DENSE MARKOV SPACES AND UNBOUNDED BERNSTEIN INEQUALITIES

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Abstract. An infinite Markov system $\{f_0, f_1, \dots\}$ of C^2 functions on $[a,b]$ has dense span in $C[a,b]$ if and only if there is an unbounded Bernstein inequality on every subinterval of $[a,b]$. That is if and only if, for each $[\alpha,\beta] \subset [a,b]$ and $\gamma > 0$, we can find $g \in \operatorname{span}\{f_0, f_1, \dots\}$ with $\|g'\|_{[\alpha,\beta]} > \gamma \|g\|_{[a,b]}$. This is proved under the assumption $(f_1/f_0)'$ does not vanish on (a,b).

Extension to higher derivatives are also considered. An interesting consequence of this is that functions in the closure of the span of a non-dense C^2 Markov system are always C^n on some subinterval.

The principal result of this paper will be a characterization of denseness of the span of a Markov system by whether or not it possesses an unbounded Bernstein Inequality. In order to make sense of this result we require the following definitions.

Definition 1 (Chebyshev System). *Let $C[a,b]$ be the collection of the real valued continuous functions on $[a,b]$. Suppose that* $\operatorname{span}\{f_0, \dots, f_n\}$ *over $\mathbb{R}$ is an $n+1$ dimensional subspace of $C[a,b]$. Then $\{f_0, \dots, f_n\}$ is called a Chebyshev system of dimension $n+1$ if any element of* $\operatorname{span}\{f_0, \dots, f_n\}$ *that has $n+1$ distinct zeros in $[a,b]$ is identically zero. If $\{f_0, \dots, f_n\}$ is a Chebyshev system, then* $\operatorname{span}\{f_0, \dots, f_n\}$ *is called a Chebyshev space.*

Definition 2 (Markov System). *We say that $\{f_0, \dots, f_n\}$ is a Markov system on $[a,b]$ if each $f_i \in C[a,b]$ and $\{f_0, \dots, f_m\}$ is a Chebyshev system for every $m \geq 0$. (We allow n to tend to $+\infty$ in which case we call the system an infinite Markov system.) If $\{f_0, \cdots, f_n\}$ is a Markov system then* $\operatorname{span}\{f_0, \dots, f_n\}$ *is called a Markov space.*

Definition 3 (Unbounded Bernstein Inequality). *Let $\mathcal{A}$ be a subset of $C^1[a,b]$. We say that $\mathcal{A}$ has an everywhere unbounded Bernstein inequality if for every $[\alpha,\beta] \subset [a,b]$, $\quad \alpha \neq \beta$*

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$$\sup\left\{\frac{\|p'\|_{[\alpha,\beta]}}{\|p\|_{[a,b]}} : p \in \mathcal{A}, p \neq 0\right\} = \infty.$$

If for some $[\alpha, \beta]$ the above sup is finite the Bernstein inequality is said to be bounded in $[\alpha, \beta]$.

Note that the collection of all polynomials of the form

$$\{x^2 p(x) : p \text{ is a polynomial}\}$$

has an everywhere unbounded Bernstein inequality on $[-1, 1]$ despite the fact that every element has derivative vanishing at zero.

We now state the main result.

Theorem 1. *Suppose $\mathcal{M} := \{f_0, f_1, f_2, \dots\}$ is an infinite Markov system on $[a, b]$ with each $f_i \in C^2[a, b]$, and suppose that $(f_1/f_0)'$ does not vanish on (a, b). Then* span $\mathcal{M}$ *is dense in $C[a, b]$ if and only if* span $\mathcal{M}$ *has an everywhere unbounded Bernstein inequality.*

The additional assumption that $(f_1/f_0)'$ does not vanish on (a, b) is quite weak. It holds, for example, for any ECT system. Note that f_1/f_0 is strictly monotone if $\mathcal{M}$ is a Markov system.

The proof requires examining the Chebyshev polynomials associated with a Chebyshev system. These we now discuss.

Suppose

$$H_n := \operatorname{span}\{f_0, \dots, f_n\}$$

is a Chebyshev space on $[a, b]$. We can define the Chebyshev polynomial

$$T_n(x) := T_n\{f_0, \dots, f_n; [a, b]\}(x)$$

associated with H_n

by

$$T_n(x) = c\left(f_n(x) - \sum_{k=0}^{n-1} a_k f_k(x)\right)$$

where the $\{a_k\}_{k=0}^{n-1}$ are chosen to minimize

$$\left\| f_n - \sum_{k=0}^{n-1} a_k f_k \right\|_{[a,b]}$$

and where c is a normalization constant chosen so that

$$\|T_n\|_{[a,b]} = 1 \quad \text{and} \quad T_n(b) > 0.$$

We will call T_n the associated Chebyshev polynomial for H_n. This is a unique "generalized" polynomial in $\operatorname{span}\{f_0, \dots, f_n\}$ that alternates between ± 1 exactly

$n+1$ times and has exactly n zeros on $[a,b]$. With $f_i := x^i$, this generates the usual Chebyshev polynomials. These equioscillating polynomials encode much of the information of how the space H_n behaves with respect to the supremum norm. See [2], [3], [4] and [6].

Suppose

$$\mathcal{M} = \{f_0, f_1, \dots\}$$

is a fixed infinite Markov system on $[a,b]$. For each n

$$H_n := \{f_0, f_1, \dots, f_n\}$$

is then a Chebyshev system. So there is a sequence $\{T_n\}$ of associated Chebyshev polynomials where, for each n, T_n is associated with H_n. These we call the associated Chebyshev polynomials for the infinite Markov system $\mathcal{M}$.

Note that

$$\{T_0, T_1, \dots\}$$

is a Markov system again with the same span as $\mathcal{M}$.

In [2] we showed that the span of a C^1 Markov system $\mathcal{M}$ is dense in $C[a,b]$ in the uniform norm (i.e. the uniform closure of span $\mathcal{M}$ on $[a,b]$ equals $C[a,b]$) if and only if the zeros of the associated Chebyshev polynomials are dense. To state this result, which we will need, we require the following notation.

Suppose T_n has zeros $a \le x_1 < x_2 < \dots < x_n \le b$, and let $x_0 := a$ and $x_{n+1} := b$. Then the mesh of T_n is defined by

$$M_n := M_n(T_n : [a,b]) := \max_{1 \le i \le n+1} |x_i - x_{i-1}|.$$

For a sequence of Chebyshev polynomials T_n from a fixed Markov system on $[a,b]$ we have

$$M_n \to 0 \quad \text{iff} \quad \underline{\lim} M_n = 0$$

as follows from the interlacing of the zeros of T_n and T_{n+1} (see [6]).

Our main result requires the following theorem from [2].

Theorem 2. *Suppose $\mathcal{M} := \{1, f_1, f_2, \dots\}$ is an infinite Markov system on $[a,b]$ with each $f_i \in C^1[a,b]$. Then span $\mathcal{M}$ is dense in $C[a,b]$ in the uniform norm if and only if*

$$M_n \to 0$$

(where M_n is the mesh of the associated Chebyshev polynomials).

The next result we need shows that in most instances the Chebyshev polynomial is close to extremal for Bernstein-type inequalities.

Theorem 3. *Let $H_n := \{1, f_1, \dots, f_n\}$ be a Chebyshev system of C^1 functions on $[a,b]$. Let T_n be the associated Chebyshev polynomial. Then*

$$\frac{|p_n'(x_0)|}{\|p_n\|_{[a,b]}} \le \frac{2}{1 - |T_n(x_0)|} |T_n'(x_0)|$$

for every $0 \neq p_n \in \operatorname{span}\{1, f_1, \dots, f_n\}$ *and every* $x_0 \in [a,b]$ *with* $|T_n(x_0)| \neq 1$.

Proof. Let $a = y_0 < y_1 < \dots < y_n = b$ denote the extreme points of T_n, so

$$T_n(y_i) = (-1)^{n-i}, \qquad i = 0, 1, \dots, n.$$

Let $y_k \leq x_0 \leq y_{k+1}$ and $0 \neq p_n \in H_n$. If $p_n'(x_0) = 0$, then there is nothing to prove. So assume that $p_n'(x_0) \neq 0$. Then we may normalize p_n so that

$$\|p_n\|_{[a,b]} = 1$$

and

$$\operatorname{sign}(p_n'(x_0) = \operatorname{sign}(p(y_{k+1}) - p(y_k)).$$

Let $\delta := |T_n(x_0)|$. Let $\epsilon \in (0,1)$ be fixed. Then there exists a constant η with $|\eta| \leq \delta + (1-\delta)/2$ so that

$$\eta + \frac{(1-\delta)}{2}(1-\epsilon)p_n(x_0) = T_n(x_0).$$

Now let

$$q_n(x) := \eta + \frac{(1-\delta)}{2}(1-\epsilon)p_n(x).$$

Then

$$\|q_n\|_{[a,b]} \leq 1,$$

$$q_n(x_0) = T_n(x_0)$$

and

$$\operatorname{sign}(q_n'(x_0)) = \operatorname{sign}(T_n'(x_0)).$$

If the desired inequality does not hold for p_n then for a sufficiently small $\epsilon > 0$

$$|q_n'(x_0)| > |T_n'(x_0)|,$$

so

$$h_n(x) := q_n(x) - T_n(x)$$

will have at least 3 zeros in (y_k, y_{k+1}). But h_n has at least one zero in each of (x_i, x_{i+1}). Hence $h_n \in H_n$ has at least $n+2$ zeros in $[a,b]$, which is a contradiction.

□

We need the following technical result concerning Chebyshev polynomials.

Lemma 1. *Suppose* $\mathcal{M} := \{1, f_1, f_2, \dots\}$ *is an infinite Markov system of* C^2 *functions on* $[a,b]$ *and* f_1' *does not vanish on* (a,b). *Suppose that the associated Chebyshev polynomials* $\{T_n\}$ *has a subsequence* $\{T_{n_i}\}$ *with no zeros on some subinterval of* $[a,b]$. *Then there exists another subinterval* $[c,d]$ *and another infinite subsequence* $\{T_{n_i}\}$ *so that for some* $\delta > 0$, $\gamma > 0$

$$\|T_{n_i}\|_{[c,d]} < 1 - \delta$$

and

$$\|T'_{n_i}\|_{[c,d]} < \gamma$$

for all n_i*.*

Proof. For both inequalities we first choose a subinterval $[c_1, d_1] \subset [a, b]$ and a subsequence $\{n_{i,1}\}$ of $\{n_i\}$ so that all oscillations of each $T_{n_{i,1}}$ take place away from $[c_1, d_1]$. We now choose a subsequence $\{n_{i,2}\}$ of $\{n_{i,1}\}$ so that either each $T_{n_{i,2}}$ is increasing or each $T_{n_{i,2}}$ is decreasing on $[c_1, d_1]$. We treat the first case, the second one is analogous. Let $[c_2, d_2]$ be the middle third of $[c_1, d_1]$. If the first inequality fails to hold with $[c_2, d_2]$ and $\{n_{i,2}\}$ then there is a subsequence $\{n_{i,3}\}$ of $\{n_{i,2}\}$ so that $\|T_{n_{i,3}}\|_{[c_2,d_2]} \to 1$ as $n_{i,3} \to \infty$. Hence, there is a subsequence $\{n_{i,4}\}$ of $\{n_{i,3}\}$ so that either

$$\max_{c_2 \le x \le d_2} T_{n_{i,4}}(x) \to 1 \quad \text{or} \quad \min_{c_2 \le x \le d_2} T_{n_{i,4}}(x) \to -1.$$

Once again we treat the first case, the second one is analogous. Since each $T_{n_{i,3}}$ is increasing on $[c_1, d_1]$,

$$\lim_{n_{i,4} \to \infty} \|1 - T_{n_{i,4}}\|_{[d_2,d_1]} = 0.$$

Now take $g := a_0 + a_1 f_1 + a_2 f_2$ so that g has two distinct zeros α_1 and α_2 in $[d_2, d_1]$, $\|g\|_{[\alpha_1,\alpha_2]} < 1$ and g is positive on (α_1, α_2). Let $\beta := \max\limits_{\alpha_1 \le x \le \alpha_2} g(x)$ and $\tilde{g} := g + 1 - \beta$. One can now deduce that $T_{n_{i,4}} - \tilde{g}$ has at least $n + 1$ distinct zeros in $[a, b]$ if $n_{i,4}$ is large enough, which is a contradiction.

For the second inequality, by [8], $\{f'_1, f'_2, \dots\}$ is a weak Markov system on $[a, b]$, and so is

$$\left\{ (T'_2/T'_1)', (T'_3/T'_1)', \dots \right\}$$

on every closed subinterval of (a, b). (In the definitions of weak Markov systems and weak Chebyshev systems we only count zeros where the sign changes.) The assumption that f'_1 does not vanish on (a, b) implies that T'_1 does not vanish on (a, b).

¿From this we deduce that each $(T'_{n_{i,2}}/T'_1)'$ has at most one sign change in $[c_2, d_2]$. Choose a subinterval $[c_3, d_3] \subset [c_2, d_2]$ and a subsequence $\{n_{i,5}\}$ of $\{n_{i,2}\}$ so that none of $(T'_{n_{i,5}}/T'_1)'$ changes sign in $[c_3, d_3]$. Choose a subsequence $\{n_{i,6}\}$ of $\{n_{i,5}\}$ so that either each $T'_{n_{i,6}}/T'_1$ is increasing or each $T'_{n_{i,6}}/T'_1$ is decreasing on $[c_3, d_3]$. We only study the first case, the second one is similar. Let $[c_4, d_4]$ be the middle third of $[c_3, d_3]$. If the second inequality fails to hold with $[c_4, d_4]$ and $\{n_{i,6}\}$ then there is a subsequence $\{n_{i,7}\}$ so that either

$$\max_{c_4 \le x \le d_4} T'_{n_{i,7}}(x) \,/\, T'_1(x) \to \infty$$

or

$$\min_{c_4 \le x \le d_4} T'_{n_{i,7}}(x) \,/\, T'_1(x) \to -\infty.$$

Again we treat only the first case, the second one is analogous. Then for every $K > 0$ there is N so that for every $n_{i,7} \ge N$ we have

$$T'_{n_{i,7}}(x) > K, \qquad x \in [d_4, d_3],$$

hence

$$K(d_3 - d_4) \le \int_{d_4}^{d_3} T'_{n_{i,7}}(x)dx = T_{n_{i,7}}(d_3) - T_{n_{i,7}}(d_4) \le 2,$$

which is a contradiction. $\square$

Lemma 2. *Suppose $\mathcal{M} := \{f_0, f_1, \dots\}$ is a $C^1[a,b]$ infinite Markov system and suppose $g \in C^1[a,b]$ and g is strictly positive on $[a,b]$. Then $\mathcal{N} = \{gf_0, gf_1, \dots\}$ is also a $C^1[a,b]$ infinite Markov system. Furthermore* span $\mathcal{M}$ *has a bounded Bernstein inequality on $[\alpha,\beta] \subset [a,b]$ if and only if* span $\mathcal{N}$ *also has bounded Bernstein inequality on $[\alpha,\beta]$.*

Proof. Consider differentiating gf with $f \in \text{span}\,\mathcal{M}$ by the product rule. If span $\mathcal{M}$ has a bounded Bernstein inequality on $[\alpha,\beta]$ then

$$\begin{aligned}\|(gf)'\|_{[\alpha,\beta]} &\le \|g'f\|_{[\alpha,\beta]} + \|gf'\|_{[\alpha,\beta]} \\ &\le c_1\|gf\|_{[\alpha,\beta]} + c_2\|gf\|_{[a,b]}\end{aligned}$$

where the first constant arises since

$$g'(x)/g(x)$$

is uniformly bounded on $[a,b]$ and the second constant comes from the bounded Bernstein inequality for f. $\square$

Proof of Theorem 1. The only if part of this theorem is obvious. A good uniform approximation to a function with uniformly large derivative on a subinterval $[\alpha,\beta] \subset [a,b]$ must have large derivative at some points in $[\alpha,\beta]$.

In the other direction we first note that by Lemma 2 we may assume $f_0 \equiv 1$. We use Theorem 2 and Lemma 1 in the following way. If span $\mathcal{M}$ is not dense then there exists a subinterval $[\alpha,\beta] \subset [a,b]$ by Theorem 2, where a subsequence of the associated Chebyshev polynomials have no zeros. By Lemma 1 from this subsequence we can pick another subsequence T_{n_i} and a subinterval $[c,d] \subset [\alpha,\beta]$ with

$$\|T_{n_i}\|_{[c,d]} < 1 - \delta$$

and

$$\|T'_{n_i}\|_{[c,d]} < \gamma$$

for some positive constants δ and γ. The result now follows from Theorem 3. $\square$

Corollary 1. *Suppose $\mathcal{M} = \{f_0, f_1, \dots\}$ is an infinite Markov system of C^2 functions on $[a,b]$ so that* span $\mathcal{M}$ *fails to be dense in $C[a,b]$ in the uniform norm. Then there exists a subinterval $[\alpha,\beta]$ of $[a,b]$ so that if g is in the uniform closure of* span $\mathcal{M}$ *then g is differentiable on $[\alpha,\beta]$.*

Proof. By Theorem 1, there exists an interval $[\alpha,\beta]$ where $\|h'\|_{[\alpha,\beta]}/\|h\|_{[a,b]}$ is uniformly bounded for every $h \in \text{span}\,\mathcal{M}$. Suppose $h_n \to g, h_n \in \text{span}\,\mathcal{M}$. Then we can choose n_i so that

$$\|g - h_{n_i}\|_{[a,b]} \le \frac{1}{2^i} \qquad i = 0, 1, 2, \dots$$

and hence

$$g = \sum_{i=1}^{\infty} (h_{n_i} - h_{n_{i-1}}) + h_{n_0}.$$

Since

$$\|(h_{n_i} - h_{n_{i-1}})'\|_{[\alpha,\beta]} \le \frac{c}{2^i}$$

for some constant c independent of i, if follows that g is differentiable on $[\alpha, \beta]$. $\square$

Suppose $\mathcal{M} = \{f_0, f_1, \dots\}$ is an extended complete Markov system of C^∞ functions on $[a, b]$ (the extra requirement being that the multiplicity of the zeros matters in the definition: so if $f := \sum_{i=0}^{n} a_i f_i$ has $n+1$ zeros by counting multiplicities then $f = 0$ identically). In this case the differential operator D defined by

$$D(f) := \left(\frac{f}{f_0} \right)'$$

maps $\mathcal{M}$ to $\mathcal{M}_D$ where

$$\mathcal{M}_D = \left\{ \left(\frac{f_1}{f_0} \right)', \left(\frac{f_2}{f_0} \right)', \dots \right\}$$

and $\mathcal{M}_D$ is once again an extended complete Markov system of C^∞ functions (see Nürnberger [5]). We define the differential operators $D^{(n)}(f)$ for n times differentiable functions f by

$$F_{i,0} := f_i, \qquad F_{i,n} := \left(\frac{F_{i+1,n-1}}{F_{0,n-1}} \right)', \qquad i = 0, 1, \dots, \;\; n = 1, 2, \dots,$$

$$D^{(0)}(f) := f, \qquad D^{(n)}(f) := \left(\frac{D^{(n-1)}(f)}{F_{0,n-1}} \right)', \qquad n = 1, 2, \dots.$$

Note that if span $\mathcal{M}_D$ is dense in $C[a, b]$ in the uniform norm then so is span $\mathcal{M}$. The "if" part of the next theorem can be proved from Theorem 1 by induction on n, while the "only if" part is obvious.

Theorem 4. *Suppose $\mathcal{M} = \{f_0, f_1, \dots\}$ is an extended complete Markov system of C^∞ functions on $[a, b]$. Let n be a fixed positive integer. Then* span $\mathcal{M}$ *is dense in $C[a, b]$ in the uniform norm if and only if*

$$\sup \left\{ \frac{\|D^{(n)}(f)\|_{[\alpha,\beta]}}{\|f\|_{[a,b]}} : f \in \text{span}\, \mathcal{M}, f \ne 0 \right\} = \infty$$

for every $[\alpha, \beta] \subset [a, b]$, $\alpha \ne \beta$.

Corollary 2. *Suppose $\mathcal{M}$ is an extended complete Markov system of C^∞ functions on $[a,b]$ so that* span $\mathcal{M}$ *fails to be dense in $C[a,b]$ in the uniform norm. Then for each n there exists an interval $[\alpha_n, \beta_n] \subset [a,b]$ of positive length where all elements of the uniform closure of* span $\mathcal{M}$ *are n times continuously differentiable.*

Proof. Use Theorem 4 as in Corollary 1. We omit the technical details. □

Suppose that $\mathcal{M}$, as in Corollary 2, has the property that span $\mathcal{M}$ fails to be dense in the uniform norm on any proper subinterval of $[a,b]$, as in the case of Müntz systems

$$\mathcal{M} := \{x^{\lambda_0}, x^{\lambda_1}, \dots\}, \qquad 0 \le \lambda_0 < \lambda_1 < \cdots, \qquad \sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty, \qquad 0 \le a < b.$$

Then the uniform closure of span $\mathcal{M}$ on $[a,b]$ contains only functions that are C^∞ on a dense subset of $[a,b]$. In this non-dense Müntz case the closure actually contains only analytic functions on (a,b) (Achiezer [1], Schwartz [7]).

We record one final corollary.

Corollary 3. *Suppose $\{\alpha_k\} \subset \mathbb{R} \setminus [-1,1]$ is a sequence of distinct numbers. Then*

$$\operatorname{span}\left\{1, \frac{1}{x-\alpha_1}, \frac{1}{x-\alpha_2}, \dots\right\}$$

is dense in $C[-1,1]$ if and only if

$$\sum_{k=1}^{\infty} \sqrt{\alpha_k^2 - 1} = \infty.$$

Proof. The inequality

$$|p'(x)| \le \frac{1}{\sqrt{1-x^2}} \sum_{k=1}^{n} \frac{\sqrt{\alpha_k^2 - 1}}{|\alpha_k - x|} \|p\|_{[-1,1]}$$

holds for any

$$p \in \operatorname{span}\left\{1, \frac{1}{x-\alpha_1}, \dots, \frac{1}{x-\alpha_n}\right\}.$$

See [3]. This together with Theorem 1 gives the "only if" part of the corollary.

In [3] the Chebyshev "polynomials" T_n (of the first kind) and U_n (of the second kind) for the Chebyshev space

$$\operatorname{span}\left\{1, \frac{1}{x-\alpha_1}, \dots, \frac{1}{x-\alpha_n}\right\}$$

are introduced. Properties of

$$\widetilde{T}_n(t) := T_n(\cos t)$$

and

$$\widetilde{U}_n(t) := U_n(\cos t)\sin t$$

established in [3] include

$$\|\widetilde{T}_n\|_{\mathbb{R}} = 1 \qquad \textit{and} \qquad \|\widetilde{U}_n\|_{\mathbb{R}} = 1, \tag{1}$$

$$\widetilde{T}_n(t)^2 + \widetilde{U}_n(t)^2 = 1, \qquad t \in \mathbb{R}, \tag{2}$$

$$\widetilde{T}_n'(t)^2 + \widetilde{U}_n'(t)^2 = \widetilde{B}_n(t)^2, \qquad t \in \mathbb{R}, \tag{3}$$

$$\widetilde{T}_n'(t) = -\widetilde{B}_n(t)\widetilde{U}_n(t), \qquad t \in \mathbb{R}, \tag{4}$$

$$\widetilde{U}_n'(t) = \widetilde{B}_n(t)\widetilde{T}_n(t), \qquad t \in \mathbb{R} \tag{5}$$

where

$$\widetilde{B}_n(t) = \sum_{k=1}^{n} \frac{\sqrt{\alpha_k^2 - 1}}{|\alpha_k - \cos t|}, \qquad t \in \mathbb{R}.$$

Suppose

$$\sum_{k=1}^{\infty} \sqrt{\alpha_k^2 - 1} = \infty.$$

Then

$$\lim_{n\to\infty} \min_{t\in[\alpha,\beta]} \widetilde{B}_n(t) = \infty, \qquad 0 < \alpha < \beta < \pi. \tag{6}$$

Assume that there is a subinterval $[a, b]$ of $(-1, 1)$ so that

$$\sup_{n\in\mathbb{N}} \|T_n'\|_{[a,b]} < \infty.$$

Let $\alpha := \arccos b$ and $\beta := \arccos a$. Then by properties (4) and (6)

$$\lim_{n\to\infty} \|\widetilde{U}_n\|_{[\alpha,\beta]} = 0$$

hence by property (2)

$$\lim_{n\to\infty} \|\widetilde{T}_n^2 - 1\|_{[\alpha,\beta]} = 0.$$

Thus by properties (5) and (6)

$$\lim_{n\to\infty} \min_{t\in[\alpha,\beta]} |\widetilde{U}_n'(t)| = \infty$$

that is

$$\lim_{n\to\infty} |\widetilde{U}_n(\beta) - \widetilde{U}_n(\alpha)| = \infty$$

which contradicts property (1). Hence

$$\sup_{n\in\mathbb{N}} \frac{\|T_n'\|_{[a,b]}}{\|T_n\|_{[-1,1]}} = \sup_{n\in\mathbb{N}} \|T_n'\|_{[a,b]} = \infty.$$

for every subinterval $[a,b]$ of $(-1,1)$ which together with Theorem 1 finishes the "if" part of the proof. □

Corollary 3 is to be found in Achieser [1, p. 255] proven by entirely different methods.

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