A SHARP BERNSTEIN-TYPE INEQUALITY FOR EXPONENTIAL SUMS

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Dedicated to Professor George G. Lorentz on the occasion of his 85-th birthday.

Abstract. A subtle Bernstein-type extremal problem is solved by establishing the equality

$$\sup_{0 \neq f \in \widetilde{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} = 2n-1\,,$$

where

$$\widetilde{E}_{2n} := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} \left(a_j e^{\lambda_j t} + b_j e^{-\lambda_j t} \right), \quad a_j, b_j, \lambda_j \in \mathbb{R} \right\}.$$

This settles a conjecture of Lorentz and others and it is surprising to be able to provide a sharp solution. It follows fairly simply from the above that

$$\frac{1}{e-1} \frac{n-1}{\min\{y-a, b-y\}} \leq \sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a, b-y\}}$$

for every $y \in (a,b)$, where

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j, \lambda_j \in \mathbb{R} \right\}.$$

The proof relies on properties of the particular Descartes system

$$(\sinh \lambda_0 t\,, \sinh \lambda_1 t\,, \ldots\,, \sinh \lambda_n t)\,, \qquad 0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n$$

for which certain comparison theorems can be proved.

Essentially sharp Nikolskii-type inequalities are also proved for E_n.

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1. Introduction

In "Nonlinear Approximation Theory", Braess [3] writes "The rational functions and exponential sums belong to those concrete families of functions which are the most frequently used in nonlinear approximation theory. The starting point of consideration of exponential sums is an approximation problem often encountered for the analysis of decay processes in natural sciences. A given empirical function on a real interval is to be approximated by sums of the form

$$\sum_{j=1}^{n} a_j e^{\lambda_j t}\,,$$

where the parameters a_j and λ_j are to be determined, while n is fixed."

The aim of this paper is to prove the "right" Bernstein-type inequality for exponential sums. This inequality is the key to proving inverse theorems for approximation by exponential sums, as we will elaborate later.

Let

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}\,, \quad a_j, \lambda_j \in \mathbb{R} \right\}.$$

So E_n is the collection of all $n+1$ term exponential sums with constant first term. Schmidt [10] proved that there is a constant $c(n)$ depending only on n so that

$$\|f'\|_{[a+\delta, b-\delta]} \le c(n)\delta^{-1}\|f\|_{[a,b]}$$

for every $p \in E_n$ and $\delta \in \left(0, \frac{1}{2}(b-a)\right)$. Lorentz [7] improved Schmidt's result by showing that for every $\alpha > \frac{1}{2}$, there is a constant $c(\alpha)$ depending only on α so that $c(n)$ in the above inequality can be replaced by $c(\alpha)n^{\alpha \log n}$ (Xu improved this to allow $\alpha = \frac{1}{2}$), and he speculated that there may be an absolute constant c so that Schmidt's inequality holds with $c(n)$ replaced by cn. We [1] proved a weaker version of this conjecture with cn^3 instead of cn.

The main result, Theorem 3.2, of this paper shows that Schmidt's inequality holds with $c(n) = 2n-1$. This result can also be formulated as

$$\sup_{0 \ne f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \le \frac{2n-1}{\min\{y-a, b-y\}}\,, \qquad y \in (a,b)\,.$$

In this Bernstein-type inequality even the pointwise factor is sharp up to a multiplicative absolute constant; the inequality

$$\frac{1}{e-1}\,\frac{n-1}{\min\{y-a, b-y\}} \le \sup_{0 \ne f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}}\,, \qquad y \in (a,b)$$

is established in Theorem 3.3. Theorem 3.2 follows easily from our other central result, Theorem 3.1. This states that the equality

$$\sup_{0 \ne f \in \widetilde{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} = 2n-1$$

holds, where

$$\widetilde{E}_{2n} := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} \left(a_j e^{\lambda_j t} + b_j e^{-\lambda_j t} \right), \quad a_j, b_j, \lambda_j \in \mathbb{R} \right\}.$$

These results complement Newman's beautiful Markov-type inequality [9], see also [2], that states

$$\frac{2}{3} \sum_{j=0}^{n} \lambda_j \leq \sup_{0 \neq f \in E_n(\Lambda)} \frac{\|f'\|_{[0,\infty)}}{\|f\|_{[0,\infty)}} \leq 9 \sum_{j=0}^{n} \lambda_j ,$$

where $E_n(\Lambda) := \operatorname{span}\{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \dots , e^{-\lambda_n t}\}$ for any sequence Λ of distinct non-negative numbers λ_j.

Denote by $\mathcal{P}_n$ the set of all polynomials of degree at most n with real coefficients. Bernstein's classical inequality states the following.

Proposition 1.1 (Bernstein's Inequality). *The inequality*

$$|p'(x)| \leq \frac{n}{\sqrt{1-x^2}} \|p\|_{[-1,1]} , \qquad -1 < x < 1$$

holds for every $p \in \mathcal{P}_n$.

This implies by substitution and scaling (though not entirely obviously) that

$$|f'(y)| \leq \frac{2n}{\min\{y-a, b-y\}} \|f\|_{[a,b]} , \qquad y \in (a,b)$$

holds for the particular exponential sums of the form

$$f(t) = a_0 + \sum_{j=1}^{n} a_j e^{jt} , \qquad a_j \in \mathbb{R} .$$

This is a very special case ($\lambda_j = j$) of our Theorem 3.1.

Bernstein-type inequalities play a central role in approximation theory via a machinery developed by Bernstein, which turns Bernstein-type inequalities into inverse theorems of approximation. See, for example Lorentz [6] and DeVore and Lorentz [4]. Roughly speaking, our Theorem 3.2 implies that inverse theorems of approximation, over large classes of functions, by the particular exponential sums f of the form

$$f(t) = a_0 + \sum_{j=1}^{n} a_j e^{jt} , \qquad a_j \in \mathbb{R}$$

are essentially the same as those of approximation by arbitrary exponential sums f with $n+1$ terms of the form

$$f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t} , \qquad a_j, \lambda_j \in \mathbb{R} .$$

So one deduces in a standard fashion [4, 6], for example, that if there is a sequence $(f_n)_{n=1}^\infty$ of exponential sums with $f_n \in E_n$ that approximates f on an interval $[a,b]$ uniformly with errors $\|f-f_n\|_{[a,b]} = o\big(n^{-m}\big)$, $m \in \mathbb{N}$, then f is m times continuously differentiable on (a,b).

The following slight improvement of Bernstein's inequality may be found in Natanson [8].

Proposition 1.2. *The inequality*

$$|p'(0)| \le (2n-1)\,\|p\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_{2n}$.

Note that Proposition 1.1 implies Proposition 1.2 only with $(2n-1)$ replaced by $2n$. The following corollary of Proposition 1.2 can be obtained by a linear transformation. It plays an important role in the proof of Theorem 3.1.

Proposition 1.3. *The inequality*

$$|p'(x)| \le \frac{2n-1}{1-|x|}\,\|p\|_{[-1,1]}\,, \qquad x \in (-1,1)$$

holds for every $p \in \mathcal{P}_{2n}$.

Of course, Proposition 1.3 gives a better result than Proposition 1.1 only if x is very close to 0. However, this is going to be exactly the case in the proof of Theorem 3.1.

Theorem 3.4 deals with an essentially sharp Nikolskii-type inequality for E_n. We show that

$$\|f\|_{[a+\delta,b-\delta]} \le 2^{2/p^2}\left(\frac{n+1}{\delta}\right)^{1/p}\|f\|_{L_p[a,b]}$$

for every $f \in E_n$, $p \in (0,2]$, and $\delta \in \big(0, \frac{1}{2}(b-a)\big)$. A weaker version of our Theorem 3.1 can be easily deduced from this showing that Schmidt's inequality holds with $c(n) = 8(n+1)^2$.

Theorems 3.1 to 3.3 of this paper trivially extend to the wider classes E_n^* defined in the next section. A less direct version of our main results in this paper and related results will eventually appear in the book [2].

2. Notation and Definitions

The notations

$$\|f\|_A := \sup_{x\in A}|f(x)|$$

and

$$\|f\|_{L_p(A)} := \left(\int_A |f|^p\right)^{1/p}$$

are used throughout this paper for measurable functions f defined on a measurable set $A \subset \mathbb{R}$ and for $p \in (0,\infty)$. If $A := [a,b]$ is an interval, then the notation $L_p[a,b] := L_p(A)$ is used.

The classes E_n and $\widetilde{E}_{2n}$ are defined in the Introduction. The classes E_n^* and E_n^{*c} are introduced in Section 8.

The space of all real-valued continuous functions defined on $A \subset \mathbb{R}$ equipped with the uniform norm is denoted by $C(A)$.

3. New Results

Theorem 3.1. *We have*

$$\sup_{0 \neq f \in \widetilde{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} = 2n-1\,.$$

Theorem 3.2. *The inequality*

$$\sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}} \leq \frac{2n-1}{\min\{y-a, b-y\}}$$

holds for every $n \in \mathbb{N}$ and $y \in (a,b)$.

Theorem 3.3. *The inequality*

$$\frac{1}{e-1} \frac{n-1}{\min\{y-a, b-y\}} \leq \sup_{0 \neq f \in E_n} \frac{|f'(y)|}{\|f\|_{[a,b]}}$$

holds for every $n \in \mathbb{N}$ and $y \in (a,b)$.

Theorem 3.4. *The inequality*

$$\|f\|_{[a+\delta, b-\delta]} \leq 2^{2/p^2} \left(\frac{n+1}{\delta}\right)^{1/p} \|f\|_{L_p[a,b]}$$

holds for every $f \in E_n$, $p \in (0,2]$, and $\delta \in \big(0, \frac{1}{2}(b-a)\big)$.

4. Chebyshev and Descartes Systems

The proof of our main result relies heavily on the observation that for every $0 < \lambda_0 < \lambda_1 < \cdots$,

$$(\sinh \lambda_0 t\,, \sinh \lambda_1 t\,, \dots)$$

is a Descartes system on $(0,\infty)$. In this section we give the definitions of Chebyshev and Descartes systems. The only result of this section that is not to be found in standard sources is the critical Lemma 4.5. The remaining theory can be found in, for example, [5] or [4].

Definition 4.1 (Chebyshev System). Let $I \subset \mathbb{R}$ be an interval. The sequence $(f_0, f_1, \dots, f_n)$ is called a (real) Chebyshev system of dimension $n+1$ on I if $f_0, f_1, \dots, f_n$ are real-valued continuous functions on I, $\operatorname{span}\{f_0, f_1, \dots, f_n\}$ over $\mathbb{R}$ is an $n+1$ dimensional subspace of $C(A)$, and any $f \in \operatorname{span}\{f_0, f_1, \dots, f_n\}$ that has $n+1$ distinct zeros on I is identically zero.

If $(f_0, f_1, \dots, f_n)$ is a Chebyshev system on I, then $\operatorname{span}\{f_0, f_1, \dots, f_n\}$ is called a Chebyshev space on I.

The following simple equivalences are well known facts of linear algebra.

Proposition 4.2. *Let $f_0, f_1, \dots, f_n$ be real-valued continuous functions on an interval $I \subset \mathbb{R}$. Then the following are equivalent.*

a] *Every $0 \neq p \in \operatorname{span}\{f_0, f_1, \dots, f_n\}$ has at most n distinct zeros on I.*

b] *If $x_0, x_1, \dots, x_n$ are distinct elements of I and $y_0, y_1, \dots, y_n$ are real numbers, then there exists a unique $p \in \operatorname{span}\{f_0, f_1, \dots, f_n\}$ so that*

$$p(x_i) = y_i\,, \qquad i = 1, 2, \dots, n\,.$$

c] *If $x_0, x_1, \dots, x_n$ are distinct points of I, then*

$$D\begin{pmatrix} f_0 & f_1 & \dots & f_n \\ x_0 & x_1 & \dots & x_n \end{pmatrix} := \begin{vmatrix} f_0(x_0) & \dots & f_n(x_0) \\ \vdots & \ddots & \vdots \\ f_0(x_n) & \dots & f_n(x_n) \end{vmatrix} \neq 0\,.$$

Definition 4.3 (Descartes System). *The system $(f_0, f_1, \dots, f_n)$ is said to be a Descartes system (or order complete Chebyshev system) on an interval I if each $f_i \in C(I)$ and*

$$D\begin{pmatrix} f_{i_0} & f_{i_1} & \dots & f_{i_m} \\ x_0 & x_1 & \dots & x_m \end{pmatrix} > 0$$

for any $0 \le i_0 < i_1 < \cdots < i_m \le n$ and for any $x_0 < x_1 < \cdots < x_m$ from I. The definition of an infinite Descartes system $(f_0, f_1, \dots)$ on I is analogous.

This is a property of the basis. It implies that any finite dimensional subspace generated by some basis elements is a Chebyshev space on I. We remark the trivial fact that a Descartes system on I is a Descartes system on any subinterval of I.

Lemma 4.4. *The system*

$$(e^{\lambda_0 t}, e^{\lambda_1 t}, \dots), \qquad \lambda_0 < \lambda_1 < \cdots$$

is a Descartes system on $(-\infty, \infty)$. In particular, it is also a Chebyshev system on $(-\infty, \infty)$.

Proof. The determinant in Definition 4.3 is a generalized Vandermonde. See, for example, [5, p. 9]. $\square$

The following lemma plays a crucial role in the proof of Theorem 3.1.

Lemma 4.5. *Suppose* $0 < \lambda_0 < \lambda_1 < \cdots$. *Then*

$$(\sinh \lambda_0 t\,,\, \sinh \lambda_1 t\,,\, \dots)$$

is a Descartes system on $(0, \infty)$.

Proof. Let $0 \le i_0 < i_1 < \cdots < i_m$ be fixed integers . First we show that

$$(\sinh \lambda_{i_0} t\,,\, \sinh \lambda_{i_1} t\,,\, \dots\,,\, \sinh \lambda_{i_m} t)$$

is a Chebyshev system on $(0, \infty)$. Indeed, let

$$0 \ne f \in \operatorname{span}\{\sinh \lambda_{i_0} t\,,\, \sinh \lambda_{i_1} t\,,\, \dots\,,\, \sinh \lambda_{i_m} t\}\,.$$

Then

$$0 \ne f \in \operatorname{span}\{e^{\pm\lambda_{i_0} t}\,,\, e^{\pm\lambda_{i_1} t}\,,\, \dots\,,\, e^{\pm\lambda_{i_m} t}\}$$

and since

$$\operatorname{span}\{e^{\pm\lambda_{i_0} t}\,,\, e^{\pm\lambda_{i_1} t}\,,\, \dots\,,\, e^{\pm\lambda_{i_m} t}\}$$

is a Chebyshev system, f has at most $2m + 1$ zeros in $(-\infty, \infty)$. Since f is odd, it has at most m zeros in $(0, \infty)$.

Since for every $0 \le i_0 < i_1 < \cdots < i_m$, $(\sinh \lambda_{i_0} t\,,\, \sinh \lambda_{i_1} t\,,\, \dots\,,\, \sinh \lambda_{i_m} t)$ is a Chebyshev system on $(0, \infty)$, the determinant

$$D\begin{pmatrix} \sinh \lambda_{i_0} t & \sinh \lambda_{i_1} t & \dots & \sinh \lambda_{i_m} t \\ x_0 & x_1 & \dots & x_m \end{pmatrix}$$

is non-zero for any $0 < x_0 < x_1 < \cdots < x_m < \infty$ by Proposition 4.2. So it only remains to prove that it is positive whenever $0 < x_0 < x_1 < \cdots < x_m < \infty$. Now let

$$D(\alpha) := D\begin{pmatrix} \sinh \lambda_{i_0} t & \sinh \lambda_{i_1} t & \dots & \sinh \lambda_{i_m} t \\ \alpha x_0 & \alpha x_1 & \dots & \alpha x_m \end{pmatrix}$$

and

$$D^*(\alpha) := D\begin{pmatrix} \frac{1}{2} e^{\lambda_{i_0} t} & \frac{1}{2} e^{\lambda_{i_1} t} & \dots & \frac{1}{2} e^{\lambda_{i_m} t} \\ \alpha x_0 & \alpha x_1 & \dots & \alpha x_m \end{pmatrix},$$

where $0 < x_0 < x_1 < \cdots < x_m < \infty$ are fixed. Since

$$(\sinh \lambda_{i_0} t\,,\, \sinh \lambda_{i_1} t\,,\, \dots\,,\, \sinh \lambda_{i_m} t)$$

and

$$(e^{\lambda_{i_0} t}\,,\, e^{\lambda_{i_1} t}\,,\, \dots\,,\, e^{\lambda_{i_m} t})$$

are Chebyshev systems on $(0, \infty)$, $D(\alpha)$ and $D^*(\alpha)$ are continuous non-vanishing functions of α on $(0, \infty)$. Now observe that

$$\lim_{\alpha \to \infty} |D(\alpha)| = \lim_{\alpha \to \infty} |D^*(\alpha)| = \infty \qquad \text{and} \qquad \lim_{\alpha \to \infty} \frac{D(\alpha)}{D^*(\alpha)} = 1\,.$$

Since

$$(e^{\lambda_{i_0} t}\,,\, e^{\lambda_{i_1} t}\,,\, \dots\,,\, e^{\lambda_{i_m} t})$$

is a Descartes system on $(-\infty, \infty)$, $D^*(\alpha) > 0$ for every $\alpha > 0$. So the above limit relations imply that $D(\alpha) > 0$ for every large enough α, hence for every $\alpha > 0$. In particular,

$$D(1) = D\begin{pmatrix} \sinh \lambda_{i_0} t & \sinh \lambda_{i_1} t & \dots & \sinh \lambda_{i_m} t \\ x_0 & x_1 & \dots & x_m \end{pmatrix} > 0\,,$$

which finishes the proof. □

The following proposition on polynomials from the span of a Descartes system with a maximal number of zeros is also needed in the next section. Its proof is standard, and can be pieced together from [5, pp. 25-36] or found in [2, p. 108].

Lemma 4.6. *Suppose $(f_0, f_1, \dots, f_n)$ is a Descartes system on $[a, b]$. Suppose*

$$a < t_1 < t_2 < \cdots < t_n < b\,,$$

Then there exists a unique

$$p = f_n + \sum_{i=0}^{n-1} a_i f_i\,, \qquad a_i \in \mathbb{R}$$

so that

(1) $p(t_i) = 0\,, \quad i = 1, 2, \dots, n\,.$

Further, this p satisfies

(2) $p(t) \neq 0\,, \quad t \notin \{t_1, t_2, \dots, t_n\}\,,$

(3) $p(t)$ *changes sign at each* $t_1, t_2, \dots, t_n\,,$

(4) $a_i a_{i+1} < 0\,, \quad i = 0, 1, \dots, n-1\,, \quad a_n := 1\,,$

(5) $p(t) > 0$ *for* $t \in (t_n, b]$, *and* $(-1)^n p(t) > 0$ *for* $t \in [a, t_1)\,,$

(6) $(-1)^{n-i} p(t) > 0\,, \quad t \in (t_i, t_{i+1})\,, \quad i = 1, 2, \dots, n-1\,.$

5. Chebyshev Polynomials for $\operatorname{span}\{\sinh \lambda_0 t\,, \sinh \lambda_1 t\,, \dots\,, \sinh \lambda_n t\}$

We study the space

$$H_n := \operatorname{span}\{\sinh \lambda_0 t\,, \sinh \lambda_1 t\,, \dots\,, \sinh \lambda_n t\}\,,$$

where

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n\,.$$

We can define the *generalized Chebyshev polynomial* T_n for H_n on $[0, 1]$ by the following three properties:

$$T_n \in \operatorname{span}\{\sinh \lambda_0 t\,, \sinh \lambda_1 t\,, \dots\,, \sinh \lambda_n t\}\,, \tag{5.1}$$

there exists an alternation sequence $(x_0 < x_1 < \cdots < x_n)$ for T_n on $(0, 1]$, that is,

$$(-1)^i T_n(x_i) = \|T_n\|_{[0,1]}\,, \qquad i = 0, 1, \ldots, n\,, \tag{5.2}$$

and

$$\|T_n\|_{[0,1]} = 1\,. \tag{5.3}$$

The existence and uniqueness of such a T_n follows from the properties of the best uniform approximation to $\sinh \lambda_0 t$ on $[\epsilon, 1]$ from an n-dimensional Chebyshev space on $[\epsilon, 1]$ ($\epsilon > 0$ is sufficiently small). See [5, p. 35], for example.

The following extremal property of the Chebyshev polynomial T_n will be needed in the next section.

Theorem 5.1. *Using the notation above, we have*

$$\sup_{0 \neq p \in H_n} \frac{|p'(0)|}{\|p\|_{[0,1]}} = \frac{T_n'(0)}{\|T_n\|_{[0,1]}} = T_n'(0)\,.$$

Proof. Suppose $p \in H_n$ with $\|p\|_{[0,1]} < 1$ and $p'(0) > 0$. Observe that $T_n - p$ has at least one zero in each of the intervals $(x_0, x_1), (x_1, x_2), \ldots (x_{n-1}, x_n)$, where $(x_0 < x_1 < \cdots < x_n)$ is the alternation sequence for T_n on $(0, 1]$. Note that $p'(0) > T_n'(0)$ would imply that $T_n - p$ has at least one zero in $(0, x_0)$, therefore $0 \neq T_n - p \in H_n$ has at least $n + 1$ zeros in $(0, 1)$, which is impossible. □

6. A Comparison Theorem

The heart of the proof of Theorem 3.1 is the following comparison theorem, which can be proved by a zero counting argument. The method of our proof is very similar to that of a comparison therem of Pinkus and Smith [11] for Descartes systems. In fact, the simple proof of Theorem 6.1 was suggested by Allan Pinkus.

Theorem 6.1. *Let*

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n \qquad \textit{and} \qquad 0 < \gamma_0 < \gamma_1 < \cdots < \gamma_n\,.$$

Suppose $\gamma_i \leq \lambda_i$ for each i. Let

$$H_n := \operatorname{span}\{\sinh \lambda_0 t\,, \sinh \lambda_1 t\,, \ldots, \sinh \lambda_n t\}$$

and

$$G_n := \operatorname{span}\{\sinh \gamma_0 t\,, \sinh \gamma_1 t\,, \ldots, \sinh \gamma_n t\}\,.$$

Then

$$\max_{0 \neq p \in H_n} \frac{|p'(0)|}{\|p\|_{[0,1]}} \leq \max_{0 \neq p \in G_n} \frac{|p'(0)|}{\|p\|_{[0,1]}}\,.$$

Proof. We have
$$\sup_{0\neq p\in H_n}\frac{|p'(0)|}{\|p\|_{[0,1]}}=\frac{|T_n'(0)|}{\|T_n\|_{[0,1]}}\,,$$
where T_n is the Chebyshev polynomial for H_n on $[0,1]$. In particular, T_n has n distinct zeros in $(0,1)$. Let
$$T_n(t)=:\sum_{j=0}^{n}c_j\sinh\lambda_j t\,,\qquad c_j\in\mathbb{R}\,.$$
By Lemma 4.6, $(-1)^jc_j>0$. Let $k\in\{1,2,\dots n\}$ be fixed. Let $(\gamma_j)_{j=0}^n$ be such that
$$\gamma_0<\gamma_1<\cdots<\gamma_n\,,\qquad \gamma_j=\lambda_j\ \text{ for }\ j\neq k\,,\qquad \lambda_{k-1}<\gamma_k<\lambda_k$$
(we let $\gamma_{-1}:=0$). To prove this theorem, it is sufficient to study the above case since the general case follows from this by a finite number of pairwise comparisons.

Let $t_1<t_2<\cdots<t_n$ be the n zeros of T_n in $(0,1)$. Pick a $t_0\in(0,x_0)$, where x_0 is the first extreme point of T_n in $(0,1)$ (see (5.2)). Choose $Q_n\in G_n$ of the form
$$Q_n(x)=\sum_{j=0}^{n}d_j\sinh\gamma_j t\,,\qquad d_j\in\mathbb{R}$$
so that
$$Q_n(t_i)=T_n(t_i)\,,\qquad i=0,1,\dots,n\,.$$
By the unique interpolation property of Chebyshev spaces, Q_n is uniquely determined, has n zeros (the points $t_1,t_2,\dots,t_n$), and is positive at t_0. By Lemma 4,6, $(-1)^jd_j>0$ for each $j=0,1,\dots,n$.

We have
$$(T_n-Q_n)(t)=c_k\sinh\lambda_k t-d_k\sinh\gamma_k t+\sum_{j=0,j\neq k}^{n}(c_j-d_j)\sinh\lambda_j t\,.$$
The function T_n-Q_n changes sign on $(0,\infty)$ strictly at the points t_i, $i=0,1,\dots,n$, and has no other zeros. Also, by Lemma 4.5,
$$(\sinh\lambda_0 t\,,\ \sinh\lambda_1 t\,,\ \dots\,,\sinh\lambda_{k-1}t\,,\ \sinh\gamma_k t\,,\ \sinh\lambda_k t\,,\ \sinh\lambda_{k+1}t\,,\ \dots\,,\sinh\lambda_n t)$$
is a Descartes system on $(0,\infty)$. Hence, by Lemma 4.6, the sequence
$$(c_0-d_0\,,\ c_1-d_1\,,\ \dots\,,\ c_{k-1}-d_{k-1}\,,\ -d_k\,,\ c_k\,,\ c_{k+1}-d_{k+1}\,,\ \dots\,,\ c_n-d_n)$$
strictly alternates in sign. Since $(-1)^kc_k>0$, this implies that
$$(-1)^n(T_n-Q_n)(t)>0\,,\qquad t>t_n\,.$$
Thus for $t\in(t_j,t_{j+1})$ we have
$$(-1)^jT_n(t)>(-1)^jQ_n(t)>0\,,\qquad j=-1,0,1,\dots,n\,,$$

where $t_{-1} := 0$ and $t_{n+1} := \infty$. In addition, we recall that $Q_n(0) = T_n(0) = 0$ and $Q_n(t_0) = T_n(t_0) > 0$.

The observations above imply that if $t_0 \in (0, x_0)$ is sufficiently close to 0, then

$$\|Q_n\|_{[0,1]} \le \|T_n\|_{[0,1]} = 1 \qquad \text{and} \qquad Q_n'(0) \ge T_n'(0) > 0\,.$$

Thus

$$\frac{|Q_n'(0)|}{\|Q_n\|_{[0,1]}} \ge \frac{|T_n'(0)|}{\|T_n\|_{[0,1]}} = \sup_{0 \ne p \in H_n} \frac{|p'(0)|}{\|p\|_{[0,1]}}\,.$$

Since $Q_n \in G_n$, the conclusion of the theorem follows from this. $\square$

7 Proof of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. First we prove that

$$|f'(0)| \le (2n-1)\,\|f\|_{[-1,1]}$$

for every $f \in \widetilde{E}_{2n}$. So let

$$f \in \operatorname{span}\{1\,,\, e^{\pm\lambda_1 t}\,,\, e^{\pm\lambda_2 t}\,,\, \dots\,,\, e^{\pm\lambda_n t}\}$$

with some non-zero real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, where, without loss of generality, we may assume that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n\,.$$

Let

$$g(t) := \tfrac{1}{2}(f(t) - f(-t))\,.$$

Observe that

$$g \in \operatorname{span}\{\sinh\lambda_1 t\,,\, \sinh\lambda_2 t\,,\, \dots\,,\, \sinh\lambda_n t\}\,.$$

It is also straightforward that

$$g'(0) = f'(0) \qquad \text{and} \qquad \|g\|_{[0,1]} \le \|f\|_{[-1,1]}\,.$$

For a given $\epsilon > 0$, let

$$G_{n,\epsilon} := \operatorname{span}\{\sinh\epsilon t\,,\, \sinh 2\epsilon t\,,\, \dots\,,\, \sinh n\epsilon t\}$$

and

$$K_{n,\epsilon} := \sup\left\{|h'(0)| : h \in G_{n,\epsilon}\,,\ \ \|h\|_{[0,1]} = 1\right\}.$$

By Theorem 6.1, it is sufficient to prove that $\inf\{K_{n,\epsilon} : \epsilon > 0\} \le 2n-1$. Observe that every $h \in G_{n,\epsilon}$ is of the form

$$h(t) = e^{-n\epsilon t} P(e^{\epsilon t})\,, \qquad P \in \mathcal{P}_{2n}\,.$$

Therefore, using Proposition 1.3 combined with a linear transformation from $[-1,1]$ to $[e^{-\epsilon}, e^{\epsilon}]$, we obtain for every $h \in G_{n,\epsilon}$ that

$$\begin{aligned}
|h'(0)| &= |\epsilon P'(1) - n\epsilon P(1)| \\
&\leq \frac{\epsilon(2n-1)}{1-e^{-\epsilon}} \|P\|_{[e^{-\epsilon}, e^{\epsilon}]} + n\epsilon \|P\|_{[e^{-\epsilon}, e^{\epsilon}]} \\
&\leq \left(\frac{\epsilon(2n-1)}{1-e^{-\epsilon}} + n\epsilon\right) e^{n\epsilon} \|h\|_{[-1,1]} \\
&= \left(\frac{\epsilon(2n-1)}{1-e^{-\epsilon}} + n\epsilon\right) e^{n\epsilon} \|h\|_{[0,1]} \,.
\end{aligned}$$

It follows that

$$K_{n,\epsilon} \leq \left(\frac{\epsilon(2n-1)}{1-e^{-\epsilon}} + n\epsilon\right) e^{n\epsilon}.$$

So $\inf\{K_{n,\epsilon} : \epsilon > 0\} \leq 2n-1$, and the result follows.

Now we prove that

$$\sup_{0 \neq f \in \widetilde{E}_{2n}} \frac{|f'(0)|}{\|f\|_{[-1,1]}} \geq 2n-1\,.$$

Let $\epsilon > 0$ be fixed. We define

$$Q_{2n,\epsilon}(t) := e^{-n\epsilon t} T_{2n-1}\left(\frac{e^{\epsilon t}}{e^{\epsilon}-1} - \frac{1}{e^{\epsilon}-1}\right),$$

where T_{2n-1} denotes the Chebyshev polynomial of degree $2n-1$ defined by

$$T_{2n-1}(x) = \cos((2n-1)\arccos x)\,, \qquad x \in [-1,1]\,.$$

It is simple to check that $Q_{2n,\epsilon} \in \widetilde{E}_{2n}$,

$$\|Q_{2n,\epsilon}\|_{[-1,1]} \leq e^{n\epsilon t}$$

and

$$|Q'_{2n,\epsilon}(0)| \geq 2n-1-n\epsilon\,.$$

Now the result follows by letting $\epsilon > 0$ tend to 0. □

Proof of Theorem 3.2. Observe that $E_n \subset \widetilde{E}_{2n}$. Hence the result follows from Theorem 3.1 by a linear substitution. □

Proof of Theorem 3.3. Let $a < b$ and $y \in (a,b)$. Suppose that $n \in \mathbb{N}$ is odd. Let T_n be the Chebyshev polynomial of degree n defined by $T_n(x) = \cos(n \arccos x)$, $x \in [-1,1]$. Let

$$Q_n(t) := T_n\left(\frac{e}{e-1}\exp\left(\frac{t-b}{b-y}\right) - \frac{1}{e-1}\right)$$

and

$$R_n(t) := T_n\left(\frac{e}{e-1}\exp\left(\frac{t-a}{a-y}\right) - \frac{1}{e-1}\right).$$

Obviously $Q_n, R_n \in E_n$ and

$$\frac{|Q_n'(y)|}{\|Q_n\|_{[a,b]}} = \frac{1}{e-1}\frac{n}{b-y}\,,$$

and

$$\frac{|R_n'(y)|}{\|R_n\|_{[a,b]}} = \frac{1}{e-1}\frac{n}{y-a}$$

for every $y \in (a,b)$. The proof is now complete. $\square$

Proof of Theorem 3.4. Without loss of generality we may assume that $\Lambda := (\lambda_j)_{j=1}^n$ is a sequence of distinct non-zero real numbers. For the sake of brevity, let

$$E_n(\Lambda) := \operatorname{span}\{1\,,\, e^{\lambda_1 t}\,,\, e^{\lambda_2 t}\,,\, \dots\,,\, e^{\lambda_n t}\}\,.$$

Take an orthonormal sequence $(L_k)_{k=0}^n$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$ satisfying

(1) $\quad L_k \in \operatorname{span}\{1\,,\, e^{\lambda_1 t}\,,\, e^{\lambda_2 t}\,,\, \dots\,,\, e^{\lambda_k t}\}\,, \qquad k = 0,1,\dots,n$

and

(2) $\quad \int_{-1/2}^{1/2} L_i L_j = \delta_{i,j}\,, \qquad 0 \le i \le j \le n\,,$

where $\delta_{i,j}$ is the Kronecker symbol. On writing $f \in E_n(\Lambda)$ as a linear combination of $L_0, L_1, \dots, L_n$, and using the Cauchy-Schwarz inequality and the orthonormality of $(L_k)_{k=0}^n$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$, we obtain in a standard fashion that

$$\max_{0\neq f\in E_n(\Lambda)} \frac{|f(t_0)|}{\|f\|_{L_2[-1/2,1/2]}} = \left(\sum_{k=0}^n L_k^2(t_0)\right)^{1/2}\,, \qquad t_0 \in \mathbb{R}\,.$$

Since

$$\int_{-1/2}^{1/2} \textstyle\sum_{k=0}^n L_k^2(x)\,dx = n+1\,,$$

there exists a $t_0 \in \left[-\frac{1}{2},\frac{1}{2}\right]$ so that

$$\max_{0\neq p\in E_n(\Lambda)} \frac{|f(t_0)|}{\|f\|_{L_2[-1/2,1/2]}} = \left(\sum_{k=0}^n L_k^2(t_0)\right)^{1/2} \le \sqrt{n+1}\,.$$

Observe that if $f \in E_n(\Lambda)$, then $g(t) := f(t-t_0) \in E_n(\Lambda)$, so

$$\max_{0\neq f\in E_n(\Lambda)} \frac{|f(0)|}{\|f\|_{L_2[-1,1]}} \le \sqrt{n+1}\,.$$

Let

$$C := \max_{0\neq f\in E_n(\Lambda)} \frac{|f(0)|}{\|f\|_{L_p[-2,2]}}\,.$$

Then

$$\max_{0\neq f\in E_n(\Lambda)} \frac{|f(y)|}{\|f\|_{L_p[-2,2]}} \le C\left(\frac{2}{2-|y|}\right)^{1/p} \le 2^{1/p}C\,, \qquad y \in [-1,1]\,.$$

Therefore, for every $f \in E_n(\Lambda)$,

$$\begin{aligned}
|f(0)| &\leq \sqrt{n+1}\,\|f\|_{L_2[-1,1]} \\
&\leq \sqrt{n+1}\left(\|f\|^p_{L_p[-1,1]}\|f\|^{2-p}_{[-1,1]}\right)^{1/2} \\
&\leq \sqrt{n+1}\left(\|f\|^p_{L_p[-1,1]}\left(2^{1/p}C\right)^{2-p}\|f\|^{2-p}_{L_p[-2,2]}\right)^{1/2} \\
&\leq \sqrt{n+1}\left(2^{1/p}C\right)^{1-p/2}\|f\|_{L_p[-2,2]} \\
&= 2^{1/p-1/2}\sqrt{n+1}\,C^{1-p/2}\|f\|_{L_p[-2,2]}\,.
\end{aligned}$$

Hence

$$C = \max_{0 \neq f \in E_n(\Lambda)} \frac{|f(0)|}{\|f\|_{L_p[-2,2]}} \leq 2^{1/p-1/2}\sqrt{n+1}\,C^{1-p/2}$$

and we conclude that $C \leq 2^{2/p^2-1/p}(n+1)^{1/p}$. Therefore

$$|f(0)| \leq 2^{2/p^2-1/p}(n+1)^{1/p}\|f\|_{L_p[-2,2]}$$

for every $f \in E_n(\Lambda)$. Now let $f \in E_n(\Lambda)$ and $t_0 \in [a+\delta, b-\delta]$. If we apply the above inequality to

$$g(t) := f\left(\tfrac{1}{2}\delta t + t_0\right) \in E_n(\Lambda)\,,$$

we obtain

$$\|f\|_{[a+\delta,b-\delta]} \leq 2^{2/p^2-1/p}(n+1)^{1/p}\left(\frac{2}{\delta}\right)^{1/p}\|f\|_{L_p[a,b]}\,,$$

and the result follows. □

8 Remarks.

Remark 8.1. *Theorem 3.4 implies a weaker version of Theorem 3.1, namely*

$$\|f'\|_{[a+\delta,b-\delta]} \leq 8(n+1)^2\delta^{-1}\|f\|_{[a,b]}$$

for every $f \in E_n$ *and* $\delta \in (0, \frac{1}{2}(b-a))$.

Proof. Note that $f \in E_n(\Lambda)$ implies $f' \in E_n(\Lambda)$. Applying Theorem 3.4 to f' with $p = 1$, we obtain

$$|f'(0)| \leq 2(n+1)\|f'\|_{L_1[-2,2]} = 2(n+1)\mathrm{Var}_{[-2,2]}(f) \leq 4(n+1)^2\|f\|_{[-2,2]}$$

for every $f \in E_n(\Lambda)$. Now if $f \in E_n(\Lambda)$ and $t_0 \in [a+\delta, b-\delta]$, then on applying the above inequality to

$$g(t) := f\left(\tfrac{1}{2}\delta t + t_0\right) \in E_n(\Lambda)\,,$$

we obtain the desired result. □

Remark 8.2. *Theorems 3.2 and 3.4 trivially extend to the classes*

$$E_n^* := \left\{ f : f(t) = \sum_{j=1}^{l} P_{k_j}(t) e^{\lambda_j t}\,, \quad \lambda_j \in \mathbb{R}\,, \quad P_{k_j} \in \mathcal{P}_{k_j}\,, \quad \sum_{j=1}^{l} (k_j + 1) = n \right\}.$$

Remark 8.3. *Theorem 3.4 extends to the classes*

$$E_n^{*c} := \left\{ f : f(t) = \sum_{j=1}^{l} P_{k_j}(t) e^{\lambda_j t}\,, \quad \lambda_j \in \mathbb{C}\,, \quad P_{k_j} \in \mathcal{P}_{k_j}^c\,, \quad \sum_{j=1}^{l} (k_j + 1) = n \right\},$$

where $\mathcal{P}_{k_j}^c$ *denotes the family of all polynomials of degree at most* k_j *with complex coefficients. This follows by trivial modifications of the proof.*

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