

Chebyshev constants for the unit circle

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ABSTRACT

It is proven that for any system of n points z_1, \dots, z_n on the (complex) unit circle, there exists another point z of norm 1, such that

$$\sum \frac{1}{|z_k - z|^2} \leq \frac{n^2}{4}.$$

Two proofs are presented: one uses a characterisation of equioscillating rational functions, while the other is based on Bernstein's inequality.

1. Introduction and results

Chebyshev constants were defined by Fekete [11] and Pólya and Szegő [17]. Since then the notion became fundamental in classical potential theory. There are several different definitions available, among which the following is the most suitable for our purposes.

DEFINITION 1. Let K be a compact set in a normed space $(X, \|\cdot\|)$. For a given point set $(x_k)_1^n$ on K , let

$$M^p(x_1, \dots, x_n) = \min_{x \in K} \sum_{k=1}^n \frac{1}{\|x - x_k\|^p}$$

for $p > 0$, and

$$M^0(x_1, \dots, x_n) = \min_{x \in K} \prod_{k=1}^n \frac{1}{\|x - x_k\|}.$$

The n th L_p Chebyshev constant of K is then given by

$$M_n^p(K) = \max_{x_1, \dots, x_n \in K} M^p(x_1, \dots, x_n).$$

Note that $M_n^0(K)$ is the reciprocal of the usual (modified) n th Chebyshev constant of K , cf. page 39 of [6]. The sum $\sum \|x - x_k\|^{-p}$ is the Riesz potential of order $(2 - p)$ of the discrete distribution with the x_j 's as atoms of weight 1, see [15].

Throughout the article, the role of K will be played by the complex unit circle T endowed with the natural norm. It is natural to expect that in this situation, the point sets maximising $M^p(z_1, \dots, z_n)$ are equally distributed on the circle. For $M_n^0(T)$, this assertion is well known, and easy to obtain by applying the method for proving the extremality of Chebyshev polynomials. We illustrate a proof that is parallel to the subsequent arguments; note that this proof is not unique, see e.g. Theorem 2 of [2].

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PROPOSITION 1. For any positive integer n , $M_n^0(T) = 1/2$.

Proof. First, we show that $M_n^0(T) \leq 1/2$. Assume on the contrary that there exists a set $(z_k)_1^n$ of complex numbers of norm 1, such that $|q(z)| < 2$ for every $z \in T$ with

$$q(z) = \prod_{k=1}^n (z - z_k). \tag{1.1}$$

Note that for $v, w \in T$,

$$(v - w)^2 = -v w \varrho \tag{1.2}$$

holds with some $\varrho \geq 0$. Thus, $q^2(z)$ can be written as

$$q^2(z) = (-1)^n \gamma z^n \varrho(z),$$

where $\gamma = \prod z_k$, and $\varrho(z)$ is a real function defined on T , taking only non-negative values. By rotation, we may assume that $\gamma = (-1)^{n-1}$, and thus

$$q^2(z) = -z^n \varrho(z). \tag{1.3}$$

According to the assumption, $0 \leq \varrho(z) < 4$ for all $z \in T$. Take now

$$Q(z) = (z^n - 1)^2 = -z^n R(z). \tag{1.4}$$

Here $R(z)$ is again a real function on T satisfying $0 \leq R(z) \leq 4$; moreover, $R(z) = 0$ or $R(z) = 4$ at exactly $2n$ points on T . Thus, counting with multiplicities, the function $\varrho(z) - R(z)$ has at least $2n$ zeroes on T , and by (1.3) and (1.4), the same holds for $q^2(z) - Q(z)$. However, this contradicts the fact that $q^2(z) - Q(z)$ is a polynomial of degree at most $2n - 1$.

To see that $M_n^0(T) \geq 1/2$, choose the n th unit roots: $z_k = \xi_k = e^{i2\pi k/n}$. □

Prior to the present article, the exact value of $M_n^p(T)$ was not known except for $p = 0$ or for small values of n . The analogous problem for $p < 0$ was considered and partly solved by Stolarsky [20]. For $p > 0$, the following observation gives an upper estimate of $M_n^p(T)$ for any $n \geq 1$.

PROPOSITION 2. The L_p Chebyshev constants of T can be estimated as

$$M_n^p(T) < \begin{cases} c_0 n/(1-p) & \text{for } 0 < p < 1 \\ c_1 n(1 + \ln n) & \text{for } p = 1 \\ c_p n^p & \text{for } p > 1 \end{cases}$$

with positive constants c_0 and c_p ($p \geq 1$).

Proof. Let $q(z)$ be the polynomial defined by (1.1), and let z_0 be a point on T where $q(z)$ attains its maximal modulus on T . According to exercise E.12 on page 237 of [6], for every $r > 0$, there are at most enr zeroes of $q(z)$ on the arc $\{z_0 e^{i\rho} : \rho \in [-r, r]\}$. Thus,

$$\sum_{k=1}^n \frac{1}{|z_0 - z_k|^p} < \sum_{k=1}^n \left(2 \sin\left(\frac{k}{2en}\right)\right)^{-p} < (3e)^p n^p \sum_{k=1}^n \left(\frac{1}{k}\right)^p, \tag{1.5}$$

where at the last inequality the constant 3 can be replaced by $1/(2e \sin(1/2e))$.

For $p > 1$, we obtain that

$$M_n^p(T) < (3e)^p \zeta(p) n^p. \tag{1.6}$$

For $p = 1$, (1.5) yields

$$M_n^1(T) < 3e n(1 + \ln n),$$

whereas for $0 < p < 1$,

$$M_n^p(T) < (3e)^p \left(\frac{n}{1-p} - \frac{p}{1-p} \right).$$

□

Taking the system of the n th unit roots, $\xi_k = e^{i2\pi k/n}$, $k = 1, \dots, n$, shows that the estimate provided by Proposition 2 is asymptotically sharp in terms of n . The complete asymptotic expansion of $M^p(\xi_1, \dots, \xi_n)$ was established by Brauchart, Hardin and Saff [8]. More precisely, they determined the asymptotic expansion of the Riesz p -energy

$$E^p(\xi_1, \dots, \xi_n) = \sum_{j=1}^n \sum_{\substack{k=1 \\ k \neq j}}^n \frac{1}{|\xi_j - \xi_k|^p}.$$

Using the notation $E_n^p = E^p(\xi_1, \dots, \xi_n)$, the expansion of $M^p(\xi_1, \dots, \xi_n)$ can be obtained by the formula

$$M^p(\xi_1, \dots, \xi_n) = \frac{E_{2n}^p}{2n} - \frac{E_n^p}{n}.$$

In comparison with Proposition 2, we note that the results of [8] yield that

$$\begin{aligned} M^p(\xi_1, \dots, \xi_n) &\approx (2^p - 1)\zeta(p)n^p && \text{for } p > 1, \\ M^p(\xi_1, \dots, \xi_n) &\approx \frac{1}{\pi} n \ln n && \text{for } p = 1, \text{ and} \\ M^p(\xi_1, \dots, \xi_n) &\approx \frac{2^{-p} \Gamma((1-p)/2)}{\sqrt{\pi} \Gamma(1-p/2)} n && \text{for } 0 < p < 1. \end{aligned}$$

That the point systems on T minimising the Riesz energies are equally distributed on T was (essentially) proved by Fejes Tóth [10]. The case of Chebyshev constants is much harder to tackle. One may even conjecture that the equally distributed case is extremal in a more general setting.

Let f be an even, 2π -periodic real function. We say that f is convex, if it is convex on $(0, 2\pi)$; in this case, f has a maximum at 0. We allow f to have a pole at 0. For $\theta_1, \dots, \theta_n \in [0, 2\pi)$, let

$$S_f(\theta) = \sum_{k=1}^n f(\theta - \theta_k),$$

and

$$M_f(\theta_1, \dots, \theta_n) = \min_{\theta \in [0, 2\pi)} S_f(\theta).$$

We say that $\theta_1, \dots, \theta_n \in [0, 2\pi)$ is uniformly distributed on $[0, 2\pi)$, if for some $\lambda \in \mathbb{R}$,

$$\{\theta_1, \dots, \theta_n\} = \left\{ \left(\lambda + \frac{2\pi k}{n} \right) \bmod 2\pi : k = 1, \dots, n \right\}.$$

CONJECTURE. For any 2π -periodic, even, convex function f , $M_f(\theta_1, \dots, \theta_n)$ is maximised when $(\theta_k)_1^n$ is uniformly distributed on $[0, 2\pi)$.

It is easy to see that for $p > 0$, the function $f(\theta) = \sin^{-p}(\theta/2)$ satisfies the above conditions. Thus, in particular, we conjecture that for any $p > 0$ and $n \geq 1$,

$$M_n^p(T) = M^p(\xi_1, \dots, \xi_n). \tag{1.7}$$

A strong indication for the validity of the conjecture is the following fact. A set of points $(\theta_j)_1^n \subset [0, 2\pi)$ is locally maximal with respect to f , if there exists $\nu > 0$, such that for any

$(\theta'_j)_1^n \subset [0, 2\pi)$ satisfying $|\theta_j - \theta'_j| < \nu \pmod{2\pi}$ for every j ,

$$M_f(\theta_1, \dots, \theta_n) \geq M_f(\theta'_1, \dots, \theta'_n).$$

Note that when f is strictly convex, locally minimal cases are of little interest: then $\theta_1 = \dots = \theta_n$. Clearly, S_f is convex on the intervals between consecutive θ_j 's, and if f is strictly convex, then S_f has exactly one local minimum on each of these intervals.

LEMMA 1. *If f satisfies the above conditions, and $(\theta_j)_1^n$ is a locally maximal set with respect to f , then all the local minima of $S_f(\theta)$ are equal.*

Proof. Let $\Theta = (\theta_j)_1^n$, and assume on the contrary that $0 < \theta_1 < \theta_2 < 2\pi$ are two consecutive points such that the local minimum of $S_f(\theta)$ on the interval $[\theta_1, \theta_2]$ is strictly larger than $M_f(\Theta)$. Let ε be a small positive number, and consider the new set of points Θ' obtained from Θ by exchanging θ_1 and θ_2 for

$$\theta'_1 = \theta_1 - \varepsilon, \quad \theta'_2 = \theta_2 + \varepsilon.$$

Let $S'_f(\theta)$ be the function determined by the point set Θ' . By the symmetry and convexity of f , it is easy to verify that for $\theta_1 \leq \theta \leq \theta_2$,

$$S_f(\theta) \geq S'_f(\theta). \tag{1.8}$$

Interchanging the role of θ_1 and θ_2 , it also follows that for $\theta \in [0, 2\pi] \setminus [\theta'_1, \theta'_2]$, the reverse of (1.8) holds. If $\varepsilon > 0$ is sufficiently small, then the global minimum of $S'_f(\theta)$ is still attained on $[0, 2\pi] \setminus [\theta'_1, \theta'_2]$. Thus, $M_f(\Theta) < M_f(\Theta')$, which contradicts the extremality of $(\theta_j)_1^n$. \square

Analogous equioscillation properties often arise in the context of minimax problems. For instance, Bernstein conjectured that for the optimal set of nodes for Lagrange interpolation, the so-called Lebesgue function of the projection is equioscillating. This long-standing problem was solved by Kilgore [13]. In this case, the equioscillation property also leads to the characterisation of the optimal nodes, see [14] and [4].

The main goal of the paper is to give a complex analytic proof for the $p = 2$ case of (1.7).

THEOREM. *For any set z_1, \dots, z_n of complex numbers of modulus 1, there exists a complex number z_0 of norm 1, such that*

$$\sum_{j=1}^n \frac{1}{|z_0 - z_j|^2} \leq \frac{n^2}{4}. \tag{1.9}$$

The inequality is sharp if and only if the numbers z_j are distinct and there exists a $c \in T$ such that $z_j^n = c$ for all $j = 1, 2, \dots, n$.

We present two proofs for the above Theorem. The first one is based on Lemma 1 and it uses the equioscillating property of the functions arising in local extremal cases. The second proof refers to Bernstein's inequality, which in turn is based again on the equioscillating property. We believe that both of these proofs are of independent interest.

We note that the problem arose in connection with the so-called polarisation problems. In fact, the Theorem is the planar case of the strong polarisation problem. There is a third proof of it following these lines; we refer the interested reader to [1]. This proof uses directly the extremal property of the Chebyshev polynomials, and thus it is a close relative of the second proof presented here. For reasons of conserving space, we do not give a detailed description of the polarisation problems, which the interested reader can find in [1] and in [16]. Also, numerous results in potential theory are related to the present problem, see e.g. [9].

2. Complex analytic tools

The following notion will play an important role; we follow Szegő [21].

DEFINITION 2. Let $g(z) = a_0 + a_1z + \dots + a_nz^n$ be a complex polynomial. Its *reciprocal polynomial of order n* is defined by

$$g^*(z) = \bar{a}_n + \bar{a}_{n-1}z + \dots + \bar{a}_0z^n.$$

If we do not specify otherwise, $g^*(z)$ will denote the reciprocal polynomial whose order is the exact degree of g . For any non-zero complex number z , let z^* denote its image under the inversion with respect to complex unit circle T : $z^* = 1/\bar{z}$. Clearly,

$$g^*(z) = z^n \overline{g(z^*)}, \tag{2.1}$$

and thus, if the non-zero roots of $g(z)$ are ζ_1, \dots, ζ_n , then the non-zero roots of $g^*(z)$ are $\zeta_1^*, \dots, \zeta_n^*$.

In particular, if all zeroes of $g(z)$ lie on T , then the zeroes of $g(z)$ and $g^*(z)$ agree, hence we obtain the following.

PROPOSITION 3. If all zeroes of the polynomial $g(z)$ have modulus 1, then

$$g^*(z) = \gamma g(z)$$

for a complex constant γ with $|\gamma| = 1$.

Lemma 1 implies that for a stationary point set, the resulting function $S_f(\theta)$ is equioscillating. In the special case that we treat, $S_f(\theta)$ can be written as the real part of a complex rational function. In light of these observations, we introduce the following concept.

DEFINITION 3. The real valued function f on T is *equioscillating of order n* , if there are $2n$ points w_1, w_2, \dots, w_{2n} on T in this order, such that

$$f(w_j) = (-1)^j \|f\|_T$$

for every $j = 1, \dots, 2n$, and $|f(z)| < \|f\|_T$ if $z \neq w_j$ for any j .

Although equioscillation in general is not a very specific property (plainly, any real valued function on T whose level sets are finite has a shifted copy which is equioscillating of some order), equioscillation of a possible maximal order is a strong condition. This becomes apparent in the context of rational functions.

Suppose that $R(z)$ is a rational function, whose numerator is of degree k and whose denominator has degree l ; then the real and imaginary parts of $R(z)$ are the quotients of two trigonometric polynomials of degrees k and l , and therefore $\Re(R(z))$ and $\Im(R(z))$ cannot be equioscillating of order larger than $\max\{k, l\}$. These simple observations are intimately tied to the right Bernstein-type inequalities for spaces of rational functions on the unit circle as well as for spaces of ratios of trigonometric polynomials on the period, see [7] and [5], respectively.

A characterisation of those rational functions whose real and imaginary parts are oscillating with the maximal possible order was given by Glader and Högnäs [12]. They showed that if $R(z)$ is a rational function with numerator and denominator degrees at most n , and $\Re(R(z))$ and $\Im(R(z))$ are equioscillating functions on T of order n , then

$$R(z) = cB(z) \text{ or } R(z) = c/B(z),$$

where c is a real constant and $B(z)$ is a finite Blaschke product of order n :

$$B(z) = \rho z^k \prod_{j=1}^{n-k} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}.$$

Here $\rho, \alpha_1, \dots, \alpha_{n-k}$ are complex numbers with $|\rho| = 1$ and $0 < |\alpha_j| < 1$.

The essence of the above result of [12] is the following statement, for which we present a simple proof.

LEMMA 2. *Suppose that $w_1, w_2, \dots, w_{2n}, w$ are different points on T in this order. There exists a complex polynomial $h(z)$ of degree n , such that*

$$\frac{h(w_k)}{h^*(w_k)} = (-1)^{k+1}$$

for each $k = 1, \dots, 2n$, and

$$\frac{h(w)}{h^*(w)} = i.$$

Proof. Taking $g_1(z) = h(z) + h^*(z)$ and $g_2(z) = h(z) - h^*(z)$, the original problem is equivalent to finding polynomials $g_1(z)$ and $g_2(z)$ of degree n with the following properties:

- (i) The zeros of g_1 are (w_{2k}) , where $1 \leq k \leq n$;
- (ii) The zeros of g_2 are (w_{2k-1}) , where $1 \leq k \leq n$;
- (iii) $g_1(z) = g_1^*(z)$
- (iv) $g_2(z) = -g_2^*(z)$
- (v) $g_1(w) + i g_2(w) = 0$.

Property (i) is fulfilled, if $g_1(z)$ has the form

$$g_1(z) = \alpha \prod_{k=1}^n (z - w_{2k}), \quad (2.2)$$

where α is a complex number of modulus 1. Proposition 3 implies that property (iii) is satisfied if the leading coefficient and the constant term of $g_1(z)$ are conjugates of each other, that is,

$$\bar{\alpha} = \alpha(-1)^n \prod w_{2k}.$$

This is achieved by choosing α such that

$$\alpha^2 = (-1)^n \prod \bar{w}_{2k}.$$

Similarly, conditions (ii) and (iv) are fulfilled if $g_2(z)$ is defined by

$$g_2(z) = c\beta \prod_{k=1}^n (z - w_{2k-1}),$$

where c is a non-zero real and β is a complex number with $|\beta| = 1$ satisfying

$$\beta^2 = (-1)^{n+1} \prod \bar{w}_{2k-1}.$$

For any $z \in T$, by (iii), (iv) and the fact $z^* = z$, (2.1) implies that

$$g_1(z) = z^n \overline{g_1(z)}$$

and

$$g_2(z) = -z^n \overline{g_2(z)}.$$

Thus, if neither g_1 nor g_2 has a zero at z , then

$$\arg g_1(z) \equiv \arg(i g_2(z)) \equiv \frac{n}{2} \arg z \pmod{\pi}. \tag{2.3}$$

In particular, choosing $z = w$, we obtain that the non-zero real c can be specified so that property (v) holds. \square

Finally, we present the variant of Bernstein’s inequality that is needed for the second proof.

An entire function f is said to be of *exponential type* τ if for any $\varepsilon > 0$ there exists a constant $k(\varepsilon)$ such that $|f(z)| \leq k(\varepsilon)e^{(\tau+\varepsilon)|z|}$ for all $z \in \mathbb{C}$. The following inequality [3], p. 102, is known as Bernstein’s inequality. It can be viewed as an extension of Bernstein’s (trigonometric) polynomial inequality [6], p. 232, to entire functions of exponential type bounded on the real axis.

LEMMA 3 (Bernstein’s inequality). *Let f be an entire function of exponential type $\tau > 0$ bounded on \mathbb{R} . Then*

$$\sup_{t \in \mathbb{R}} |f'(t)| \leq \tau \sup_{t \in \mathbb{R}} |f(t)|.$$

The reader may find another proof of the above Bernstein’s inequality in [18], pp. 512–514, where it is also shown that an entire function f of exponential type τ satisfying

$$|f'(t_0)| = \tau \sup_{t \in \mathbb{R}} |f(t)|$$

at some point $t_0 \in \mathbb{R}$ is of the form

$$f(z) = ae^{i\tau z} + be^{-i\tau z}, \quad a \in \mathbb{C}, b \in \mathbb{C}, \quad |a| + |b| = \sup_{t \in \mathbb{R}} |f(t)|. \tag{2.4}$$

3. First approach - Equioscillating functions

Proof of the Theorem. We may assume that $(z_j)_1^n$ is a locally maximal set, and hence it consists of n different points. Setting

$$m = 2\sqrt{M^2(z_1, \dots, z_n)},$$

the inequality (1.9), that we wish to prove, is equivalent to $m \leq n$.

Using that for any z and z_j on T ,

$$|z - z_j|^2 = -\frac{(z - z_j)^2}{z z_j},$$

we obtain that

$$\sum_{j=1}^n \frac{1}{|z - z_j|^2} = R^{-1}(z),$$

where $R(z)$ is the rational function given by

$$R(z) = \frac{\prod_{j=1}^n (z - z_j)^2}{-z \sum_{j=1}^n z_j \prod_{k \neq j} (z - z_k)^2}. \tag{3.1}$$

The degrees of the numerator and the denominator of $R(z)$ are $2n$ and at most $2n - 1$, respectively. The zeroes are $(z_j)_1^n$ with multiplicity 2, and $R(z)$ assigns real values on the unit circle. Moreover, Lemma 1 implies that the function

$$R(z) - \frac{2}{m^2},$$

which is a rational function as well, oscillates equally between $-2/m^2$ and $2/m^2$ of order n . Let w_1, \dots, w_{2n} be the equioscillation points such that $w_{2k} = z_k$ for every $k = 1, \dots, n$, and let w be a further point on T satisfying $R(w) = 2/m^2$. Applying Lemma 2 yields a polynomial $h(z)$ of degree n , such that

$$R(z) - \frac{2}{m^2} = \frac{2}{m^2} \Re \left(\frac{h(z)}{h^*(z)} \right) \tag{3.2}$$

for every $z = w_1, \dots, w_{2n}, w$. Moreover, both functions assign real values on T , and they have local extrema at the points $(w_j)_1^{2n}$, therefore their derivatives vanish at these places.

Since $|h(z)| = |h^*(z)|$ on the unit circle,

$$\frac{2}{m^2} + \frac{2}{m^2} \Re \left(\frac{h(z)}{h^*(z)} \right) = \frac{1}{m^2} \left(2 + \frac{h(z)}{h^*(z)} + \frac{h^*(z)}{h(z)} \right) = \frac{(h(z) + h^*(z))^2}{m^2 h(z)h^*(z)}.$$

Thus, from (3.2) we deduce that the rational function

$$R(z) - \frac{(h(z) + h^*(z))^2}{m^2 h(z)h^*(z)}$$

has double zeroes at all the points w_1, \dots, w_{2n} , and it also vanishes at w . On the other hand, its numerator is of degree at most $4n$. Hence, it must be identically 0, and using (3.1), we obtain that

$$\frac{\prod_{j=1}^n (z - z_j)^2}{-z \sum_{j=1}^n z_j \prod_{k \neq j} (z - z_k)^2} = \frac{(h(z) + h^*(z))^2}{m^2 h(z)h^*(z)}. \tag{3.3}$$

This equation is the crux of the proof.

As before, let $g_1(z) = h(z) + h^*(z)$ and $g_2(z) = h(z) - h^*(z)$. Then by (2.2),

$$g_1(z) = \alpha \prod_{j=1}^n (z - z_j), \tag{3.4}$$

with a complex number α of norm 1. According to properties (iii) and (iv),

$$\begin{aligned} g_1(z) &= \alpha z^n + \dots + \bar{\alpha}, \\ g_2(z) &= \beta z^n + \dots - \bar{\beta}, \end{aligned} \tag{3.5}$$

where now $\beta \in \mathbb{C} \setminus \{0\}$. Substituting $g_1(z)$ and $g_2(z)$, equation (3.3) transforms to

$$\frac{\prod_{j=1}^n (z - z_j)^2}{-z \sum_{j=1}^n z_j \prod_{k \neq j} (z - z_k)^2} = \frac{g_1(z)^2}{\frac{m^2}{4} (g_1(z)^2 - g_2(z)^2)}. \tag{3.6}$$

Since the degree of the denominator on the left hand side is at most $2n - 1$, (3.5) implies that

$$\alpha = \pm \beta. \tag{3.7}$$

The quotient of the leading coefficients of the numerators on the two sides of (3.6), which is α^2 , is the same as the quotient of those of the denominators. Therefore

$$-\alpha^2 z \sum_{j=1}^n z_j \prod_{k \neq j} (z - z_k)^2 = \frac{m^2}{4} (g_1(z)^2 - g_2(z)^2).$$

Let $1 \leq j \leq n$ be arbitrary. Substituting $z = z_j$ and taking square roots yields

$$\alpha z_j \prod_{k \neq j} (z_j - z_k) = \pm \frac{m}{2} g_2(z_j),$$

which, by (3.4), is equivalent to

$$z_j g_1'(z_j) = \varepsilon_j \frac{m}{2} g_2(z_j), \tag{3.8}$$

where $\varepsilon_j = \pm 1$.

LEMMA 4. For all j and k , $\varepsilon_j = \varepsilon_k$.

Proof. First, for any j ,

$$\begin{aligned} \arg g_1'(z_j) &= \lim_{\delta \rightarrow 0^+} (\arg g_1(z_j e^{i\delta}) - \arg(z_j e^{i\delta} - z_j)) \\ &= \lim_{\delta \rightarrow 0^+} \arg g_1(z_j e^{i\delta}) - \arg z_j - \frac{\pi}{2} \end{aligned}$$

and therefore

$$\arg(z_j g_1'(z_j)) = \lim_{\delta \rightarrow 0^+} \arg g_1(z_j e^{i\delta}) - \frac{\pi}{2}. \tag{3.9}$$

Second, if $z \in T$ with $g_1(z) \neq 0$ and $g_2(z) \neq 0$, then by (2.3),

$$\arg \frac{g_1(z)}{g_2(z)} \equiv \frac{\pi}{2} \pmod{\pi}.$$

Since $g_1(z)$ and $g_2(z)$ are polynomials with single zeroes only, their arguments change continuously on T apart from their zeroes, where a jump of π occurs. Observe that the zeroes of $g_1(z)$ and $g_2(z)$ are alternating on T (as the zeroes of g_2 are the local maximum places of h/h^*). Hence,

$$\lim_{\delta \rightarrow 0^+} \arg \frac{g_1(z_j e^{i\delta})}{g_2(z_j e^{i\delta})}$$

is the same for every j . Now (3.9) yields that

$$\arg \frac{z_j g_1'(z_j)}{g_2(z_j)}$$

does not depend on j either, and by (3.8), the same is true for ε_j . □

Let $\varepsilon_j = \varepsilon = \pm 1$. From (3.8), we conclude that the polynomial

$$z g_1'(z) - \varepsilon \frac{m}{2} g_2(z)$$

of degree n attains 0 at all $(z_j)_1^n$, and hence its zeroes agree with those of $g_1(z)$. Therefore there exists a complex number γ , such that

$$z g_1'(z) - \varepsilon \frac{m}{2} g_2(z) = \gamma g_1(z),$$

and thus

$$\varepsilon \frac{m}{2} g_2(z) = z g_1'(z) - \gamma g_1(z). \tag{3.10}$$

Equating the leading coefficients, referring to (3.5), gives

$$\varepsilon \frac{m}{2} \beta = (n - \gamma) \alpha, \tag{3.11}$$

which, with the aid of (3.7), yields that $\gamma \in \mathbb{R}$.

Finally, by comparing the leading coefficients and the constant terms in (3.10) and using the form (3.5), we deduce that $(n - \gamma)\alpha = \gamma\alpha$ and, since $\alpha \neq 0$,

$$\gamma = \frac{n}{2}.$$

Taking absolute values in (3.11) and referring to (3.7), we arrive at $m = n$, which proves (1.9).

Note that the proof gives more than the desired inequality: it shows that every locally maximal set is actually a maximal set.

Next, we have to show that a set is locally extremal if and only if it is equally distributed. First, assume that for some $c \in T$, $z_j^n = c$ for every $j = 1, 2, \dots, n$. Choosing z_0 to be the

midpoint of the smaller arc between two consecutive z_j 's, the sharpness of (1.9) follows from setting $t = \pi/n$ in the identity

$$\sum_{j=1}^n \sin^{-2} \left(\frac{t}{2} - \frac{j\pi}{n} \right) = \frac{2n^2}{1 - \cos nt}, \tag{3.12}$$

which can be proved using Fejér kernels or Chebyshev polynomials; it also follows from (4.2) by setting

$$Q(t) = \sin \frac{nt}{2} = (-1)^n 2^{n-1} \prod_{j=1}^n \sin \left(\frac{t}{2} - \frac{j\pi}{n} \right).$$

Finally, we prove that any locally extremal set is equally distributed, based on an idea of L. Fejes Tóth [10]. Let $(z_j)_1^n$ be a maximal set with $z_j = e^{it_j}$, and let $M^2(z_1, \dots, z_n)$ be attained at the points e^{is_j} , $j = 1, \dots, n$. Assume that $0 \leq \theta_1 < s_1 < t_2 < s_2 < \dots < t_n < s_n < 2\pi$. Then, by Lemma 1,

$$\sum_{j=1}^n \sum_{k=1}^n \sin^{-2} \left(\frac{s_j - t_k}{2} \right) = n^3. \tag{3.13}$$

For the sake of simplicity, the indices of the t_j 's will be understood cyclically, i.e. $t_k = t_j$ for $j \equiv k \pmod n$. Then, by Jensen's inequality and (3.12),

$$\begin{aligned} & 2 \sum_{k=1}^n \sum_{j=1}^n \sin^{-2} \left(\frac{s_j - t_k}{2} \right) \\ &= \sum_{k=1}^n \sum_{j=1}^n \sin^{-2} \left(\frac{s_j - t_{j+k}}{2} \right) + \sin^{-2} \left(\frac{s_j - t_{j-k+1}}{2} \right) \\ &\geq \sum_{k=1}^n 2n \sin^{-2} \left(\frac{(2k-1)\pi}{2n} \right) = 2n^3. \end{aligned}$$

Thus, by (3.13), equality holds in Jensen's inequality at all instances. Hence, the strict convexity of $\sin^{-2}(t/2)$ implies that $s_j - t_j = t_{j+1} - s_j = \pi/(2n)$ for every j . \square

We remark that starting from an arbitrary point set $(z_j)_1^n \subset T$, defining $g_1(z)$ by (3.4), and taking $m = n$ and $\gamma = n/2$, the function $g_2(z)$ given by (3.10) has its zeroes where the modulus of $g_1(z)$ is locally maximal. Thus, by (3.6), the proof implicitly shows that in the extremal cases, $\sum |z - z_j|^{-2}$ and $\prod |z - z_j|^{-1}$ have the same local minimum places on T .

4. Second approach - Derivatives

Proof of the Theorem. Associated with $z_j \in T$ we write $z_j = e^{it_j}$, $t_j \in [0, 2\pi)$, $j = 1, 2, \dots, n$. We define

$$Q(t) = \prod_{j=1}^n \sin \frac{t - t_j}{2}. \tag{4.1}$$

Then

$$\frac{Q'(t)}{Q(t)} = \frac{1}{2} \sum_{j=1}^n \cot \frac{t - t_j}{2}$$

and

$$\begin{aligned} \frac{Q''(t)Q(t) - (Q'(t))^2}{(Q(t))^2} &= \left(\frac{Q'(t)}{Q(t)} \right)' \\ &= -\frac{1}{4} \sum_{j=1}^n \csc^2 \frac{t-t_j}{2} = -\frac{1}{4} \sum_{j=1}^n \sin^{-2} \frac{t-t_j}{2}. \end{aligned} \tag{4.2}$$

Observe that Q and Q' are entire functions of type $n/2$ (in fact they are trigonometric polynomials of degree $n/2$ if n is even), so by Bernstein's inequality we have

$$\max_{t \in \mathbb{R}} |Q'(t)| \leq \frac{n}{2} \max_{t \in \mathbb{R}} |Q(t)|$$

and

$$\max_{t \in \mathbb{R}} |Q''(t)| \leq \left(\frac{n}{2} \right)^2 \max_{t \in \mathbb{R}} |Q(t)|. \tag{4.3}$$

Let $t_0 \in \mathbb{R}$ be chosen so that

$$|Q(t_0)| = \max_{t \in \mathbb{R}} |Q(t)|.$$

Then $Q'(t_0) = 0$. Hence combining (4.2) and (4.3) we obtain

$$\frac{1}{4} \sum_{j=1}^n \sin^{-2} \frac{t_0 - t_j}{2} = \frac{|Q''(t_0)|}{|Q(t_0)|} \leq \frac{n^2}{4}.$$

Introducing $z_0 := e^{it_0}$, we arrive at the desired inequality:

$$\sum_{j=1}^n \frac{1}{|z_0 - z_j|^2} = \frac{1}{4} \sum_{j=1}^n \sin^{-2} \frac{t_0 - t_j}{2} \leq \frac{n^2}{4}.$$

Suppose now that the inequality (1.9) is sharp. Then equality holds in Bernstein's inequality for Q , that is, Q is of the form (2.4) with $\tau = n/2$. Here $a \neq 0$ otherwise $|Q(t)| = |b|$ identically for $t \in \mathbb{R}$, a contradiction. Then the zeros of $Q(z)$ satisfy $e^{inz} = -b/a$. Since the zeros of Q are real, we have $|-b/a| = 1$, and each $z_j = e^{it_j}$ satisfies the equation $z^n = -b/a$. Obviously the zeros $t_j, j = 1, 2, \dots, n$, of Q on the period are distinct, hence $z_j = e^{it_j}, j = 1, 2, \dots, n$, are also distinct.

Now suppose that the numbers $z_j = e^{it_j}$ are distinct and there is a number $c \in T$ such that $z_j^n = c$ for all $j = 1, 2, \dots, n$. Then Q is of the form (2.4) with $\tau = n/2$. Choosing $z_0 = e^{it_0}$ to be the midpoint of the smaller arc between two consecutive points z_j on T , we obtain $Q'(t_0) = 0$. Hence (4.2) implies that

$$\frac{1}{4} \sum_{j=1}^n \sin^{-2} \frac{t_0 - t_j}{2} = \sum_{j=1}^n \frac{1}{|z_0 - z_j|^2} = \frac{|Q''(t_0)|}{|Q(t_0)|} = \frac{n^2}{4},$$

where in the last equality we used that Q is of the form (2.4). □

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