = \int_0^1 t^n h(t)\, dt = 0\,, \qquad n = 0, 1, 2, \dots\,.$$

Now the Weierstrass Approximation Theorem yields that

$$\int_0^1 u(t)h(t)\, dt = 0$$

for every $u \in C[0,1]$. This implies

$$\int_0^x h(t)\, dt = 0$$

for all $x \in [0,1]$, so $h(x) = 0$ almost everywhere on $[0,1]$, a contradiction. □

Proof of Theorem 3.1. Let

$$\lambda_j^* = \lambda_j + (1/p) - (\alpha/2)\,,$$

where the assumptions on Λ insure that $\lambda_j^* > (\alpha/4)$ for all sufficiently large j . Let $m \geq (\alpha/2)$. Then by Theorem 2.1 (Full Müntz Theorem in $C[0,1]$), for every $\varepsilon > 0$ there is $Q_\varepsilon \in \operatorname{span}\{x^{\lambda_1^*}, x^{\lambda_2^*}, \dots\}$ such that

$$\|x^{m-(\alpha/2)+(1/p)} - Q_\varepsilon\|_{[0,1]} < \varepsilon\,.$$

Let

$$R_\varepsilon(x) := x^{(\alpha/2)-(1/p)} Q_\varepsilon(x) \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}\,.$$

Then

$$\begin{aligned}
\int_0^1 |x^m - R_\varepsilon(x)|^p\, dx &= \int_0^1 \left| x^{(\alpha/2)-(1/p)} \left(x^{m-(\alpha/2)+(1/p)} - Q_\varepsilon(x) \right) \right|^p dx \\
&\leq \left(\int_0^1 x^{p(\alpha/2)-1}\, dx \right) \left\| x^{m-(\alpha/2)+(1/p)} - Q_\varepsilon(x) \right\|_{L_\infty[0,1]}^p \\
&\leq \frac{\varepsilon^p}{p(\alpha/2)}\,.
\end{aligned}$$

Hence the monomials x^m are in the $L_p[0,1]$ closure of $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ for all sufficiently large m. Now Theorem 2.1 (Full Müntz Theorem in $C[0,1]$) implies that the elements f of $C[0,1]$ with $f(0) = 0$ are contained in the $L_p[0,1]$ closure of $\operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$, and since all such functions form a dense set in $L_p[0,1]$, the theorem is proved. □

Proof of Theorem 3.2. The case $p \in [1, \infty)$ is handled by Theorem 2.2 (the part of Theorem 2.2 needed here is proved in the beginning of this section). So in the rest of the proof we assume that $p \in (0,1)$.

Step 1. For $t > 0$ we define $f_t(x) := x^t(1-\log x)^b$, $x \in (0,1]$, and $f_t(0) := 0$. Let $b \in [1,\infty)$. We show that

$$\mathrm{span}\{1 \cup \{f_t : t > 0\}\}$$

is dense in $C[0,1]$. To see this, for a given $\varepsilon > 0$ we get a polynomial P so that

$$\|(1-\log x)^{-b} - P(x)\|_{L_\infty[0,1]} < \varepsilon\,.$$

This can be done by the Weierstrass Theorem. For $m \geq 1$ multiply through by the factor $x^m(1-\log x)^b$ to see that

$$\|x^m - x^m(1-\log x)^b P(x)\|_{L_\infty[0,1]} < \varepsilon \|x(1-\log x)^b\|_{L_\infty[0,1]}\,.$$

Step 2. It is elementary calculus to show that for $x \in (0,1], a \in (0,1)$, and $b \in [1,\infty)$, we have

$$x^a(1-\log x)^b \leq (b/a)^b\,.$$

Step 3. Suppose $(\gamma_n)_{n=1}^\infty$ is a strictly decreasing sequence tending to 0. Suppose

$$\sum_{j=1}^\infty \gamma_j = \infty \qquad \text{and} \qquad b \in [1,\infty)\,.$$

Let

$$f_0 := 1 \qquad \text{and} \qquad f_n(x) := x^{\gamma_n}(1-\log x)^b\,, \quad x \in (0,1]\,, \quad f_n(0) := 0\,, \quad n = 1,2,\ldots\,.$$

We show that $\mathrm{span}\{f_n : n = 0,1,2,\ldots\}$ is dense in $C[0,1]$.

Suppose to the contrary that $\mathrm{span}\{f_n : n = 0,1,2,\ldots\}$ is not dense in $C[0,1]$. Then by the Hahn Banach Theorem and the Riesz Representation Theorem there is a nonzero finite signed measure μ on $[0,1]$ so that for each $n = 0,1,2,\ldots$ we have

$$\int_0^1 f_n(x)\,d\mu(x) = 0\,.$$

For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ we define

$$F(z) = \int_0^1 x^z(1-\log x)^b\,d\mu(x)\,.$$

So F is analytic and bounded on

$$\{z \in \mathbb{C} : \mathrm{Re}(z) > a\}$$

for all $a > 0$. Now for any z in the open unit disk, we define $g(z) := (1+z)^{2b}F(z+1)$. Observe that $g(\gamma_n - 1) = 0$ for each $n = 1,2,\ldots$. Step 2 implies that g is bounded on the open unit disk, so by Blaschke's Theorem and the hypothesis on γ_n we conclude that $g = 0$ on the open unit disk, hence $F(z) = 0$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ by the Unicity Theorem of analytic functions.

Step 4. Now assume that $\Lambda := (\lambda_j)_{j=1}^{\infty}$ satisfies the assumptions of the theorem. Up to now $b \in [1, \infty)$ was arbitrary. Now take $b > 1/p$, so that $x^{-(1/p)}(1 - \log x)^{-b}$ is in $L_p[0,1]$. Let

$$\gamma_j := \lambda_j + (1/p) , \qquad j = 1, 2, \dots .$$

For $m \geq 1$ and $\varepsilon > 0$ we use Step 3 to get an $n \in \mathbb{N}$ and coefficients $a_1, a_2, \dots, a_n \in \mathbb{R}$ so that

$$\left\| x^{m+(1/p)}(1 - \log x)^b - \sum_{j=1}^{n} a_j x^{\lambda_j + (1/p)}(1 - \log x)^b \right\|_{L_p[0,1]} < \varepsilon .$$

Then

$$\left\| x^m - \sum_{j=1}^{n} a_j x^{\lambda_j} \right\|_{L_p[0,1]} \leq \varepsilon \| x^{-(1/p)}(1 - \log x)^{-b} \|_{L_p[0,1]} .$$

Hence x^m is in the $L_p[0,1]$ closure of $\mathrm{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ for every integer $m \geq 1$. Since the polynomials with constant term 0 form a dense set in $L_p[0,1]$, the theorem is proved. $\square$

Proof of Theorem 3.7. Let $m = -(1/p) + \alpha$ with $\alpha > 0$. Let $k = k(n)$ be such that

$$\lambda_k = \min_{1 \leq j \leq n} \lambda_j .$$

For $j = 1, 2, \dots, n$ let

$$\lambda_j^* := \lambda_j + (1/p) > 0 , \qquad \mu_j^* = \lambda_j^* - (\lambda_k^*/2) > 0 , \qquad \widetilde{\mu}_j := \mu_j^* - (1/2) > -(1/2) .$$

Note that

$$0 < \lambda_j^*/2 \leq \mu_j^* \leq \lambda_j^* \tag{4.2}$$

for every $j = 1, 2, \dots$. Assume that $\lambda_k + (1/p) = \lambda_k^* \leq \alpha$. By [Bo-Er3, page 173], there is a

$$P_n \in \mathrm{span}\{x^{\widetilde{\mu}_1}, x^{\widetilde{\mu}_2}, \dots, x^{\widetilde{\mu}_n}\}$$

such that

$$
\begin{aligned}
&\left\| x^{m-(\lambda_k^*/2)+(1/p)-(1/2)} - P_n(x) \right\|_{L_2[0,1]} \\
&\leq \frac{1}{\sqrt{2m-\lambda_k^*+(2/p)}} \left| \prod_{j=1}^{n} \frac{(m-(\lambda_k^*/2)+(1/p)-(1/2)) - \widetilde{\mu}_j}{(m-(\lambda_k^*/2)+(1/p)-(1/2)) + \widetilde{\mu}_j + 1} \right| \\
&\leq \frac{1}{\sqrt{2\alpha-\lambda_k^*}} \left| \prod_{j=1}^{n} \left(1 - \frac{2\widetilde{\mu}_j + 1}{(m-(\lambda_k^*/2)+(1/p)-(1/2)) + \widetilde{\mu}_j + 1} \right) \right| \\
&\leq \frac{1}{\sqrt{\alpha}} \left| \prod_{j=1}^{n} \left(1 - \frac{2\mu_j^*}{\alpha + \mu_j^* - (\lambda_k^*/2)} \right) \right| \leq \frac{c_1(\Lambda,\alpha)}{\sqrt{\alpha}} \left| \prod_{\substack{j=1 \\ \mu_j^* \leq \alpha/2}}^{n} \left(1 - \frac{2\mu_j^*}{2\alpha} \right) \right| \\
&\leq \frac{c_1(\Lambda,\alpha)}{\sqrt{\alpha}} \left| \prod_{\substack{j=1 \\ \mu_j^* \leq \alpha/2}}^{n} \left(1 - \frac{\lambda_j^*}{2\alpha} \right) \right| \leq c_2(\Lambda,\alpha) \exp\left(-\frac{1}{2\alpha} \sum_{\substack{j=1 \\ \mu_j^* \leq \alpha/2}}^{n} \lambda_j^* \right) \\
&\leq c_3(\Lambda,\alpha) \exp\left(-\frac{1}{2\alpha} \sum_{j=1}^{n} \lambda_j^* \right) ,
\end{aligned}
$$

where $c_1(\Lambda,\alpha)$, $c_2(\Lambda,\alpha)$, and $c_3(\Lambda,\alpha)$ are constants depending only on Λ and α . Now let

$$
Q_n(x) := x^{1/2} P_n(x) \in \operatorname{span}\{x^{\mu_1^*}, x^{\mu_2^*}, \ldots, x^{\mu_n^*}\} .
$$

Then, combining the Nikolskii-type inequality of [Bo-Er3, page 281] (see Theorem 2.7 of this paper) and the above $L_2[0,1]$ estimate, we obtain

$$
\begin{aligned}
&\left\| x^{m-(\lambda_k^*/2)+(1/p)} - Q_n(x) \right\|_{L_\infty[0,1]} \\
&= \left\| \left(x^{m-(\lambda_k^*/2)+(1/p)-(1/2)} - P_n(x) \right) x^{1/2} \right\|_{L_\infty[0,1]} \\
&\leq 6\sqrt{2} \left(\alpha + \sum_{j=1}^{n} (\widetilde{\mu}_j + (1/2)) \right)^{1/2} \left\| x^{m-(\lambda_k^*/2)+(1/p)-(1/2)} - P_n(x) \right\|_{L_2[0,1]} \\
&\leq 6\sqrt{2} \left(\alpha + \sum_{j=1}^{n} (\widetilde{\mu}_j + (1/2)) \right)^{1/2} c_3(\Lambda,\alpha) \exp\left(-\frac{1}{2\alpha} \sum_{j=1}^{n} \lambda_j^* \right) \\
&= 6\sqrt{2} \left(\alpha + \sum_{j=1}^{n} \mu_j^* \right)^{1/2} c_3(\Lambda,\alpha) \exp\left(-\frac{1}{2\alpha} \sum_{j=1}^{n} \lambda_j^* \right)
\end{aligned}
$$

$$\leq 6\sqrt{2}\left(\alpha + \sum_{j=1}^{n} \lambda_j^*\right)^{1/2} c_3(\Lambda, \alpha) \exp\left(-\frac{1}{2\alpha}\sum_{j=1}^{n} \lambda_j^*\right)$$
$$\leq c_4(\Lambda, \alpha) \exp\left(-\left(\frac{1}{2\alpha} - \frac{1}{2}\right)\sum_{j=1}^{n} \lambda_j^*\right)$$

with a constant $c_4(\Lambda, \alpha) > 0$ depending only on Λ and α. Now we define

$$R_n(x) = x^{(\lambda_k^*/2)-(1/p)} Q_n(x) \in \operatorname{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_n}\}.$$

Then

$$\begin{aligned}
\int_0^1 |x^m - R_n(x)|^p \, dx &= \int_0^1 \left| x^{(\lambda_k^*/2)-(1/p)} \left(x^{m-(\lambda_k^*/2)+(1/p)} - Q_n(x)\right)\right|^p dx \\
&\leq \left(\int_0^1 x^{p(\lambda_k^*/2)-1} \, dx\right) \left\| x^{m-(\lambda_k^*/2)+(1/p)} - Q_n(x)\right\|_{L_\infty[0,1]}^p \\
&\leq \frac{c_4(\Lambda, \alpha)^p}{p(\lambda_k^*/2)} \exp\left(-p\left(\frac{1}{2\alpha} - 1\right)\sum_{j=1}^{n} \lambda_j^*\right),
\end{aligned}$$

and the theorem is proved $\square$

5. Proof of Theorems 3.3 and 3.6

Proof of Theorem 3.3. Let $p \in (0, \infty)$. Suppose $(\lambda_j)_{j=1}^\infty$ is a sequence of distinct real numbers greater than $-(1/p)$. Suppose

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} < \infty.$$

Then $\{\lambda_j : j = 1, 2, \ldots\} = \{\gamma_j : j = 1, 2, \ldots\} \cup \{\delta_j : j = 1, 2, \ldots\}$, where $(\gamma_j)_{j=1}^\infty$ is a strictly decreasing sequence of distinct real numbers greater than $-(1/p)$ satisfying

$$\sum_{j=1}^{\infty} (\gamma_j + (1/p)) < \infty$$

and $(\delta_j)_{j=1}^\infty$ is a strictly increasing sequence of positive numbers satisfying

$$\sum_{j=1}^{\infty} \frac{1}{\delta_j} < \infty.$$

Let $A \subset [0,1]$ be a compact set with lower density $\delta > 0$ at 0. Choose $b \in (0,1]$ such that $m(A \cap [0,\beta]) \geq \delta\beta$ for every $\beta \in [0,b]$. Then choose $N \in \mathbb{N}$ such that

$$\sum_{j=N+1}^{\infty} (\gamma_j + (1/p)) =: \eta \leq \delta b/36\,.$$

Let U be the $L_p(A)$ closure of

$$\mathrm{span}\{\{x^{\lambda_1}, x^{\lambda_2}, \dots\} \setminus \{x^{\gamma_1}, x^{\gamma_2}, \dots, x^{\gamma_N}\}\}\,,$$

Let V be the $L_p(A)$ closure of

$$\mathrm{span}\{x^{\gamma_{N+1}}, x^{\gamma_{N+2}}, \dots\}\,,$$

and let W be the $L_p(A)$ closure of $\mathrm{span}\{x^{\delta_1}, x^{\delta_2}, \dots\}$. Then by Theorem 2.3 every $f \in W$ can be represented as an analytic function on

$$D_{r_A} := \{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < r_A\}$$

restricted to $A \cap (0, r_A)$. Further, by Corollary 2.10 every $f \in V$ can be represented as an analytic function on $\mathbb{C} \setminus (-\infty, 0]$ restricted to $A \setminus \{0\}$. Finally, by Theorems 2.4 and 2.8, W and V satisfy the assumptions of Theorem 3.5. Hence $W + V$ is closed in $L_p[0,1]$, and every function from $W + V$ can be represented as an analytic function on D_{r_A}. Since $U \subset W + V$, every function from U can be represented as an analytic function on D_{r_A}. Now let Y be the $L_p(A)$ closure of

$$\mathrm{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots\}\,.$$

Since

$$Z := \mathrm{span}\{x^{\lambda_1}, x^{\lambda_2}, \dots x^{\lambda_N}\}$$

is a finite-dimensional vector space, we have $Y = U + Z$, hence every function from Y can be represented as an analytic function on D_{r_A}. This finishes the proof. $\square$

Proof of Theorem 3.6. If

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} < \infty\,,$$

then the theorem follows from Theorem 3.3. If

$$\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty\,,$$

then the theorem follows from Tietze's Extension Theorem and from Theorems 2.1 and 3.2. We omit the trivial details. $\square$

6. Proof of Theorems 3.4 and 3.5

In this section we prove Theorem 3.4. Since the ideas that underly the proof led to some new results about quasi-Banach spaces that may be useful elsewhere, we present some general results that include more information than what is needed for the proof of Theorem 3.4. We thank Nigel Kalton for several very useful and illuminating e-discussions about the contents of this section and related matters.

A *quasi-norm* is a real valued function $\|\cdot\|$ on a (real or complex) vector space X which satisfies the axioms for a norm except that the triangle inequality is replaced by the condition

$$\|x+y\| \leq k(\|x\| + \|y\|)$$

for some constant k. The smallest such k is called the *modulus of concavity* of the quasi-norm. For $0 < p \leq 1$, a quasi-norm $\|\cdot\|$ is *p-subadditive* provided

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all vectors x and y. A *p-subadditive* quasi-norm is called a *p-norm.* A quasi-norm $\|\cdot\|$ with modulus of concavity k is equivalent to a p-norm with $2^{1/p} = 2k$. A p-norm is obviously also a q-norm for all $0 < q < p$.

These and many other basic facts about quasi-norms and p-norms are discussed in the first few sections of [K-P-R]. This book also contains much of the deeper theory of p-normed spaces.

In this section all spaces are p-normed spaces for some fixed $0 < p \leq 1$. $B(X,Y)$ denotes the space of bounded (same as continuous for p-normed spaces) linear operators, p-normed by $\|T\| := \sup\{\|Tx\|_Y : \|x\|_X \leq 1\}$.

We recall that a linearly independent sequence $\{x_n\}_{n=1}^{\infty}$ in a p-normed space is *basic* provided that the natural partial sum projections P_n from the linear span of $\{x_n\}_{n=1}^{\infty}$ onto the span of $\{x_k\}_{k=1}^{n}$ are uniformly bounded. A sequence $\{y_n\}_{n=1}^{\infty}$ of nonzero vectors is called a *block basis* of $\{x_n\}_{n=1}^{\infty}$ provided that there is a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers so that for each k, y_k is in $\text{span}\{x_j : n_k \leq j < n_{k+1}\}$. A block basis of a basic sequence is again a basic sequence.

Just as for normed spaces, basic sequences play an important role in studying the structure theory of quasi-normed spaces (see [K-P-R, I.5ff]). However, in quasi-normed spaces it typically is difficult to construct basic sequences.

The main functional analytical concept we study in this section is that of *strictly singular operator.* An operator T in $B(X,Y)$ is called strictly singular provided that for every infinite dimensional subspace X_0 of X, the restriction $T_{|X_0}$ of T to X_0 is not an isomorphism. Here it is convenient to work with nonclosed subspaces but the definition is obviously equivalent if we add "closed" before "subspace". The space of all strictly singular operators from X to Y is denoted by $SS(X,Y)$.

Lemma 6.1. *Assume that T, S are in $SS(X,Y)$. Then $T+S$ is strictly singular provided that either*

(1) *Every infinite dimensional closed subspace of X contains a basic sequence;*

or
(2) $\quad$ *X is complete and* $\ker T = \{0\}$.

Proof of Lemma 6.1. The proof of (1) is just like the proof when X is a normed space (of course, every normed space X satisfies the hypothesis of (1); see [Li-Tz 1.a.5]): Consider any closed subspace X_0 of X which has a basis $\{x_n\}_{n=1}^{\infty}$. Since for every N the restriction of T to span$\{x_n : n > N\}$ is not an isomorphism, get a normalized block basis $\{y_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ so that $\|Ty_n\| \to 0$ arbitrarily quickly. Using then the strict singularity of S, get a normalized block basis $\{z_n\}_{n=1}^{\infty}$ of $\{y_n\}_{n=1}^{\infty}$ so that $\|Sz_n\| \to 0$. If $\|Ty_n\| \to 0$ fast enough, then necessarilly $\|Tz_n\| \to 0$, so that $T + S$ is not an isomorphism on X_0.

Part 2 is not needed in the sequel, so we present the proof at the end of this section. $\square$

Remark 6.1. Something is needed to guarantee that the sum of strictly singular operators is strictly singular. Suppose that X contains a subspace E with $\dim E = 2$ so that every closed infinite dimensional subspace of X contains E. Then for some Y there exist T, S in $SS(X,Y)$ with $T+S$ an isomorphic embedding. (Take $Q_{X_1}\colon X \to X/X_1$, $Q_{X_2}\colon X \to X/X_2$, $Y = X/X_1 \oplus X/X_2$, $T = Q_{X_1} \oplus \{0\}$, $S = \{0\} \oplus Q_{X_2}$. Here $\dim X_1 = \dim X_2 = 1$ with $X_1 \cap X_2 = \{0\}$ and $X_1 \cup X_2 \subset E$. Q_Z is the quotient map from X to X/Z.) There exists such a strange space X: In Theorem 5.5 of [Ka] Kalton constructs for every n a p-Banach space X and an n dimensional subspace E so that every closed infinite dimensional subspace of X contains E.

Definition 6.1. We say that X has property (B) if every infinite dimensional subspace of X contains a basic sequence.

Remark 6.2. If the completion of a p-normed space X has a basic sequence, then so does X (the usual normed space perturbation argument [Li-Tz, 1.a.9] works). Thus if every infinite dimensional closed subspace of X contains a basic sequence, then X has property (B).

Definition 6.2. Given a sequence $\{x_n\}_{n=1}^{\infty}$ in X, say that $\{x_n\}_{n=1}^{\infty}$ has *a lower ∞-estimate* if there is $\delta > 0$ which satisfies

$$\left\|\sum a_k x_k\right\| \geq \delta \max_k |a_k|$$

for all finitely nonzero sequences $\{a_n\}_{n=1}^{\infty}$ of scalars.

Obviously a normalized basic sequence has a lower ∞-estimate. This was used implicitly in the proof of Lemma 6.1.

Remark 6.3. Obviously the following are equivalent.

(i) $\{x_n\}_{n=1}^{\infty}$ has a lower ∞-estimate.

(ii) $x_n \mapsto e_n$ extends to a bounded linear operator from span$\{x_n\}_{n=1}^{\infty}$ into c_0.

(iii) There is an equicontinuous sequence $\{x_n^*\}_{n=1}^{\infty} \subset (\text{span}\{x_n\}_{n=1}^{\infty})^*$ so that $\{x_n, x_n^*\}_{n=1}^{\infty}$ is biorthogonal.

(iv) There is a bounded linear operator T from span$\{x_n\}_{n=1}^{\infty}$ into some space Y so that $\{Tx_n\}_{n=1}^{\infty}$ has a lower ∞-estimate.

For Banach spaces, the next lemma is a standard exercise in text books. The extension to the p-normed setting is routine.

Lemma 6.2. *Let X be a p-Banach space and W, V closed subspaces with $W \cap V = \{0\}$. Then $W + V$ is closed if and only if $dist(S_W, V) > 0$, where $S_W := \{w \in W \colon \|w\| = 1\}$.*

Proof of Lemma 6.2. Assume that $W + V$ is not closed. Take $w_n \in W$, $v_n \in V$ with $w_n + v_n \to z \notin W + V$. If $\sup \|w_n\| = \infty$, then without loss of generality $\|w_n\| \to \infty$, so $\left\| \frac{w_n}{\|w_n\|} + \frac{v_n}{\|w_n\|} \right\| \to 0$ and hence $\operatorname{dist}(S_W, V) = 0$. If $\sup \|w_n\| \neq \infty$, then still $\{w_n\}_{n=1}^{\infty}$ cannot have a Cauchy subsequence (else z would be in $W+V$), so we can assume that there exists $\delta > 0$ that $\delta < \|w_n - w_m\| < C$ for $n \neq m$. Then $\left\| \frac{w_n - w_{n+1}}{\|w_n - w_{n+1}\|} + \frac{v_n - v_{n+1}}{\|w_n - w_{n+1}\|} \right\| \to 0$, hence again $\operatorname{dist}(S_W, V) = 0$.

The other direction is even easier (and anyway is not needed in the sequel). □

Proposition 6.3. *Let X be a p-Banach space and W, V closed subspaces. If $W + V$ is not closed then there exist $\{w_n\}_{n=1}^{\infty} \subset W$, $\{v_n\}_{n=1}^{\infty} \subset V$ so that*
(1) $\|w_n\| = 1$
(2) $\|w_n + v_n\| \to 0$
(3) $\{w_n\}_{n=1}^{\infty}$ *has a lower ∞-estimate.*

Proof of Proposition 6.3. First assume that $W \cap V = \{0\}$. Under the assumptions of the lemma, by Lemma 6.2 we can pick $\{w_n\}_{n=1}^{\infty} \subset W$ and $\{v_n\}_{n=1}^{\infty} \subset V$ with $\|w_n\| = 1$ and $\|w_n + v_n\| \to 0$. Define a p-norm on W by $|w|^p = \operatorname{dist}(w, V) = \|Q_V w\|^p$. This is a p-norm since $W \cap V = \{0\}$ is weaker than $\|\cdot\|$, so by [K-P-R, Theorem 4.7], $\{w_n\}_{n=1}^{\infty}$ has a subsequence which has a lower ∞-estimate.

In the general case pass to $X/(W \cap V)$. $Q_{W\cap V} W$ is closed there since it is isometric to $W/(W \cap V)$ and similarly for $Q_{W \cap V} V$. Also $W + V = Q_{W\cap V}^{-1}(Q_{W\cap V} W + Q_{W\cap V} V)$, so since $W + V$ is not closed, neither is $Q_{W\cap V} W + Q_{W\cap V} V$. Thus we get $\{w_n\}_{n=1}^{\infty} \subset W$ and $\{v_n\}_{n=1}^{\infty} \subset V$ so that $\|Q_{W\cap V} w_n + Q_{W\cap V} v_n\| \to 0$, $\|Q_{W\cap V} w_n\| = 1$, and $\{Q_{W\cap V} w_n\}_{n=1}^{\infty}$ has a lower ∞-estimate. By adding some $z_n \in W \cap V$ to w_n and subtracting z_n from v_n we can assume that $\|w_n\| \to 1$. Pick $x_n \in W \cap V$ so that $\|w_n + v_n + x_n\| \to 0$. Then $v_n + x_n \in V$ and $\{w_n\}_{n=1}^{\infty}$ has a lower ∞-estimate since $\{Q_{W\cap V} w_n\}_{n=1}^{\infty}$ does. □

Proposition 6.4. *Assume the p-Banach space X has property (B), W, V are closed subspaces, and there are $T, S \in B(X, X)$ so that $T_{|W}$ and $S_{|V}$ are strictly singular and $I = T + S$, where I is the identity operator on X. Then $W + V$ is closed.*

Proof of Proposition 6.4. Suppose $W + V$ is not closed. Then get $\{w_n\}_{n=1}^{\infty} \subset W$ and $\{v_n\}_{n=1}^{\infty} \subset V$ by Proposition 6.3 and take $\delta > 0$ so that for all finitely nonzero sequences of scalars $\{a_n\}_{n=1}^{\infty}$,

$$\left\| \sum a_n w_n \right\| \geq \delta \max |a_n| .$$

By passing to a subsequence, assume that

$$\sum_{j=n}^{\infty} \|w_j + v_j\|^p < \frac{1}{n} .$$

Let V_0 be an infinite dimensional subspace of $\operatorname{span}\{v_n\}_{n=1}^{\infty}$. Since $T_{|W}$ is strictly singular, we can get $x_n = \sum_{k_n+1}^{k_{n+1}} a_j w_j$, $\|x_n\| = 1$, with $y_n := \sum_{k_n+1}^{k_{n+1}} a_j v_j \in V_0$ so that $\|T x_n\| \to 0$.

Then

$$
\begin{aligned}
1-\|y_n\|^p \le \|x_n+y_n\|^p &\le \sum_{j=k_n+1}^{k_{n+1}} |a_j|^p\|w_j+v_j\|^p \\
&\le \Big(\max_{k_n+1\le j\le k_{n+1}} |a_j|^p\Big) \sum_{j=k_n+1}^{\infty} \|w_j+v_j\|^p \le \delta^{-p}n^{-1}
\end{aligned}
$$

so that $1-\delta^{-p}n^{-1} \le \|y_n\|^p$. But

$$
\|Ty_n\|^p \le \|Tx_n\|^p + \|T\|^p\|x_n+y_n\|^p \to 0\,.
$$

So $T_{|V_0}$ is not an isomorphism. This proves that the restriction of T to $V_1 := \operatorname{span}\{v_n\}_{n=1}^{\infty}$ is strictly singular, hence $I_{|V_1} = T_{|V_1} + S_{|V_1}$ is strictly singular by Lemma 6.1, a contradiction. $\square$

Theorem 3.4 is a corollary of Proposition 6.4.

Corollary 6.5. *Suppose W, V are closed subspaces of $L_p := L_p[0,1]$, $0<p<\infty$, and*

$$
\begin{aligned}
\|1_{(0,1/2)}f\|_{L_\infty[0,1]} &\le C\|f\|_{L_p[0,1]}\,, \qquad f\in W\\
\|1_{(1/2,1)}f\|_{L_\infty[0,1]} &\le C\|f\|_{L_p[0,1]}\,, \qquad f\in V\,.
\end{aligned}
$$

Then $W+V$ is closed in L_p.

Proof of Corollary 6.5. The formal identity mapping $I_{\infty,p} : L_\infty[0,1] \to L_p[0,1]$ is strictly singular. When $p=2$, this is contained in elementary text books (see e.g. [Ro, Chapter 10, # 41, # 55]). The case $p<2$ follows formally from this, and the case $p>2$ follows via a simple extrapolation argument (see e.g. [Jo-Li, Section 10]). Thus if T: $L_p[0,1]\to L_p[0,1]$, S: $L_p[0,1]\to L_p[0,1]$ are defined by

$$
Tf = 1_{(0,1/2)}f\,, \qquad Sf = 1_{(1/2,1)}f\,,
$$

we have that $T_{|W}$ and $S_{|V}$ are strictly singular. Also, every closed infinite dimensional subspace of $L_p[0,1]$ contains an isomorphic copy of ℓ_r for some $r<2$ by Bastero's [Ba] extension of the Krivine-Maurey stable theory. Thus $L_p[0,1]$ has property (B) by Remark 6.2 and Proposition 6.4 applies. An easier proof of the fact that $L_p[0,1]$ has property (B) was given by Tam [Ta]. His proof uses Dvoretzky's theorem rather than the theory of stable spaces. $\square$

Remark 6.4. The case $1<p<\infty$ in Corollary 6.5 was proved jointly with G. Schechtman several years ago. Since this case is much simpler than the case $0<p<1$, we present here the proof.

Proof of Remark 6.4. Recall [Wo, III.D] that a weakly compact operator from L_∞ (or any other $C(S)$-space) is *completely continuous;* that is, sends weakly compact sets to norm compact sets. Thus the composition of two weakly compact operators with the middle space $L_\infty[0,1]$ must be a compact operator. Therefore we get from the hypotheses that the operators $T : W \to L_p[0,1]$ and $S : V \to L_p[0,1]$ are compact, where $Tf := f1_{(0,1/2)}$ and $Sf := f1_{(1/2,1)}$. Thus if you fix $0 < \delta < 1/2$, there are closed, finite codimensional subspaces $W_0 \subset W$ and $V_0 \subset V$ so that $\|T_{W_0}\| < \delta$ and $\|S_{V_0}\| < \delta$. This implies that $W_0 + V_0$ is closed in $L_p[0,1]$ (check that the unit spheres are a positive distance apart), and hence $W + V$ is also closed. $\square$

We turn now to the proof of Part 2 of Lemma 6.1.

Proof of Part 2 of Lemma 6.1. For $x \in X$ define $|x|_T := \|Tx\|_Y$. This is a p-norm on X because T is one-to-one. $|\cdot|_T$ is strictly weaker than $\|\cdot\|_X$ on every infinite dimensional subspace of X because T is strictly singular. Let X_0 be an infinite dimensional closed subspace of X and let $\{x_n\}_{n=1}^{\infty}$ be a normalized sequence in X_0 so that $|x_n|_T \to 0$. By [K-P-R, Theorem 4.7], by passing to a subsequence we can assume that $\{x_n\}_{n=1}^{\infty}$ has a lower ∞-estimate and that $\sum_{n=1}^{\infty} \|Tx_n\|_Y^p < \infty$. Since S is strictly singular, there exists for each n a unit vector $z_n \in \operatorname{span}\{x_k\}_{k=n}^{\infty}$ so that $\|Sz_n\|_Y \to 0$. Since $\{x_n\}_{n=1}^{\infty}$ has a lower ∞-estimate, the coefficients in the expansions of the y_n's in terms of the x_n's are uniformly bounded, and hence $\|Ty_n\|_Y \to 0$ because $\sum_{n=1}^{\infty} \|Tx_n\|_Y^p < \infty$ and $|\cdot|_T^p$ satisfies the triangle inequality. Thus $T + S$ is not an isomorphism on X_0. $\square$

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