MARKOV–BERNSTEIN TYPE INEQUALITIES ON COMPACT SUBSETS OF R

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Dedicated to Professor F. Móricz on his 60th birthday

1. Introduction

The classical Markov inequality

$$\max_{a\le x\le b}|p_n'(x)|\le\frac{2n^2}{b-a}\max_{a\le x\le b}|p_n(x)| \tag{1.1}$$

where $p_n\in\Pi_n$ (=the set of algebraic polynomials of degree at most n), as well as the Bernstein inequality

$$|p_n'(x)|\le\frac{n}{\sqrt{(x-a)(b-x)}}\max_{a\le y\le b}|p_n(y)|\qquad(a<x<b) \tag{1.2}$$

play an important role in approximation theory. Various generalizations in several directions are well-known; for a survey of these results see the recent monograph of P. Borwein and T. Erdélyi [2]. A number of papers study possible extensions of Markov-type inequalities to compact sets $K\subset\mathbf{R}$ when the geometry of K is known apriori (Cantor type sets, finitely many intervals, etc.; cf. W. Pleśniak [7] and the references therein, as well as Borwein and Erdélyi [1], Totik [9], [10]), and this determines the approach to the above mentioned inequalities.

In this paper we consider another possible path of generalizations. Instead of the knowledge of the geometry of the set, we define some density functions of the set in the neighborhood of a point, and estimate the derivative at this point. With this approach we will be able to settle the problem for many interesting sets, and when this method breaks down then we use an interpolation theoretic approach. In both situations, we distinguish Markov type inequalities (when we use information about the polynomial only on one side of the point), and Bernstein type inequalities (when information is provided on both sides of the point). Although our results are formulated for one point, with a proper modification of the density functions we could establish uniform estimates on the whole set.

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2. A density function measuring the "longest gaps"

Let K be an arbitrary compact set on the real line, and define the one-sided density function of K at 0 as

$$\varphi_K(t) = \frac{1}{t} \max\{b - a : (a, b) \subset [0, t], (a, b) \cap K = \emptyset\} \qquad (t > 0).$$

We shall suppose that this continuous function is positive for all $t > 0$ (otherwise K would be dense in a right neighborhood of 0 and the problem is trivial). Since in general, it is difficult to determine the density function exactly, and since it is not necessarily an increasing function, we shall work with its strictly increasing majorants. Also let

$$M_n(K) = \sup\{|p_n'(0)| : p_n \in \Pi_n, \sup_{x \in K} |p_n(x)| \le 1\}$$

be the n-th Markov-factor of the set K at 0.

First we present a Markov type inequality.

Theorem 1. *Assume that*

$$\lim_{t \to 0+} \varphi_K(t) = 0, \tag{2.1}$$

and consider any strictly monotone increasing function $f_K(t)$ for $t \ge 0$ such that $f_K(0) = 0$ and $\varphi_K(t) \le f_K(t)$ $(t \ge 0)$. Then

$$M_n(K) \le \frac{36n^2}{f_K^{-1}\left(\frac{1}{5n}\right)}. \tag{2.2}$$

Proof. Let $p_n \in \Pi_n$ be an arbitrary polynomial such that

$$\sup_{x \in K} |p_n(x)| \le 1. \tag{2.3}$$

(2.1) implies $0 \in K$, thus without loss of generality we may assume that $p_n(0) = 0$. Consider

$$q_n(x) = p_n(x) \left(1 - \frac{x}{h_n}\right)^n \in \Pi_{2n}, \tag{2.4}$$

where

$$h_n = f_K^{-1}\left(\frac{1}{5n}\right), \quad \text{i.e.} \quad \varphi_K(t) \le f_K(t) \le \frac{1}{5n} \quad \text{if} \quad t \le h_n. \tag{2.5}$$

This is a so-called incomplete polynomial at h_n of type 1/2. A fundamental result concerning incomplete polynomials says that

$$A_n = \max_{0\le x\le h_n} |q_n(x)| = \max_{0\le x\le 3h_n/4} |q_n(x)|$$

(cf. M. von Golitschek, G. G. Lorentz and Y. Makovoz [4], Hc. 3, Corollary 1.4). Suppose A_n is attained at $\xi_n \in (0, 3h_n/4]$. Then by the definition of the density function there exists $x_n \in K$, $0 \le x_n < \xi_n$ such that $0 < \xi_n - x_n \le \xi_n \varphi_K(\xi_n)$. Using the Mean Value Theorem in the interval $[x_n, \xi_n]$, as well as the Bernstein inequality (1.2) in the interval $[0, h_n]$ we obtain by (2.3)

$$A_n - 1 \le |q_n(\xi_n) - q_n(x_n)| = (\xi_n - x_n)|q_n'(\eta_n)| \le \frac{\xi_n \varphi_K(\xi_n) 2nA_n}{\sqrt{\eta_n(h_n-\eta_n)}} \le$$

$$\le \frac{2nA_n\xi_n\varphi_K(\xi_n)}{\sqrt{x_n h_n/4}} \qquad (x_n < \eta_n < \xi_n).$$

Hence by $x_n \ge \xi_n(1-\varphi_K(\xi_n))$ and $h_n \ge 4\xi_n/3$

$$A_n - 1 \le \frac{2\sqrt{3}nA_n\varphi_K(\xi_n)}{\sqrt{1-\varphi_K(\xi_n)}}.$$

Here by (2.5) and $\xi_n \le h_n$ we have $\varphi_K(\xi_n) \le \frac{1}{5n} \le 1/5$, whence $A_n - 1 \le \frac{\sqrt{15}}{5}A_n$, i.e. $A_n \le 9/2$. Thus applying the Markov inequality (1.1) for $q_n(x)$ in the interval $[0, h_n]$ we get by (2.5)

$$|p_n'(0)| = |q_n'(0)| \le \frac{8n^2}{h_n}A_n \le \frac{36n^2}{f_K^{-1}\left(\frac{1}{5n}\right)}.$$

Now we state a Bernstein type inequality. For this purpose we define a corresponding two-sided density function of the compact set K at 0 as

$$\Phi_K(t) = \frac{1}{t}\max\{b-a : (a,b) \subset [-t,t], (a,b)\cap K = \emptyset\} \qquad (t>0).$$

THEOREM 2. *Assume that* $\lim_{t\to 0+}\Phi_K(t) = 0$, *and consider any strictly monotone increasing function* $F_K(t)$ *for* $t \ge 0$ *such that* $F_K(0) = 0$ *and* $\Phi_K(t) \le F_K(t)$ $(t \ge 0)$. *Then*⋆

(2.6)
$$M_n(K) = O\left(\frac{n}{F_K^{-1}\left(\frac{1}{7n}\right)}\right).$$

⋆ Here and in what follows the O always refers to $n \to \infty$. The implied constants are independent of n but may depend on other parameters.

The proof is very similar to that of Theorem 1, so we only sketch it. We take a polynomial $p_n \in \Pi_n$ such that (2.3) holds, and instead of (2.4) we define

$$q_n(x) = p_n(x)\left(1 - \frac{x^2}{h_n^2}\right)^n \in \Pi_{3n},$$

where

$$h_n = F_K^{-1}\left(\frac{1}{7n}\right).$$

Now this is an incomplete polynomial at $\pm h_n$ of type $1/3$ and therefore

$$B_n = \max_{|x|\le h_n} |q_n(x)| = \max_{|x|\le 7h_n/9} |q_n(x)|.$$

Again, we can prove that $B_n = O(1)$, and applying Bernstein's inequality (1.2) for q_n at 0 on the interval $[-h_n, h_n]$ we get the statement of the theorem.

A natural question is the sharpness of the above estimates. Unfortunately, in this generality we are unable to answer this question. Nevertheless in the special case when K is a monotone sequence of points we shall provide some lower bounds showing that (2.2) and (2.6) are sharp, in general.

Theorem 3. *If $K = \{x_k\}_{k=1}^{\infty} \cup \{0\}$ with $x_k \downarrow 0$ as $k \to \infty$, then we have*

$$M_n(K) \ge \frac{1}{2}\max\left(\frac{n^2}{x_{[n/2]}}, \frac{1}{x_n}\right). \tag{2.7}$$

Remark. This result shows that the Markov factor can be arbitrarily large depending on the sequence K.

Proof. Let $m = [n/2] + 1$ and

$$p_n(x) = T_m(x)\prod_{k=1}^{m-2}\left(1 - \frac{x}{x_k}\right) \in \Pi_n,$$

where $T_m(x)$ is the transformed Chebyshev polynomial of degree m on the interval $[0, x_{[n/2]}]$ normalized so that

$$T_m(0) = 0 \qquad \text{and} \qquad 0 \le T_m(x) \le 1 \quad (0 \le x \le x_{[n/2]}).$$

Then evidently $\sup_{x\in K} |p_n(x)| \le 1$, and

$$p_n'(0) = T_m'(0) = \frac{2([n/2]+1)^2}{x_{[n/2]}} \ge \frac{n^2}{2x_{[n/2]}}.$$

This proves the first statement in (2.7). The second statement easily follows by considering the polynomial $p_n(x) = \prod_{k=1}^{n}\left(1 - \frac{x}{x_k}\right) \in \Pi_n$.

The analogue of the above result for oscillating sequences is the following:

THEOREM 4. *Let* $K = \{-x'_k\}_{k=1}^{\infty} \cup \{x_k\}_{k=1}^{\infty} \cup \{0\}$ *where* $x'_k, x_k \downarrow 0$ *as* $k \to \infty$, *and set* $y_n = \max\{x'_n, x_n\}$. *Then we have*

$$M_n(K) \geq \frac{1}{6} \max\left(\frac{n}{y_{[n/3]}}, \frac{1}{y_{[n/2]}}\right). \tag{2.8}$$

PROOF. Let

$$p_n(x) = T_m(x) \prod_{k=1}^{[n/3]-1} \left(1 - \frac{x}{x_k}\right)\left(1 + \frac{x}{x'_k}\right) \in \Pi_n,$$

where $m = 2[n/6] + 1$ is odd, and $T_m(x)$ is the transformed Chebyshev polynomial of degree $m < n/3$ on the interval $[-y_{[n/3]}, y_{[n/3]}]$ normalized to have uniform norm 1 on this interval and $T_m(0) = 0$. This polynomial shows the first inequality in (2.8). The second can be seen by using the polynomial

$$p_n(x) = \frac{x}{y_{[n/2]}} \prod_{k=1}^{[n/2]-1} \left(1 - \frac{x}{x_k}\right)\left(1 + \frac{x}{x'_k}\right) \in \Pi_n.$$

Now we state a sharp result as a corollary of the above estimates. In what follows $a_n \sim b_n$ means that there exists a constant $c > 0$ independent of n such that $c \leq a_n/b_n \leq 1/c$. Also, for $x_k \downarrow 0$ as $k \to \infty$ set $\Delta x_k = x_{k-1} - x_k$ $(k \geq 2)$.

COROLLARY 1. *If* $K = \{x_k\}_{k=1}^{\infty} \cup \{0\}$ *with* $x_k \downarrow 0$ *as* $k \to \infty$ *is such that*

$$M = \sup_{k \geq 2} \frac{k \Delta x_k}{x_k} < \infty \tag{2.9}$$

holds, then

$$M_n(K) \sim \frac{n^2}{x_n}.$$

PROOF. The lower estimate follows from the first part of Theorem 3. To see this we have to prove that under the condition (2.9), $u \sim v$ implies $x_u \sim x_v$. Namely if e. g. $u > v$ then

$$x_u \geq \frac{x_{u-1}}{1 + \frac{M}{u}} \geq \frac{x_{u-2}}{\left(1 + \frac{M}{u}\right)\left(1 + \frac{M}{u-1}\right)} \geq \ldots \geq \frac{x_v}{\left(1 + \frac{M}{v}\right)^{u-v}} \geq e^{-M\left(\frac{u}{v}-1\right)} x_v. \tag{2.10}$$

In order to prove the upper estimate we will apply Theorem 1. For this purpose define the strictly monotone increasing function $f_K(t)$ by

$$f_K(x_k) = \frac{M}{k} \qquad (k = 1, 2, \ldots), \tag{2.11}$$

and let $f_K(t)$ be linear in each interval $[x_{k+1}, x_k]$ $(k = 1, 2 \ldots)$. Let $t > 0$ be arbitrary, $t \in [x_n, x_{n-1}]$, say. Then by the definition of the density function φ_K there exists a $k \geq n$ such that

$$\varphi_K(t) \leq \frac{\Delta x_k}{t} \leq \frac{\Delta x_k}{x_k}.$$

Hence and by (2.9) and (2.11) we get

$$\varphi_K(t) \leq \frac{M}{k} \leq \frac{M}{n} = f_K(x_n) \leq f_K(t),$$

since f_K is monotone increasing. Using again (2.11)

$$f_K(x_{5([M]+1)n}) = \frac{M}{5([M]+1)n} < \frac{1}{5n},$$

i.e. $x_{5([M]+1)n} < f_K^{-1}\left(\frac{1}{5n}\right)$. Thus (2.2) and (2.10) yield

$$M_n(K) \leq \frac{36n^2}{x_{5([M]+1)n}} \leq \frac{cn^2}{x_n}.$$

Similarly, for oscillating sequences we can prove the following (using Theorems 2 and 4):

Corollary 2. *If $K = \{-x_k\}_{k=1}^{\infty} \cup \{x_k\}_{k=1}^{\infty} \cup \{0\}$ where $x_k \downarrow 0$ as $k \to \infty$ and* (2.9) *holds, then*

$$M_n(K) \sim \frac{n}{x_n}.$$

We now present some examples.

Example 1. We have

$$M_n(K) \sim n^2 \log^{\alpha} n \quad \text{if} \quad K = \{\log^{-\alpha} k\}_{k=1}^{\infty} \cup \{0\} \qquad (\alpha > 0).$$

Namely, in this case (2.9) holds and Corollary 1 applies. Similarly, Corollary 2 implies

$$M_n(K) \sim n \log^{\alpha} n \quad \text{if} \quad K = \{-\log^{-\alpha} k\}_{k=1}^{\infty} \cup \{\log^{-\alpha} k\}_{k=1}^{\infty} \cup \{0\} \qquad (\alpha > 0).$$

Example 2. We have

$$M_n(K) \sim n^{2+\alpha} \quad \text{if} \quad K = \{k^{-\alpha}\}_{k=1}^{\infty} \cup \{0\} \qquad (\alpha > 0).$$

Again, this follows from Corollary 1. Similarly, Corollary 2 implies

$$M_n(K) \sim n^{1+\alpha} \quad \text{for} \quad K = \{-k^{-\alpha}\}_{k=1}^{\infty} \cup \{k^{\alpha}\}_{k=1}^{\infty} \cup \{0\} \qquad (\alpha > 0).$$

EXAMPLE 3. For $K = \{e^{-\log^{\alpha} k}\}_{k=1}^{\infty} \cup \{0\}$ we have

$$e^{\log^{\alpha} n} \leq M_n(K) \leq e^{(1+o(1))\log^{\alpha} n} \qquad (\alpha > 1).$$

Here the lower estimate is a consequence of the second part of Theorem 3. To see the upper estimate, we use a Taylor expansion for $\log^{\alpha} x$ and e^x to obtain

$$\varphi_K(t) \leq \frac{\Delta x_k}{x_k} = e^{\log^{\alpha} k - \log^{\alpha}(k-1)} - 1 = O\left(\frac{\log^{\alpha-1} k}{k}\right) = O\left(\left(\log \frac{1}{t}\right)^{1-1/\alpha} e^{-\log^{1/\alpha}(1/t)}\right)$$

for some k with $e^{-\log^{\alpha} k} \leq t$. Thus we can choose $f_K^{-1}(t) \sim e^{-(1+o(1))\log^{\alpha}(1/t)}$, and Theorem 1 yields the desired result.

Now we consider a case when K is a monotone sequence of intervals.

EXAMPLE 4. If

$$K_{\alpha,\beta} = \bigcup_{k=1}^{\infty} \left[\frac{1}{(k+1)^{\alpha}} + \frac{\alpha}{4(k+1)^{\beta}}, \frac{1}{k^{\alpha}}\right] \bigcup \{0\} \qquad (1 < \alpha + 1 < \beta), \tag{2.12}$$

then

$$M_n(K_{\alpha,\beta}) = O\left(n^{2+\frac{\alpha}{\beta-\alpha}}\right).$$

Namely, we obtain for $k^{-\alpha} \leq t < (k-1)^{-\alpha}$ that

$$\varphi_K(t) = \frac{\alpha}{4(k+1)^{\beta} t} \leq \frac{\alpha}{4} t^{\beta/\alpha - 1},$$

and thus Theorem 1 with $f_K^{-1}(t) \sim t^{\frac{\alpha}{\beta-\alpha}}$ yields the result. Later we will see that this estimate can be improved for some α, β.

3. An interpolatory method

When condition (2.1) is not satisfied then we cannot apply Theorem 1. We now present a result applicable for a wider family of sets.

THEOREM 5. *Assume that* $K \supset \{x_k\}_{k=1}^{\infty}$ *with* $x_k \downarrow 0$ *as* $k \to \infty$*, and let*

$$d_n = \max_{2\le k\le n} \frac{x_k}{\Delta x_k} \qquad (n = 2, 3, \ldots). \tag{3.1}$$

Then

$$M_n(K) \le \frac{(cd_n)^{d_n}}{x_n}$$

with an absolute constant $c > 0$*.*

Combining this with (2.7) yields the next

COROLLARY 3. *If* $K = \{x_k\}_{k=1}^{\infty} \cup \{0\}$ *with* $x_k \downarrow 0$ *as* $k \to \infty$ *and* $\sup_{k\ge 2} \frac{x_k}{\Delta x_k} < \infty$*, then*

$$M_n(K) \sim \frac{1}{x_n}.$$

PROOF OF THEOREM 5. Again, let $p_n \in \Pi_n$ be an arbitrary polynomial such that (2.3) holds. Evidently, we may assume that $p_n(0) = 0$. Then using Lagrange interpolation on the nodes $0, x_1, \ldots, x_n$ we obtain

$$p_n(x) = x \sum_{j=1}^{n} \frac{p_n(x_j)}{x_j} \prod_{\substack{k=1\\k\ne j}}^{n} \frac{x - x_k}{x_j - x_k}, \tag{3.2}$$

whence

$$|p_n'(0)| = \left| \sum_{j=1}^{n} \frac{p_n(x_j)}{x_j} \prod_{\substack{k=1\\k\ne j}}^{n} \frac{x_k}{x_j - x_k} \right| \le \sum_{j=1}^{n} \frac{1}{x_j} \prod_{\substack{k=1\\k\ne j}}^{n} \frac{x_k}{|x_j - x_k|} \le \frac{1}{x_n} \sum_{j=1}^{n} \prod_{\substack{k=1\\k\ne j}}^{n} \frac{x_k}{|x_j - x_k|}. \tag{3.3}$$

Using the definition (3.1) of d_n we get

$$\frac{x_j}{x_k} = \prod_{s=k+1}^{j} \left(\frac{x_{s-1}}{x_s}\right)^{-1} = \prod_{s=k+1}^{j} \left(1 + \frac{\Delta x_s}{x_s}\right)^{-1} \le \left(1 + \frac{1}{d_n}\right)^{k-j} \qquad (k < j),$$

which coupled with the inequalities

$$(1+a)^l \ge 1 + la + \frac{l(l-1)}{2} a^2 \quad (a > 0,\ l \ge 1 \text{integer}) \qquad \text{and} \qquad 1 + u < e^u,$$

yields

$$\prod_{k<j}\frac{x_k}{x_k-x_j}=\prod_{k<j}\frac{1}{1-\frac{x_j}{x_k}}\le\prod_{k<j}\frac{1}{1-\left(1+\frac{1}{d_n}\right)^{k-j}}=\prod_{k<j}\frac{\left(1+\frac{1}{d_n}\right)^{j-k}}{\left(1+\frac{1}{d_n}\right)^{j-k}-1}<$$

$$<\prod_{k<j}\frac{1+\frac{j-k}{d_n}+\frac{(j-k)(j-k-1)}{2d_n^2}}{\frac{j-k}{d_n}+\frac{(j-k)(j-k-1)}{2d_n^2}}=\prod_{k<j}\left(1+\frac{d_n}{(j-k)\left(1+\frac{j-k-1}{2d_n}\right)}\right)\le$$

$$\le e^{d_n\sum_{k<j}\frac{1}{(j-k)\left(1+\frac{j-k-1}{2d_n}\right)}}.$$

Here, in case $j\le[d_n]+2$,

$$\sum_{k<j}\frac{1}{(j-k)\left(1+\frac{j-k-1}{2d_n}\right)}\le\sum_{k=1}^{j-1}\frac{1}{j-k}<\log j\le\log(d_n+2),$$

while in case $j\ge[d_n]+3$,

$$\sum_{k<j}\frac{1}{(j-k)\left(1+\frac{j-k-1}{2d_n}\right)}\le 2d_n\sum_{k=1}^{j-[d_n]-2}\frac{1}{(j-k-1)^2}+\sum_{k=j-[d_n]-1}^{j-1}\frac{1}{j-k}\le$$

$$\le\frac{2d_n}{[d_n]+1}+\log([d_n]+1)\le 2+\log(d_n+1).$$

Thus

$$\prod_{k<j}\frac{x_k}{x_k-x_j}\le e^{2d_n+d_n\log(d_n+2)}.$$

Similarly, for $k>j$

$$\frac{x_j}{x_k}=\prod_{s=j+1}^{k}\frac{x_{s-1}}{x_s}=\prod_{s=j+1}^{k}\left(1+\frac{\Delta x_s}{x_s}\right)\ge\left(1+\frac{1}{d_n}\right)^{k-j}\ge 1+\frac{k-j}{d_n},$$

and hence

$$\prod_{k>j}\frac{x_k}{x_j-x_k}=\prod_{k>j}\frac{1}{\frac{x_j}{x_k}-1}\le d_n^{n-j}\prod_{k>j}\frac{1}{k-j}=\frac{d_n^{n-j}}{(n-j)!}.$$

Substituting these estimates into (3.3) we get

$$|p_n'(0)|\le\frac{e^{2d_n+d_n\log(d_n+2)}}{x_n}\sum_{j=1}^{n}\frac{d_n^{n-j}}{(n-j)!}<\frac{e^{3d_n+d_n\log(d_n+2)}}{x_n}.$$

Theorem 6. *If* $K \supset (\{-x_k\}_{k=1}^{\infty} \cup \{x_k\}_{k=1}^{\infty})$ *with* $x_k \downarrow 0$ *as* $k \to \infty$*, then*

$$M_n(K) \le \frac{(cd_{[n/2]})^{d_{[n/2]}}}{x_{[n/2]}}$$

with an absolute constant $c > 0$.

For the proof one has to interpolate $p_n \in \Pi_{2n}$ at the nodes $0, \pm x_1, \dots, \pm x_n$.

Example 5. For $K = \{e^{-k^{\alpha}}\}_{k=1}^{\infty} \cup \{0\}$ we have

$$e^{n^{\alpha}} \le M_n(K) \le \begin{cases} e^{cn^{\frac{\alpha}{1-\alpha}}} & \text{if } 0 < \alpha \le \frac{3-\sqrt{5}}{2}, \\ n^{cn^{1-\alpha}} e^{n^{\alpha}} & \text{if } \frac{3-\sqrt{5}}{2} < \alpha < 1, \\ ce^{n^{\alpha}} & \text{if } \alpha \ge 1, \end{cases}$$

where $c > 0$ depends only on α. Here the lower estimate follows from the second inequality in Theorem 3. The first upper estimate follows from Theorem 1, since by (2.11) we have $\varphi_K(t) = O(k^{\alpha-1})$ for some k with $e^{-k^{\alpha}} \le t$, i.e. we can choose $f_K(t) = c \log^{1-1/\alpha} \frac{1}{t}$ with a suitable constant $c > 0$. The second and third upper estimates follow from Theorem 5, since by (3.1)

$$d_n \sim \begin{cases} n^{1-\alpha} & \text{if } 0 < \alpha < 1, \\ 1 & \text{if } \alpha \ge 1. \end{cases}$$

Note that in case $\alpha \ge 1$ the lower and upper estimates are of the same magnitude.

4. Another density function

While the previous results settle the problem fairly well when K is "thin", for denser sets (like an infinite sequence of intervals) they are less satisfactory. In particular, it should be noted that applying Theorems 1 or 2 we can never achieve the bounds $M_n(K) = O(n^2)$ and $M_n(K) = O(n)$, respectively. Therefore we define another right handed density function which will yield better estimates for some sets. Let

$$\psi_K(t) = \sup_{h \le t} \frac{m([0,h] \setminus K)}{h} \qquad (t > 0),$$

where m is the Lebesgue measure. Again we assume that this is a positive function for $t > 0$. In connection with the density function defined in Section 2 we note that $\varphi_K(t) = O(\psi_K(t))$.

We will make use of the following theorem of Remez (cf. e.g. [4], Theorem 7.1 and inequality (7.6) in Ch. 2): if I is an interval and $H \subset I$, $m(H) \le \frac{1}{2} m(I)$ is a measurable set then

$$\max_{x \in I} |p(x)| \le e^{\lambda n \sqrt{m(H)/m(I)}} \max_{x \in I \setminus H} |p(x)| \qquad (p \in \Pi_n),$$

where $\lambda > 0$ is an absolute constant. This λ will appear in the formulation of the next theorem.

THEOREM 7. *If K is such that*

$$\lim_{t\to 0+} \psi_K(t) = 0,$$

then

$$M_n(K) = O(n^2)\left(\int_{\psi_K^{-1}\left(\frac{1}{\lambda^2 n^2}\right)}^{1} \frac{\sqrt{\psi_K(t)}}{t^{3/2}}\,dt\right)^2. \tag{4.1}$$

PROOF. Let the polynomial $p_n \in \Pi_n$ satisfy (2.3). Without loss of generality we may assume that $p_n(0) = 0$. By the definition of the density function and the above mentioned theorem of Remez applied to $I = [0, x]$ and $H = I \setminus K$ we get

$$|p_n(x)| \le e^{\lambda n\sqrt{\psi_K(x)}} \max_{y\in K\cap I} |p_n(y)| \le e^{\lambda n\sqrt{\psi_K(x)}} \qquad (0 < x \le \psi_K^{-1}(1/2)). \tag{4.2}$$

Now we apply a fundamental result concerning fast decreasing polynomials (see e.g. Totik [11], p. 79): If $\Omega(x)$ is an even, right continuous and increasing function on [0,1] such that $\Omega(0) \le 0$, then there exist polynomials $q_m \in \Pi_m$ such that

$$q_m(0) = 1, \qquad |q_m(x)| \le e^{-\Omega(\sqrt{x})} \qquad (0 \le x \le 1), \tag{4.3}$$

and

$$m \le 12 \sup_{\Omega^{-1}(0)\le y<\min(\Omega^{-1}(1),1/2)} \sqrt{\frac{\Omega(y)}{y^2}} \tag{4.4}$$

$$+6\int_{\min(\Omega^{-1}(1),1/2)}^{1/2} \frac{\Omega(y)}{y^2}\,dy + 6 \sup_{1/2\le y<1} \frac{\Omega(y)}{-\log(1-y)} + 6.$$

(This is a slightly transformed form of the original statement which gives bounds for the polynomial on $[-1, 1]$. Using symmetry and substituting x for x^2 yields the above formulation.)

We use this result with

$$\Omega(x) = \lambda n\sqrt{\psi_K(x^2)} - 1.$$

In this setting $\min(\Omega^{-1}(1), 1/2) = \sqrt{\psi_K^{-1}\left(\frac{4}{\lambda^2 n^2}\right)}$ for sufficiently large n's, and (4.2)-(4.4) yield for the polynomial $r(x) = p_n(x)q_m(x) \in \Pi_{n+m}$ that $|r(x)| \le e$ for all $x \in$

$[0, \psi_K^{-1}(1/2)]$, and using $\Omega^{-1}(0) = \sqrt{\psi_K^{-1}\left(\frac{1}{\lambda^2 n^2}\right)}$, as well as a substitution $t = x^2$ in the integral we get

$$m \le \frac{12}{\sqrt{\psi_K^{-1}\left(\frac{1}{\lambda^2 n^2}\right)}} + 3\lambda n \int_{\psi_K^{-1}\left(\frac{4}{\lambda^2 n^2}\right)}^{1/4} \frac{\sqrt{\psi_K(t)}}{t^{3/2}}\, dt + \frac{6\lambda n}{\log 2}\psi_K(1) + 6 =$$

$$= O(n) \int_{\psi_K^{-1}\left(\frac{1}{\lambda^2 n^2}\right)}^{1} \frac{\sqrt{\psi_K(t)}}{t^{3/2}}\, dt$$

for sufficiently large n's. Finally, applying Markov's inequality (1.1) to $r(x)$ at 0 on the interval $[0, \psi_K^{-1}(1/2)]$ we get $|p_n'(0)| = |r'(0)| = O(m^2)$, which is equivalent to the statement of the theorem.

Similarly, introducing a corresponding symmetric density function

$$\Psi_K(t) = \sup_{0<h\le t} \frac{m([-h,h] \setminus K)}{h} \qquad (t > 0)$$

and applying a stronger version of Remez's inequality (which holds for *inner* subsets) yields the next

THEOREM 8. *If K is such that $\lim_{t\to 0+} \Psi_K(t) = 0$, then*

$$M_n(K) = O(n) \int_{\frac{1}{2}\Psi_K^{-1}\left(\frac{1}{cn}\right)}^{1} \frac{\Psi_K(t)}{t^2}\, dt$$

with some absolute constant $c > 0$.

COROLLARY 4. *If K is such that*

$$\int_0^1 \frac{\sqrt{\psi_K(t)}}{t^{3/2}}\, dt < \infty \qquad \text{or} \qquad \int_0^1 \frac{\Psi_K(t)}{t^2}\, dt < \infty,$$

then

$$M_n(K) = O(n^2) \qquad \text{or} \qquad M_n(K) = O(n),$$

respectively.

PROOF OF THEOREM 8. It is known (cf. Erdélyi [3]) that if

$$m(x \in [-1,1] : |p_n(x)| > 1) \le \varepsilon \qquad (0 < \varepsilon \le 1),$$

then

$$\max_{|x|\le 1/2} |p_n(x)| \le e^{c_1 n\varepsilon}$$

with some absolute constant $c_1 > 0$. Hence similarly to (4.2) we have

$$|p_n(x)| \le e^{c_2 n \Psi_K(2x)} \qquad (|x| \le 1/2).$$

Now the proof can be completed by using the fast decreasing polynomials (4.3)-(4.4) and Bernstein's inequality on $[-1,1]$.

Example 6. For the set $K_{\alpha,\beta}$ defined in (2.12) of Example 4, Theorem 7 yields

$$M_n(K_{\alpha,\beta}) = \begin{cases} O\left(n^{\frac{2\alpha}{\beta-\alpha-1}}\right) & \text{if } 1<\alpha+1<\beta<2\alpha+1, \\ O(n^2\log^2 n) & \text{if } 1<2\alpha+1=\beta, \\ O(n^2) & \text{if } 1<2\alpha+1<\beta. \end{cases} \tag{4.5}$$

Namely, in this case $\psi_{K_{\alpha,\beta}}(t) = O\left(t^{\frac{\beta-\alpha-1}{\alpha}}\right)$, and evaluating the integral in (4.1) we obtain the statement.

The estimate (4.5) is weaker than (2.10) if β is close to $\alpha+1$. However, for some values of α,β (4.5) is better (we omit the exact calculations here). In fact, $M_n(K_{\alpha,\beta}) = O(n^2)$ is the optimal value (i.e. in this case $M_n(K_{\alpha,\beta}) \sim n^2$).

Next we show that, under additional restriction on α,β, this optimal estimate of the Markov factor holds not only for the point 0 but for the whole set $K_{\alpha,\beta}$:

Proposition 1. *If $\beta \ge 5(\alpha+1)$ then*

$$\sup_{x\in K_{\alpha,\beta}} |p'(x)| \le c_1 n^2 \sup_{x\in K_{\alpha,\beta}} |p(x)|$$

for every $p \in \Pi_n$ ($c_1 = c_1(\alpha,\beta)$).

Proof. We need the following inequality proved in [1]:

Lemma. *Let $0 < a \le 1$, and let A be a closed subset of $[0,1]$ with Lebesgue measure $m(A) \ge 1-a$. Then there is an absolute constant $c_2 > 0$ such that*

$$\max_{x\in I} |p'(x)| \le c_2 n^2 \max_{x\in A} |p(x)|$$

for every $p\in\Pi_n$ and for every subinterval I of A with length at least a.

Let

$$I_k = \left[\frac{1}{(k+1)^\alpha} + \frac{\alpha}{4(k+1)^\beta}, \frac{1}{k^\alpha}\right] \quad \text{and} \quad J_k = \left[\frac{1}{(k+1)^\alpha}, \frac{1}{(k+1)^\alpha} + \frac{\alpha}{4(k+1)^\beta}\right]$$

$$(k = 1,2,\ldots).$$

Let $p\in\Pi_n$ satisfy

$$\sup_{x\in K_{\alpha,\beta}} |p(x)| \le 1. \tag{4.6}$$

Let $y \in K_{\alpha,\beta}$, that is $y \in I_k$ for some $k = 1, 2, \ldots$. By Chebyshev's Inequality (see e.g. [1], p. 235) and $m(J_j) = \frac{\alpha}{4}(j+1)^{-\beta}$ we obtain for $j = 1, 2, \ldots$

$$\max_{x \in J_j} |p(x)| \le e^{3n\sqrt{\frac{2m(J_j)}{m(I_j)}}} \max_{x \in I_j} |p(x)| \le e^{c_4 n j^{-\frac{\beta-\alpha-1}{2}}}. \tag{4.7}$$

Now let

$$g(x) = (1-(x-y)^2)^{cn} p(x) \in \Pi_{(1+2c)n}$$

where the positive integer c will be chosen later. Let

$$B_k = \left[0, \frac{1}{(k+2)^\alpha}\right] \cup \left[\frac{1}{(k-1)^\alpha}, 1\right] \cup I_{k-1} \cup I_k \cup I_{k+1} \qquad (k = 1, 2, \ldots) \tag{4.8}$$

where $I_0 = \emptyset$. We show that with a large enough c we have

$$|g(x)| \le 1 \qquad (x \in B_k). \tag{4.9}$$

Since $[0,1] \setminus B_k = J_{k+1} \cup J_k \cup J_{k-1}$, it suffices to consider the following three cases.

Case 1: $x \in K_{\alpha,\beta}$. Then (4.9) follows from (4.6) and the trivial inequality

$$0 < (1-(x-y)^2)^{cn} \le 1 \qquad (x \in B_k \subset [0,1]).$$

Case 2: $x \in J_j$ for some $j = k+2, k+3, \ldots$. Then

$$|x-y| > m(I_{k+1}) \ge c_3 k^{-\alpha-1}.$$

Combining this with (4.7), with c large enough we obtain by $\beta \ge 5(\alpha+1)$ that

$$|g(x)| \le e^{-cn(x-y)^2} e^{c_4 n j^{-\frac{\beta-\alpha-1}{2}}} \le e^{-cc_3 n k^{-2\alpha-2} + c_4 n j^{-2\alpha-2}} \le e^{(c_4 - cc_3)k^{-2\alpha-2}} \le 1.$$

Case 3: $x \in J_j$ for some $j = 1, 2, \ldots, k-2$. Then

$$|x-y| > m(I_{j+1}) \ge c_3 j^{-\alpha-1}.$$

Combining this with (4.7), for a large enough c we obtain by $\beta \ge 5(\alpha+1)$ again that

$$|g(x)| \le e^{-cn(x-y)^2} e^{c_4 n j^{-\frac{\beta-\alpha-1}{2}}} \le e^{(c_4 - cc_3) j^{-2\alpha-2} n} \le 1.$$

Now we can apply the Lemma with

$$A = B_k, \qquad \text{and} \qquad a = m(I_k) \ge \frac{c_3}{k^{\alpha+1}},$$

and conclude by (4.9) that

$$|p'(y)| = |g'(y)| \le \max_{x\in I_k} |g'(x)| \le c_2((1+2c)n)^2 \max_{x\in B_k} |g(x)| \le c_1 n^2,$$

and the proposition is proved.

Except when $M_n(K) \sim n^2$, (4.5) is only an upper estimate; we do not know about the sharpness of this result. In general, we can ask: if K is a sequence of intervals on one side of 0, what is the range of increase of $M_n(K)$? Obviously in this case always

$$\liminf_{n\to\infty} \frac{M_n(K)}{n^2} > 0,$$

and we just have seen that this lower bound indeed can be attained. (In contrast to the case when K is a sequence of points, where the above liminf is always infinity; cf. Theorem 3.) On the other hand, if again K is a sequence of intervals, then evidently

$$\limsup_{n\to\infty} M_n(K)^{1/n} < \infty.$$

We now show that this upper bound is also sharp, in general. This will be seen by constructing a sequence of *nonempty* intervals with exponentially growing Markov factors. Note that when the underlying set is of measure 0, examples of this nature were known before (see, e.g. Privalov [8], Theorem 2.3). But our objective is to construct a "fat" set K (i.e. the closure of its interior coincides with K). Such fat sets can be applied for constructing counter-examples in the study of multivariate Markov factors (see Kroó–Szabados [6]).

Proposition 2. *For any $d > 1$ there exists a sequence of intervals $K \subset [0,1]$ such that 0 is in the closure of K and $\liminf_{n\to\infty} M_n(K)^{1/n} \ge d$.*

Proof. Let $d > 1$ be arbitrary, and

$$K = \bigcup_{k=1}^{\infty} [d^{-2^k} - d^{-14^k}, d^{-2^k}],$$

and

$$p_n(x) = \prod_{j=1}^{s} (1 - d^{2^j} x)^{m_{jn}} \in \Pi_n,$$

where

$$m_{jn} = \left[\frac{2n}{4^j}\right] + 1 \qquad (j = 1, \dots, s), \tag{4.10}$$

with $s = [\log n / \log 2] + 1$. First we show that this polynomial is bounded in the intervals $X_k = [d^{-2^k} - d^{-14^k}, d^{-2^k}]$ $(k = 1, 2, \dots)$. Evidently, it is sufficient to show this for $1 \le k \le s-1$, since in $[0, d^{-2^s})$ the polynomial is positive, monotone increasing and $p_n(0) = 1$.

So let $x \in X_k$, $1 \le k \le s-1$. Then for $1 \le j < k$, $0 < 1 - d^{2^j} x < 1$, and for $j = k$, $0 \le 1 - d^{2^k} x \le d^{2^k - 14^k} \le d^{-12^k}$. Thus by (4.10)

$$|p_n(x)| \le d^{-12^k m_{kn}} \prod_{j=k+1}^{s} d^{2^j m_{jn}} \le d^{-12^k \frac{2n}{4^k}} \prod_{j=k+1}^{s} d^{\frac{n}{2^{j-1}} + 2^j} \le$$

$$\le d^{-2 \cdot 3^k n + \frac{n}{2^{k-1}} + 2^{s+1}} < d^{-6n+n+4n} = d^{-n} < 1.$$

Thus p_n is indeed bounded on K. On the other hand,

$$|p_n'(0)| = \sum_{j=1}^{s} m_{jn} d^{2^j} > d^{2^s} > d^n.$$

Similarly, the polynomial

$$p_n(x) = x d^{2^s} \prod_{j=1}^{s} (1 - d^{2^{j+1}} x^2)^{m_{jn}} \in \Pi_{2n+1}$$

is bounded on $K = \cup_{k=1}^{\infty} (X_k \cup -X_k)$, and the same relation for $M_n(K)$ holds.

Finally, we consider a completely different type of application of our results: a Cantor type set.

EXAMPLE 7. Set $K_0 = [0,1]$, and let K_j $(j \ge 1)$ consist of 2^j isometric intervals obtained by deleting from each interval in K_{j-1} an open interval of length $3^{-(\alpha+1)j}$ $(\alpha > 0)$. Let $K = \cap_{j=1}^{\infty} K_j$. It is shown in A. Jonsson [5] that for any connected subinterval I_j of K_j

$$m(I_j) \ge c_1 2^{-j} \quad \text{and} \quad m(I_j \setminus K) = c_2 3^{-(\alpha+1)j}.$$

Now let $h > 0$ and choose s so that $I_{s+1} \subset [0,h] \subset I_s$. Then by the above estimates

$$\frac{m([0,h] \setminus K)}{h} \le \frac{m(I_s \setminus K)}{m(I_{s+1})} \le c_3 \left(\frac{2}{3^{\alpha+1}} \right)^s,$$

where clearly $h \ge c_1 2^{-s-1}$, i.e. $s \ge \frac{1}{\log 2} \log \frac{c_2}{h}$. This easily implies that $\psi_K(t) = O(t^\gamma)$ with $\gamma = (\alpha+1) \frac{\log 3}{\log 2} - 1$. Thus we obtain by (4.1) that

$$M_n(K) = \begin{cases} O(n^2), & \text{if } \gamma < 1, \\ O(n^2 \log^2 n), & \text{if } \gamma = 1, \\ O(n^{2/\gamma}), & \text{if } \gamma > 1. \end{cases}$$

Thus, even for Cantor type sets the Markov factors can be of order $O(n^2)$. Moreover, using Theorem 8 and properly modifying K we can achieve the magnitude $O(n)$.

Finally, we mention that V. Totik constructed a set of measure 0 such that the Markov factor of the whole set is $O(n^2)$ (private communication).

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