MARKOV-TYPE INEQUALITIES ON CERTAIN IRRATIONAL ARCS AND DOMAINS

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Abstract. Let $\mathcal{P}_n^d$ denote the set of real algebraic polynomials of d variables and of total degree at most n. For a compact set $K \subset \mathbb{R}^d$ set

$$\|P\|_K = \sup_{x \in K} |P(x)| \,.$$

Then the Markov factors on K are defined by

$$M_n(K) := \max \left\{ \|D_\omega P\|_K : \;\; P \in \mathcal{P}_n^d, \;\; \|P\|_K \leq 1 \,, \;\; \omega \in S^{d-1} \right\} \,.$$

(Here, as usual, S^{d-1} stands for the Euclidean unit sphere in $\mathbb{R}^d$.) Furthermore, given a smooth curve $\Gamma \subset \mathbb{R}^d$, we denote by $D_T P$ the tangential derivative of P along Γ (T is the unit tangent to Γ). Correspondingly, consider the tangential Markov factor of Γ given by

$$M_n^T(\Gamma) := \max \left\{ \|D_T P\|_\Gamma : \;\; P \in \mathcal{P}_n^d \,, \;\; \|P\|_\Gamma \leq 1 \right\} \,.$$

Let $\Gamma_\alpha := \{(x, x^\alpha) : \;\; 0 \leq x \leq 1\}$. We prove that for every irrational number $\alpha > 0$ there are constants $A, B > 1$ depending only on α such that

$$A^n \leq M_n^T(\Gamma_\alpha) \leq B^n$$

for every sufficiently large n.

Our second result presents some new bounds for $M_n(\Omega_\alpha)$, where

$$\Omega_\alpha := \left\{ (x, y) \in \mathbb{R}^2 : \;\; 0 \leq x \leq 1 \,; \;\; \frac{1}{2} x^\alpha \leq y \leq 2x^\alpha \right\}$$

($d = 2, \alpha > 1$). We show that for every $\alpha > 1$ there exists a constant $c > 0$ depending only on α such that

$$M_n(\Omega_\alpha) \leq n^{c \log n} \,.$$

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1. Introduction

Recent years have seen an increased activity in the study of Markov-Bernstein type inequalities for the derivatives of multivariate polynomials. These inequalities provide estimates on the size of the directional derivatives $D_\omega P$ of multivariate polynomials P under some normalization. Let $\mathcal{P}_n^d$ denote the set of real algebraic polynomials of d variables and of total degree at most n. For a compact set $K \subset \mathbb{R}^d$ set

$$\|P\|_K = \sup_{x\in K} |P(x)|\,.$$

Then the Markov factors on K are defined by

$$M_n(K) := \max\left\{\|D_\omega P\|_K : \ P \in \mathcal{P}_n^d, \ \|P\|_K \le 1\,, \ \omega \in S^{d-1}\right\}\,.$$

(Here, as usual, S^{d-1} stands for the Eucledean unit sphere in $\mathbb{R}^d$.) Furthermore, given a smooth curve $\Gamma \subset \mathbb{R}^d$, we denote by $D_T P$ the tangential derivative of P along Γ (T is the unit tangent to Γ). Correspondingly, consider the tangential Markov factor of Γ given by

$$M_n^T(\Gamma) := \max\left\{\|D_T P\|_\Gamma : \ P \in \mathcal{P}_n^d\,, \ \|P\|_\Gamma \le 1\right\}\,.$$

It was shown by Bos et. al. [3] that $M_n^T(\Gamma)$ is of order n^2 when Γ is algebraic. In another paper [4] the authors show that for the curve

$$\Gamma_\alpha := \{(x, x^\alpha) : \ 0 \le x \le 1\} \subset \mathbb{R}^2$$

with a rational exponent $\alpha = p/q \ge 1$ (p and q are relative primes), $M_n^T(\Gamma_\alpha)$ is of precise order n^{2q}, while for an irrational exponent $\alpha > 1$, $M_n^T(\Gamma_\alpha)$ grows faster than any power of n. In this paper we shall generalize the latter statement by showing that $M_n^T(\Gamma_\alpha)$ is of exponential order of magnitude for irrational exponents $\alpha > 0$.

The Markov factors $M_n(K)$ of a domain $K \subset \mathbb{R}^d$ have been widely investigated when K admits a polynomial parametrization (see [2], [7], [6]) or an analytic parametrization (see [5],[8]), that is, points of K can be connected to the interior of K by polynomial or analytic curves, respectively. For instance, if

$$\Omega_\alpha := \left\{(x,y) \in \mathbb{R}^2 : \ 0 \le x \le 1\,; \ \frac{1}{2}x^\alpha \le y \le 2x^\alpha\right\}$$

($d = 2, \alpha > 1$), then it follows from Theorem 2 in [6] that for a rational exponent $\alpha = p/q$ (p and q are positive integers) we have $M_n(\Omega_\alpha) = O(n^{2p})$. The method of analytic (or polynomial) parametrization does not apply to Ω_α when $\alpha > 1$ is irrational. Using a new approach we shall show below that for irrational exponents $\alpha > 1$ we have

$$M_n(\Omega_\alpha) \le n^{c \log n}$$

with some constant $c > 1$ depending only on α. The growth of this upper bound is faster than polynomial growth (which holds for rational exponents α), but substantially smaller than exponential growth which will be shown to hold for $M_n^T(\Gamma_\alpha)$ when $\alpha > 0$ is irrational.

2. New results

Our first result shows that the magnitude of $M_n^T(\Gamma_\alpha)$ is of exponential order when $\alpha > 0$ is irrational.

Theorem 2.1. *For every irrational number $\alpha > 0$ there are constants $A, B > 1$ depending only on α such that*

$$A^n \le M_n^T(\Gamma_\alpha) \le B^n \,.$$

By using a different method the following local version of Theorem 2.1 is obtained in [9]: for every irrational number $\alpha > 0$ there are constants $A, B > 1$ depending only on α such that

$$A^n \le \max\left\{|D_T P(0,0)| : \quad P \in \mathcal{P}_n^2 \,, \quad \|P\|_{\Gamma_\alpha} \le 1\right\} \le B^n \,,$$

where $D_T P(0,0)$ is the tangential derivative of P along Γ_α at $(0,0)$. This result was then built in Theorem 2 of [9] where the dependence on α is not discussed as explicitly as it is seen from our demonstrations here.

Our second result presents some new bounds for $M_n(\Omega_\alpha)$.

Theorem 2.2. *For every $\alpha > 1$ there exists a constant $c > 0$ depending only on α such that*

$$M_n(\Omega_\alpha) \le n^{c \log n} \,.$$

The question of verifying lower bounds for $M_n(\Omega_\alpha)$ faster than polynomial order of magnitude remains open. (Applying Theorem 2 in [6] yields $M_n(\Omega_\alpha) \ge cn^{2\alpha}$.) In this respect we conjecture that for every irrational exponent $\alpha > 1$ we have

$$\limsup_{n \to \infty} \frac{\log M_n(\Omega_\alpha)}{\log n} = \infty \,,$$

that is, $M_n(\Omega_\alpha)$ increases faster than any power of n. Our next theorem shows that the above conjecture would provide a best possible lower bound, that is, a stronger lower bound cannot hold, in general.

Theorem 2.3. *Let (β_n) be an arbitrary increasing sequence of positive numbers tending to ∞. Then there exists an irrational number $\alpha > 1$ so that*

$$\liminf_{n \to \infty} M_n(\Omega_\alpha) n^{-\beta_n} < \infty \,.$$

3. Lemmas for Theorem 2.1

Our first lemma is the "Distance Formula" (see part c] of E.2 on page 177 in [1]).

Lemma 3.1. *Let μ_j, $j = 0, 1, \ldots, m$, and μ be distinct real numbers greater than $-\frac{1}{2}$. Then*

$$\min_{b_j \in \mathbb{C}} \left\| x^\mu - \sum_{j=0}^{m} b_j x^{\mu_j} \right\|_{L_2[0,1]} = \frac{1}{\sqrt{1+2\mu}} \prod_{j=0}^{m} \frac{|\mu - \mu_j|}{\mu + \mu_j + 1} \,.$$

Let $\alpha > 1$ be an irrational number. For a fixed $n \in \mathbb{N}$ let $\nu := \nu(n) = (n+1)^2 - 1$. We define the numbers $\lambda_0 < \lambda_1 < \cdots < \lambda_\nu$ by

$$\{\lambda_0, \lambda_1, \ldots, \lambda_\nu\} = \{j + k\alpha,\ \ j,k \in \{0,1,\ldots,n\}\} . \tag{3.1}$$

Note that $\lambda_0 := 0$ and $\lambda_1 := 1$. Let $M_{\nu,\alpha} := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_\nu}\}$. Associated with $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_\nu$ defined by (3.1), we define $\mu_j := \lambda_{j+1} - 1$, $j = 0,1,\ldots,\nu-1$, where $0 = \mu_0 < \mu_1 < \cdots < \mu_{\nu-1}$. We also define $M'_{\nu,\alpha} := \operatorname{span}\{x^{\mu_0}, x^{\mu_1}, \ldots, x^{\mu_{\nu-1}}\}$. Note that if $P \in M_{\nu,\alpha}$, then $P' \in M'_{\nu,\alpha}$.

Lemma 3.2. *Let $\alpha > 1$ be irrational. Then there is a constant $c_1 > 1$ depending only on α such that if $0 < \delta < c_1^{-n}$, then*

$$\|P\|_{[0,1]} \le 2\|P\|_{[\delta,1]} , \qquad P \in M'_{\nu,\alpha} .$$

To prove Lemma 3.2 we need first the following lemma.

Lemma 3.3. *Let $\alpha > 2$. Then there is an absolute constant $c > 1$ such that*

$$|P'(0)| \le \frac{\alpha+1}{\alpha-2}\, c^n \|P\|_{L_2[0,1]} , \qquad P \in M'_{\nu,\alpha} .$$

Proof. Let

$$A'_{\nu,\alpha} := \sup_{P \in M'_{\nu,\alpha}} \frac{|P'(0)|}{\|P\|_{L_2[0,1]}} .$$

Using Lemma 3.1 with $\{\mu_0, \mu_1, \ldots, \mu_m\} = \{\lambda_0, \lambda_2, \lambda_3, \ldots, \lambda_\nu\}$ and $\mu = \lambda_1 = 1$, we obtain

$$\begin{aligned}
A'_{\nu,\alpha} &= 2\sqrt{3}\prod_{j=2}^{\nu} \frac{\mu_j + 2}{\mu_j - 1} = 2\sqrt{3}\prod_{j=2}^{\nu}\left(1 + \frac{3}{\mu_j - 1}\right) = 2\sqrt{3}\prod_{j=3}^{\nu}\left(1 + \frac{3}{\lambda_j - 2}\right) \\
&= 2\sqrt{3}\prod_{j=3}^{n}\left(1 + \frac{3}{j-2}\right)\prod_{k=1}^{n}\left(1 + \frac{3}{k\alpha - 2}\right)\prod_{j=1}^{n}\prod_{k=1}^{n}\left(1 + \frac{3}{j + k\alpha - 2}\right) \\
&\le 2\sqrt{3}\,\frac{\alpha+1}{\alpha-2}\exp\left(\sum_{j=3}^{n}\frac{3}{j-2}\right)\exp\left(\sum_{k=2}^{n}\frac{3}{k\alpha-2}\right)\exp\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{3}{j+k\alpha-2}\right) \\
&\le \frac{\alpha+1}{\alpha-2}\, c^n
\end{aligned}$$

with a suitable absolute constant $c > 1$. $\square$

Proof of Lemma 3.2. First we assume that $\alpha > 2$. We will use the concept of the Chebyshev "polynomial" $T_{\nu-1}$ for a given ν-dimensional Chebyshev space, see Section 3.3 of [1], for instance. Let $T_{\nu-1} \in M'_{\nu,\alpha}$ be the Chebyshev "polynomial" for $M'_{\nu,\alpha}$ on $[\eta,1]$, where $\eta \in (0,1)$ is chosen so that $|T_{\nu-1}(0)| = 2$. So $T_{\nu-1} \in M'_{\nu,\alpha}$, $\|T_{\nu-1}\|_{[\eta,1]} = 1$, $|T_{\nu-1}(1)| = 1$, and $T_{\nu-1}$ equioscillates between -1 and 1 on $[\eta,1]$ the maximum number of times, that is, ν times. Note that $1, x \in M'_{\nu,\alpha}$. By Lemma 3.3 we have

$$|T'_{\nu-1}(0)| \leq \frac{\alpha+1}{\alpha-2}\, c^n$$

with a suitable absolute constant $c > 1$. Observe that $1, x \in M'_{\nu,\alpha}$ and the fact that $T_{\nu-1}$ equioscillates on $[\eta,1]$ $n+1$ times imply that $T''_{\nu-1}$ does not vanish on $[0,\eta]$, hence $|T'_{\nu-1}|$ is decreasing on $[0,\eta]$. Therefore

$$1 = |T_{\nu-1}(0) - T_{\nu-1}(\eta)| = \eta |T'_{\nu-1}(x))| \leq \eta |T'_{\nu-1}(0)| \leq \eta \frac{\alpha+1}{\alpha-2}\, c^n\,, \qquad x \in [0,\delta]\,. \tag{3.2}$$

Now using the fact that the Chebyshev polynomial $T_{\nu-1} \in M'_{\nu,\alpha}$ on $[\eta,1]$ has the property

$$2 \geq |T_{\nu-1}(y)| = \frac{|T_{\nu-1}(y)|}{\|T_{\nu-1}\|_{[\eta,1]}} = \max_{P \in M'_{n,\alpha}} \frac{|P(y)|}{\|P\|_{[\eta,1]}}$$

for every fixed $y \in [0,\eta)$, we can deduce from (3.2) that

$$\|P\|_{[0,1]} \leq 2\|P\|_{[\eta,1]}$$

for every $P \in M'_{\nu,\alpha}$, where

$$\eta \geq \frac{\alpha-2}{\alpha+1}\, c^{-n}\,.$$

This finishes the case when $\alpha > 2$.

We show now that the theorem remains valid for all $\alpha > 1$. To see this we can use the "Comparison Theorem" formulated by part g] of E.4 on page 120-121 in [1]. Observe that if $\alpha > 1$, then

$$j + k(\alpha+1) - 1 \leq \frac{\alpha}{\alpha-1}(j + k\alpha - 1)$$

holds for all nonnegative integers j and k. Now let η be chosen for $\alpha+1 > 2$ as in the first part of the proof. Then

$$\eta^* := \eta^{\alpha/(\alpha-1)}$$

is a suitable choice for $\alpha > 1$. □

Lemma 3.4. *Let $\alpha > 1$ be irrational. Then there is a constant $c > 1$ depending only on α such that*

$$\|P'\|_{[0,1]} \leq c^n \|P\|_{[0,1]}$$

for every $P \in M_{\nu,\alpha}$.

Proof. We need to prove that

$$|P'(y)| \leq c_2^n \|P\|_{[0,1]} \tag{3.3}$$

for every $P \in M_{\nu,\alpha}$ and for every $y \in (0,1]$, where $c_2 > 1$ is a constant depending only on α. By Newman's inequality (see Theorem 6.1.1 on page 276 in [1]), we have

$$\begin{aligned} |P'(y)| &\leq \frac{9}{y} \left(\sum_{j=0}^{\nu} \lambda_j \right) \|P\|_{[0,1]} \leq 9(n+1)^2 n(1+\alpha) c_1^n \|P\|_{[0,1]} \\ &\leq c_2^n \max_{x \in [0,1]} |P(x)| \, . \end{aligned}$$

for every $P \in M_{\nu,\alpha}$ and $y \in [c_1^{-n}, 1]$, where c_1 is a constant coming from Lemma 3.2, and $c_2 > 1$ is a suitable constant depending only on α. Since (3.3) is proved for every $y \in [c_1^{-n}, 1]$, we can apply Lemma 3.2 to see that (3.3) is true for all $y \in [0,1]$ with c_2^n replaced by $2c_2^n$. $\square$

Lemma 3.5. *Let $\alpha > 1$ be irrational. Then there is an absolute constant $c > 0$ so that for some $P \in M_{\nu,\alpha}$ with $\|P\|_{[0,1]} = 1$ we have*

$$|P'(0)| \geq \exp\left(\frac{cn}{\alpha}\right) .$$

Proof. Let

$$B_{\nu,\alpha} = \frac{1}{\min \left\| x^{1/2} - \sum_{j=2}^{\nu} a_j x^{\lambda_j - 1/2} \right\|_{L_2[0,1]}} ,$$

where the minimum is taken for all

$$(a_2, a_3, \ldots, a_\nu) \in \mathbb{R}^{\nu-1} .$$

By the "Distance Formula" of Lemma 3.1 we have for $n \geq 6$

$$\begin{aligned} B_{\nu,\alpha} &= \sqrt{2} \prod_{j=2}^{\nu} \frac{\lambda_j + 1}{\lambda_j - 1} = \sqrt{2} \prod_{j=2}^{\nu} \left(1 + \frac{2}{\lambda_j - 1} \right) \\ &\geq \sqrt{2} \prod_{k=2}^{n} \prod_{j=2}^{n} \left(1 + \frac{2}{j + k\alpha - 1} \right) \geq \sqrt{2} \exp\left(\sum_{k=2}^{n} \sum_{j=2}^{n} \frac{1}{j + k\alpha - 1} \right) \\ &\geq \sqrt{2} \exp\left((n-1)^2 \frac{1}{(1+\alpha)n} \right) \geq \sqrt{2} \exp\left(\frac{n}{3\alpha} \right) . \end{aligned}$$

Therefore there is a Müntz polynomial Q of the form

$$Q(x) = x^{1/2} + \sum_{j=2}^{\nu} a_j x^{\lambda_j - 1/2} , \qquad a_j \in \mathbb{R} ,$$

such that

$$\|Q\|_{L_2[0,1]} \leq \frac{1}{\sqrt{2}} \exp\left(-\frac{n}{3\alpha} \right) . \tag{3.4}$$

Now let $P \in M_{\nu,\alpha}$ be defined by

$$P(x) = x^{1/2} Q(x) .$$

Using the Nikolskii-type inequality of Theorem 6.1.3 on page 281 in [1] and combining it with (3.4), we obtain that $|P'(0)| = 1$ and

$$\|P\|_{[0,1]} \leq \sqrt{72} \left(\sum_{j=1}^{\nu} \lambda_j \right)^{1/2} \|Q\|_{L_2[0,1]} \leq c n^{3/2} \sqrt{\alpha} \exp\left(-\frac{n}{3\alpha}\right)$$

with an absolute constant $c > 0$. $\square$

4. Proof of Theorems 2.1, 2.2, and 2.3

Proof of Theorem 2.1. The theorem follows immediately from Lemmas 3.4 and 3.5. Observe that, by symmetry, we may assume that $\alpha > 1$. $\square$

Proof of Theorem 2.2. It is well-known that for any $m \in \mathbb{N}$ there exist $p_m, q_m \in \mathbb{N}$ with $1 \leq q_m \leq m$ and

$$\left| \alpha - \frac{p_m}{q_m} \right| \leq \frac{1}{m q_m} . \tag{4.1}$$

Set $r_m := p_m / q_m$. Obviously $r_m < 2\alpha$ if m is sufficiently large. In the sequel let m be so large that $r_m < 2\alpha$ is satisfied. We shall assume that $r_m > \alpha > 1$ (the case $r_m < \alpha$ is analogous). In addition, set

$$m := \lfloor 6 \log_2 n \rfloor + 1 , \qquad \delta_n := n^{-3m} , \tag{4.2}$$

and

$$\Omega_{\alpha,\delta_n} := \{(x,y) \in \Omega_\alpha : \ 0 \leq x \leq \delta_n\} .$$

Assume that $P \in \mathcal{P}_n^2$ and $\|P\|_{\Omega_\alpha} \leq 1$. First we consider the simple case when $\|D_\omega P\|_{\Omega_\alpha} = |D_\omega P(x_0, y_0)|$ with some $(x_0, y_0) \in \Omega_\alpha \setminus \Omega_{\alpha,\delta_n}$. Clearly, for $(x_0, y_0) \in \Omega_\alpha \setminus \Omega_{\alpha,\delta_n}$ there exist horizontal and vertical segments of length at least $c\,\delta_n^\alpha$ passing through (x_0, y_0) and imbedded into Ω_α. If we apply Markov's inequality (see Theorem 5.1.8 on page 233 in [1]) transformed linearly to these line segments, we obtain that

$$\left| \frac{\partial P}{\partial x}(x_0, y_0) \right| + \left| \frac{\partial P}{\partial y}(x_0, y_0) \right| \leq \frac{4n^2}{c\,\delta_n^\alpha} \leq \exp(c_1 \log^2 n)$$

with a suitable positive constant c_1 depending only on α.

Now we may assume that $\|D_\omega P\|_{\Omega_\alpha} = D_\omega P(x_0, y_0)$, where $(x_0, y_0) \in \Omega_{\alpha,\delta_n}$, that is,

$$0 \leq x_0 \leq \delta_n , \quad \frac{1}{2} x_0^\alpha \leq y_0 \leq 2 x_0^\alpha .$$

Consider the curve

$$\left\{\gamma(t) := (x,y) := (x_0 + t^{q_m}, y_0 + t^{p_m}) : \ \ 0 \le t \le t_0 = (1-x_0)^{1/q_m}\right\} .$$

Clearly, $\gamma(0) = (x_0, y_0)$. Set

$$\xi := 2^{-1/(4\alpha)}, \qquad c := \frac{\xi}{1-\xi} > 2^{1/\alpha} . \tag{4.3}$$

We claim that if $t > c/n^3$, then $\gamma(t) \in \Omega_\alpha$. Assume to the contrary that for some $t > c/n^3$ we have $\gamma(t) \notin \Omega_\alpha$, that is, either

$$y_0 + t^{p_m} = y_0 + (x-x_0)^{r_m} > 2x^\alpha ,$$

or

$$y_0 + t^{p_m} = y_0 + (x-x_0)^{r_m} < \frac{1}{2}\, x^\alpha .$$

Consider the first possibility. Then

$$2x^\alpha < y_0 + (x-x_0)^{r_m} \le 2x_0^\alpha + x^{r_m} \le 2\delta_n^\alpha + x^\alpha ,$$

that is, $x < 2^{1/\alpha}\delta_n$. But then we have

$$t = (x-x_0)^{1/q_m} \le x^{1/q_m} \le x^{1/m} \le \left(2^{1/\alpha}\delta_n\right)^{1/m} \le \frac{2^{1/\alpha}}{n^3}$$

contradicting the choice $t > c/n^3$.

It remains to consider the case when for some $t = (x-x_0)^{1/q_m} > c/n^3$ we have

$$y_0 + (x-x_0)^{r_m} < \frac{1}{2} x^\alpha .$$

Clearly, using that $1 > \xi > 1/2$, that is, $\xi/(1-\xi) > 1$, we have

$$(x-x_0)^{1/q_m} > \frac{c}{n^3} \ge \frac{\xi}{1-\xi}\, \frac{1}{n^3} = \frac{\xi}{1-\xi}\, \delta_n^{1/m} \ge \frac{\xi}{1-\xi}\, \delta_n^{1/q_m} \ge \left(\frac{\xi}{1-\xi}\, \delta_n\right)^{1/q_m} ,$$

and hence

$$x - x_0 \ge \frac{\xi}{1-\xi}\, \delta_n \ge \frac{\xi}{1-\xi}\, x_0 .$$

This yields that

$$x \ge \frac{\xi}{1-\xi}\, x_0 + x_0 = \frac{x_0}{1-\xi} .$$

Therefore $x - x_0 \ge \xi x$. Thus, recalling that $r_m < 2\alpha$, we have

$$\frac{1}{2}\, x^\alpha > y_0 + (x-x_0)^{r_m} > (\xi x)^{r_m} ,$$

that is, by (4.3)

$$x^{r_m-\alpha} < \frac{1}{2}\,\xi^{-r_m} < \frac{1}{2}\,\xi^{-2\alpha} = \frac{1}{\sqrt{2}}\,.$$

Using (4.1), we obtain

$$x < \left(2^{-1/2}\right)^{1/(r_m-\alpha)} < \left(2^{-1/2}\right)^{mq_m}\,,$$

that is,

$$t = (x-x_0)^{1/q_m} \le x^{1/q_m} < 2^{-m/2} \le 2^{-3\log_2 n} = \frac{1}{n^3}\,,$$

which contradicts that $t > c/n^3 > 1/n^3$. Now we have completed the proof of our claim that $\gamma(t) \in \Omega_\alpha$ whenever $t > c/n^3$. Furthermore, for $t > c/n^3$ we have by (4.2)

$$x = x_0 + t^{q_m} \ge \left(\frac{c}{n^3}\right)^{q_m} \ge \left(\frac{c}{n^3}\right)^{m} \ge \exp(-c_2\log^2 n)$$

with a constant c_2 depending only on α. As it was noted at the beginning of the proof, for $(x,y) \in \Omega_\alpha$ with $x \ge \exp(-c_2\log^2 n)$ we have

$$\left|\frac{\partial P}{\partial x}(x,y)\right| + \left|\frac{\partial P}{\partial y}(x,y)\right| \le \exp(c_3\log^2 n) \tag{4.4}$$

with a suitable positive constant c_3 depending only on α. Consider now, for instance, the univariate polynomial

$$G(t) := \frac{\partial P}{\partial y}\left(x_0 + t^{q_m}, y_0 + t^{p_m}\right).$$

By (4.4) we have that

$$|G(t)| \le \exp(c_3\log^2 n)$$

for every $t > c/n^3$. Moreover, by (4.2)

$$\deg(G) \le c_4 n q_m \le c_4 n m \le c_5 n \log n$$

with suitable positive constants c_4 and c_5 depending only on α. Thus by the Chebyshev (or Remez) inequality (see page 235 or 393 in [1], for example) we conclude that

$$\|G\|_{[0,c/n^3]} \le \exp(c_6\log^2 n)\,,$$

with a suitable positive constants c_6 depending only on α. Now we obtain

$$\left|\frac{\partial P}{\partial y}(x_0,y_0)\right| \le \exp(c_6\log^2 n)$$

by setting $t = 0$. We can estimate $(\partial P/\partial x)(x_0,y_0)$ in the same way. The proof of the theorem is now completed. $\Box$

Proof of Theorem 2.3. The proof of this theorem is somewhat similar to that of Theorem 2.2, so we give only a sketch of the proof. Clearly, given an increasing function $\varphi(x)$ tending to ∞ as $x \to \infty$, there exists an irrational number $\alpha > 1$ such that with some $p_m, q_m \in \mathbb{N}$, $q_m \to \infty$, we have

$$0 < \frac{p_m}{q_m} - \alpha < \frac{1}{q_m \varphi(q_m)}\,, \qquad m \in \mathbb{N}\,. \tag{4.5}$$

Set

$$n := \lfloor 2^{\varphi(q_m)/6} \rfloor\,, \qquad \delta_n := n^{-3q_m}\,. \tag{4.6}$$

Then, as in the proof of Theorem 2.2, it can be shown that whenever $P \in \mathcal{P}_n^2$, $\|P\|_{\Omega_\alpha} \leq 1$, and $(x_0, y_0) \in \Omega_\alpha$ with $x_0 \geq \delta_n$ we have

$$|D_\omega P(x_0, y_0)| \leq n^{cq_m}\,, \qquad \omega \in S^1\,,$$

for some $c > 0$ depending only on α. Now let $(x_0, y_0) \in \Omega_\alpha$ and $0 \leq x_0 \leq \delta_n$. Consider the curve

$$\{\gamma(t) := (x_0 + t^{q_m}, y_0 + t^{p_m});\ \ 0 \leq t \leq t_0\}\ ,$$

where $t_0 := (1 - x_0)^{1/q_m}$. Similarly to the proof of Theorem 2.2 it can be shown that $\gamma(t)$ stays below the curve $y = 2x^\alpha$ if $2/n^3 \leq t \leq t_0$. Now we prove that $\gamma(t)$ is located above the curve $y = \frac{1}{2}x^\alpha$ whenever $t > c_0/n^3$ with a properly chosen absolute constant $c_0 > 1$. Set

$$x := x_0 + t^{q_m}; \quad y := y_0 + t^{p_m}; \quad r_m := \frac{p_m}{q_m}\,.$$

Again, using that $t > c_0/n^3$ and (4.6), we have

$$x - x_0 = t^{q_m} > c_0 n^{-3q_m} = c_0 \delta_n \geq c_0 x_0\,,$$

that is, $x - x_0 \geq \xi x$ provided that $c_0 > \xi(1-\xi)^{-1}$, $\xi := 2^{-1/(4\alpha)}$. Assume now that $\gamma(t)$ is below the curve $y = \frac{1}{2}x^\alpha$ for some $t > c_0/n^3$. Then

$$\frac{1}{2}x^\alpha > y_0 + (x - x_0)^{r_m} \geq (x - x_0)^{r_m} \geq (\xi x)^{r_m}\,,$$

that is, since $r_m < 2\alpha$ for sufficiently large values of m, we have

$$x^{r_m - \alpha} \leq \frac{1}{2}\xi^{-r_m} \leq \frac{1}{2}\xi^{-2\alpha} = \frac{1}{\sqrt{2}}\,.$$

Therefore, by (4.5)

$$x \leq \left(\frac{1}{\sqrt{2}}\right)^{1/(r_m - \alpha)} \leq \left(\frac{1}{\sqrt{2}}\right)^{q_m \varphi(q_m)}\,,$$

hence using (4.6), we conclude

$$t \le x^{1/q_m} \le \left(\frac{1}{\sqrt{2}}\right)^{\varphi(q_m)} \le 2^{-\varphi(q_m)/2} \le \frac{1}{n^3}\,.$$

Evidently, this contradicts our choice $t > c_0/n^3, c_0 > 1$. Hence $\gamma(t) \in \Omega_\alpha$ whenever $t > c_0/n^3$, and similarly to the proof of Theorem 2.2, we obtain that

$$M_n(\Omega_\alpha) \le n^{c_1 q_m}$$

with some absolute constant $c_1 > 0$ and $n = \lfloor 2^{\varphi(q_m)/6} \rfloor$. Note that $\varphi(q_m) < c_2 \log n$, where the increasing φ can be chosen to have arbitrarily fast growth to ∞ as $x \to \infty$. This completes the proof of Theorem 2.3. $\square$

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