BERNSTEIN INEQUALITIES FOR POLYNOMIALS WITH CONSTRAINED ROOTS

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Abstract. We prove Bernstein type inequalities for algebraic polynomials on the finite interval $I := [-1, 1]$ and for trigonometric polynomials on $\mathbf{R}$ when the roots of the polynomials are outside of a certain domain of the complex plane. The case of real vs. complex coefficients are handled separately. In case of trigonometric polynomials with real coefficients and root restriction, the L_p-situation will also be considered. In most cases, the sharpness of the estimates will be shown.

1. Introduction

Let $\mathcal{P}_n$ and $\mathcal{P}_n^c$ denote the set of all algebraic polynomials of degree at most n with real and complex coefficient, respectively. By making appropriate restrictions on these sets of polynomials, there are many ways of improving the classical Markov–Bernstein inequalities

$$|p_n'(x)| \le \min\left(n^2, \frac{n}{\sqrt{1-x^2}}\right) ||p_n||_I \qquad (x \in I,\ p_n \in \mathcal{P}_n^c).$$

(Here $||\cdot||_I$ means supremum norm on the interval $I = [-1, 1]$. All variables and arguments in this paper will be real, except for z and ζ which denote complex numbers.) For example, the sharp inequalities

$$|p_n'(x)| \le c \min\left(n, \sqrt{\frac{n}{1-x^2}}\right) ||p_n||_I \qquad (x \in I) \tag{1.1}$$

and

$$|p_n'(x)| \le cn \log \min\left(n, \frac{e}{1-x^2}\right) ||p_n||_I \qquad (x \in I) \tag{1.2}$$

with an absolute constant $c > 0$ are valid for all $p_n \in \mathcal{P}_n$ and $p_n \in \mathcal{P}_n^c$, respectively, whose roots are outside the open unit disk $|z| < 1$ (cf. G. G. Lorentz [6], P. Borwein and T. Erdélyi [2], and T. Erdélyi [4]).

[1] Research supported, in part, by NSF Grant No. DMS-9623156.
[2] Research supported by OTKA No. T022943 and T023441.

In this paper we generalize (1.1) – (1.2) for the following subset of algebraic polynomials $\mathcal{P}_n$ of degree at most n:

$$(1.3) \qquad \mathcal{P}_n(\varepsilon) := \{p \in \mathcal{P}_n \,|\, p(z) \neq 0 \text{ if } x^2 + y^2/\varepsilon^2 < 1 \ (z = x + iy)\}.$$

Although the ellipse appearing in this definition is a simple homogeneous transformation of the unit disk, it will turn out that the corresponding results are by no means easy consequences of the inequalities (1.1) – (1.2). In most cases, the inequalities to be presented prove to be sharp.

Also, with respect to the classical Bernstein inequality

$$||t'||_{\mathbf{R}} \leq n||t||_{\mathbf{R}}$$

valid for all $\mathcal{T}_n$(=the set of all trigonometric polynomials of order n with real coefficients), we will consider the subset

$$\mathcal{T}_n(r) := \{t \in \mathcal{T}_n \,|\, t(z) \neq 0 \text{ for } |\mathrm{Im}\, z| < r\} \qquad (0 < r < 1),$$

and prove a Bernstein-type inequality in L_p-metric, including the case $p = \infty$.

2. Trigonometric polynomials with real coefficients

Let us introduce the notation

$$(2.1) \qquad \Delta = \max\left(\frac{1}{n}, r\right) \qquad (0 < r \leq 1,\ n = 1, 2, \ldots).$$

Theorem 1. *We have*

$$(2.2) \qquad \sup_{0 \neq t \in \mathcal{T}_n(r)} \frac{||t'||_{\mathbf{R}}}{||t||_{\mathbf{R}}} \sim \sqrt{\frac{n}{\Delta}}.$$

Here and in what follows, $\sim$ means that the ratio of the left and right hand sides remains between two positive absolute constants.

For the proof we need a lemma in connection with the class $\mathcal{T}_n(r)$. In what follows, c will denote positive absolute constants, not necessarily the same at different occurrences.

Lemma 1. *We have*

$$|t_n(x+iy)| \leq c||t_n||_{\mathrm{R}} \quad \textit{for all} \quad t_n \in \mathcal{T}_n(r),\ x, y \in \mathbf{R},\ |y| \leq c\sqrt{\frac{r}{n}},\ \frac{1}{n} \leq r \leq 1.$$

Proof. It is sufficient to prove the lemma for $x = 0$, since for $x \neq 0$ we can consider the polynomial $T_n(\xi) := t_n(\xi + x) \in \mathcal{T}_n(r)$ and apply the result for $\xi = 0$. Evidently,

$$q_n(z) := t_n(z)t_n(-z) \in \mathcal{T}_{2n}(r)$$

is an even trigonometric polynomial. Let

$$p_n(z) := q_n(\arccos z) \in \mathcal{P}_{2n}.$$

Denote $\arccos z = u + ir$, then $z = \cos u \cosh r + i \sin u \sinh r := x + iy$, i.e. p_n has no roots in the ellipse

$$\frac{x^2}{\cosh^2 r} + \frac{y^2}{\sinh^2 r} = 1.$$

An easy calculation shows that this ellipse contains the disks

$$(x - 1 + 2\tanh^2 r)^2 + y^2 < 4\tanh^4 r \quad \text{and} \quad (x + 1 - 2\tanh^2 r)^2 + y^2 < 4\tanh^4 r,$$

and hence, according to Lemma 4.1 in [5],

$$|p_n(x)| \leq \exp\left(\frac{16(|x| - 1)n}{\sqrt{2}\tanh r}\right) ||p_n||_I \quad \text{for} \quad 1 \leq |x| \leq 1 + 2\tanh^2 r.$$

Using this with $x = \cosh y$ we get

$$|t_n(iy)|^2 = q_n(iy) = |p_n(\cos iy)| = |p_n(\cosh y)| \leq \exp\left(\frac{16(\cosh y - 1)n}{\sqrt{2}\tanh r}\right) ||p_n||_I \leq$$

$$\leq c||p_n||_I \leq c||t_n||_{\mathbf{R}}^2 \qquad (|y| \leq c\sqrt{r/n} \leq cr).$$

Proof of Theorem 1. Using Cauchy's integral formula and Lemma 1 we get

$$|t_n'(x)| \leq \frac{1}{2\pi}\left|\oint_{|\zeta - x| = c\sqrt{\frac{r}{n}}} \frac{t_n(\zeta)}{(\zeta - x)^2}\, d\zeta\right| \leq c\sqrt{\frac{n}{r}}||t_n||_{\mathbf{R}} \qquad (x \in \mathbf{R}),$$

which proves the upper estimate in (2.2) for $r < 1/n$. For $r \geq 1/n$ it follows from the classical Bernstein inequality.

In order to prove the lower estimate, consider the trigonometric polynomial

$$t_n(x) = (\cos mx + 2)^{[n\Delta]}, \tag{2.3}$$

where $m = [1/\Delta]$. First we show that $t_n \in \mathcal{T}_n(r)$. Indeed, for the roots $z = x + iy$ of (2.3) we have

$$\cos m(x+iy)+2=\cos mx\cosh my+2+i\sin mx\sinh my=0,$$

whence $\cosh my \geq 2$, i.e. $|y| > \frac{1}{m} > r$. Now, if $r < 1/n$ then (2.3) takes the form $t_n(x) = \cos nx + 2$, and here $||t_n'||_{\mathbf{R}} = n = \frac{n}{3}||t_n||_{\mathbf{R}}$ indeed. If $r \geq 1/n$, then let $x_0 \in \mathbf{R}$ be such that $\cos mx_0 = 1 - \frac{1}{nr}$. Then $\sin mx_0 \geq \frac{1}{\sqrt{nr}}$, and hence

$$t_n'(x_0) = [nr]\cdot m\sin mx_0\left(3-\frac{1}{nr}\right)^{[nr]-1} \geq c\sqrt{\frac{n}{r}}||t_n||_{\mathbf{R}}.$$

3. Algebraic polynomials with real coefficients

In analogy with (2.1), let us introduce the notation

$$\delta = \max\left(\frac{1}{n}, \varepsilon\right) \qquad (0 < \varepsilon \leq 1,\ n = 1, 2, \ldots). \tag{3.1}$$

Theorem 2. *We have*

$$\sup_{0\neq p\in\mathcal{P}_n(\varepsilon)} \frac{|p'(x)|}{||p||_I} \leq \begin{cases} c\sqrt{\frac{n}{\delta(1-x^2)}} & \textit{if} \quad x^2 \leq 1-\delta^2, \\ c\frac{\sqrt{n}}{(1-x^2)^{3/4}} & \textit{if} \quad 1-\delta^2 \leq x^2 < 1-\frac{\delta^{4/3}}{n^{2/3}}, \\ c\frac{n}{\delta} & \textit{if} \quad 1-\frac{\delta^{4/3}}{n^{2/3}} \leq x^2 \leq 1. \end{cases} \tag{3.2}$$

Proof. In case $0 < \varepsilon < \frac{1}{n}$ the statements of the theorem are identical with the classical Bernstein inequality (the second possibility does not arise in this case). Thus we may assume that $\frac{1}{n} \leq \varepsilon \leq 1$. Let $p_n \in \mathcal{P}_n(\varepsilon)$, $||p_n||_I = 1$, and consider the even trigonometric polynomial

$$t_n(z) := p_n(a\cos z) \in \mathcal{T}_n, \tag{3.3}$$

where the parameter $0 < a < 1$ will be determined later. Let $z = x + iy$, and suppose that $a\cos z = u + iv$ is on the ellipse $u^2 + \frac{v^2}{\varepsilon^2} = 1$. This means that

$$\cos^2 x\cosh^2 y + \frac{\sin^2 x\sinh^2 y}{\varepsilon^2} = \frac{1}{a^2}.$$

Hence

$$\cosh^2 y = \frac{1}{a^2}\cdot\frac{\varepsilon^2 + a^2\sin^2 x}{\varepsilon^2 + (1-\varepsilon^2)\sin^2 x}.$$

Calculating the extrema of this rational function of the variable $\sin^2 x$, we obtain

$$\cosh^2 y \geq \begin{cases} 1+\frac{\varepsilon^2}{a^2} & \text{if } a^2 < 1-\varepsilon^2, \\ \frac{1}{a^2} & \text{if } 1-\varepsilon^2 \leq a^2 < 1. \end{cases}$$

Hence the trigonometric polynomial $t_n(z)$ has no roots in the strip

$$|y| \leq c\min(\varepsilon, \sqrt{1-a^2}).$$

By (3.3), $t_n'(z) = -a\sin z \cdot p_n'(a\cos z)$, whence by Theorem 1 we obtain

$$|p_n'(x)| \leq \frac{||t_n'||_{\mathbf{R}}}{\sqrt{a^2-x^2}} \leq c\frac{\sqrt{\frac{n}{\min(\varepsilon,\sqrt{1-a^2})}}}{\sqrt{a^2-x^2}} \qquad (|x|<a). \tag{3.4}$$

First let $x^2 \leq 1-\varepsilon^2$. Then choosing $a^2 = 1-\frac{\varepsilon^2}{2}$, (3.4) yields the corresponding estimate in (3.2). Now if $x^2 > 1-\varepsilon^2$, then $a^2 = \frac{x^2+2}{3}$ results in the second estimate in (3.2). However, this bound becomes worse than the last estimate in (3.2) if $1-\frac{\varepsilon^{4/3}}{n^{2/3}} \leq x^2 \leq 1$. This last estimate follows from a more general theorem of the first named author. Namely, if $p_n \in \mathcal{P}_n(\varepsilon)$ then it is easy to see that p_n has no roots in the circles with diameters $[-1,-1+\varepsilon^2]$ and $[1-\varepsilon^2, 1]$, and Theorem 1 of [3] applies.

We now show that in most part of the interval I, the estimates of Theorem 2 are sharp.

Theorem 3. *With the notation* (3.1) *we have*

$$\sup_{0\neq p\in\mathcal{P}_n(\varepsilon)} \frac{|p'(x)|}{||p||_I} \geq \begin{cases} c\sqrt{\frac{n}{\delta(1-x^2)}} & \textit{if} \quad x^2 \leq 1-\delta^2, \\ c\frac{n}{\delta} & \textit{if} \quad 1-\frac{\delta}{n} \leq x^2 \leq 1. \end{cases} \tag{3.5}$$

Proof. Let first $0 \leq x \leq 1/2$ be fixed, and consider the polynomial

$$p_n(y) = \left\{T_m\left(\frac{y-x}{2}+\xi\right)+4\right\}^{[n\delta]} \in \mathcal{P}_n, \tag{3.6}$$

where $m = [1/\delta]$, $\quad T_m(y) = \cos(m\arccos y)$ is the Chebyshev polynomial and $\xi \leq x$ is the nearest point to x such that

$$T_m(\xi) = 1-\frac{1}{n\delta}. \tag{3.7}$$

Then $\left|\frac{y-x}{2}+\xi\right| \leq 1$ $(|y| \leq 1)$, $||p_n||_I = 5^{[n\delta]}$ and $p_n \in \mathcal{P}_n(\delta) \subset \mathcal{P}_n(\varepsilon)$. (3.7) implies that $\sin(m\arccos\xi) \geq \frac{1}{\sqrt{n\delta}}$, whence

(3.8)
$$|p_n'(x)| \geq cn\delta m \frac{\sin(m \arccos \xi)}{\sqrt{1-\xi^2}} \{T_m(\xi)+4\}^{[n\delta]-1} \geq$$

$$\geq c\sqrt{\frac{n}{\delta}} \left(5 - \frac{1}{n\delta}\right)^{[n\delta]} \geq c\sqrt{\frac{n}{\delta}} ||p_n||_I,$$

which proves the first estimate in (3.5) when $0 \leq x \leq 1/2$.

Now let $1/2 < x \leq 1$. Define $\xi \leq x$ and m as above and consider the polynomial

(3.9)
$$p_n(y) = \left\{T_m\left(\frac{\xi}{x} y\right) + 4\right\}^{[n\delta]},$$

Again, it is easily seen that $p_n \in \mathcal{P}_n(\varepsilon)$, and similarly to (3.8) we obtain

(3.10)
$$|p_n'(x)| \geq c \frac{\xi}{x} \frac{\sqrt{\frac{n}{\delta}} ||p_n||_I}{\sqrt{1-\xi^2}} \geq c \frac{\sqrt{\frac{n}{\delta}} ||p_n||_I}{\sqrt{1-\xi^2}}.$$

Here, since $m \geq \frac{1}{2\delta}$,

$$1 - \xi^2 \leq 1 - x^2 + 2(x - \xi) \leq 1 - x^2 + \frac{c}{m}(\sqrt{1-x^2} + 1/m) \leq c(1-x^2)$$

provided $1/2 < x \leq \sqrt{1-\delta^2}$. Substituting this into (3.10) we obtain the first estimate in (3.5). Finally, by the Mean Value Theorem for $\cos \frac{\pi}{2m} \leq \xi \leq 1$ we get

$$c\delta^2 \leq \frac{c}{m^2} \leq T_m'\left(\cos \frac{\pi}{2m}\right) \leq \frac{T_m(1) - T_m(\xi)}{1 - \xi} \leq T_m'(1) = m^2 \leq \frac{1}{\delta^2},$$

whence and from (3.7)

(3.11)
$$\frac{\delta}{n} \leq \frac{1}{m^2 n\delta} \leq 1 - \xi \leq \frac{c\delta}{n}.$$

Thus if $x^2 \geq 1 - \frac{\delta}{n} \geq \cos \frac{\pi}{2m}$ then the ξ defined for x is indeed in the interval $\cos \frac{\pi}{2m}, x]$ and (3.10)–(3.11) imply the second lower estimate in (3.5). In case $1 - \frac{\delta}{n} \leq \cos \frac{\pi}{2m}$ we have $\delta \leq c/n$ and $1 - \frac{\delta}{n} \geq \cos \frac{3\pi}{2m}$, whence $1 - \xi^2 \geq \frac{c}{m^2} \geq \frac{c}{n^2}$, and (3.10) yields $|p_n'(x)| \geq cn^2$, which proves the second estimate in (3.5) in this case.

4. Algebraic polynomials with complex coefficients

In analogy with (1.3), define

$$\mathcal{P}_n^c(\varepsilon) := \{p \in \mathcal{P}_n \,|\, p(z) \neq 0 \text{ if } x^2 + y^2/\varepsilon^2 < 1 \ (z = x + iy)\}.$$

In this setting, using the notation (3.1), we have the following sharp result:

Theorem 4. *We have*

$$\sup_{0 \neq p \in \mathcal{P}_n^c(\varepsilon)} \frac{|p'(x)|}{||p||_I} \sim \begin{cases} \frac{n}{\sqrt{1-x^2}} & \text{if } x^2 \leq 1 - \delta^2, \\ \frac{n}{\delta} \log \frac{2\sqrt{2}\delta}{\sqrt{1-x^2}+1/n} & \text{if } 1 - \delta^2 \leq x^2 \leq 1. \end{cases} \tag{4.1}$$

In particular, this result implies the relation

$$\sup_{0 \neq p \in \mathcal{P}_n^c(\varepsilon)} \frac{||p'||_I}{||p||_I} \leq c \frac{n \log(2\sqrt{3}n\delta)}{\delta},$$

which is equivalent to the upper estimate from Theorem 2.2 of [5].

Proof. If $\varepsilon < 1/n$, then (4.1) is equivalent to

$$\sup_{0 \neq p \in \mathcal{P}_n^c(\varepsilon)} \frac{|p'(x)|}{||p||_I} \sim \begin{cases} \frac{n}{\sqrt{1-x^2}} & \text{if } x^2 \leq 1 - 1/n^2, \\ n^2 & \text{if } 1 - 1/n^2 \leq x^2 \leq 1, \end{cases} \tag{4.2}$$

which is nothing but a weaker version (with respect to the constants) of the combined Bernstein–Markov inequality for any algebraic polynomial. In this case the lower estimate in (4.2) can be easily seen by considering the polynomial $T_n(x) + 2$ (with real coefficients!) and its modifications obtained by linear transformations of the variable, similarly to the proof of Theorem 3.

Now if $1/n \leq \varepsilon (\leq 1)$, then normalize $p \in \mathcal{P}_n^c$ such that $||p||_I \leq e^{-8}$, and use Nevanlinna's inequality

$$\log |p(x + iy)| \leq \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\log |p(t)|}{(t - x)^2 + y^2} \, dt$$

(cf. Boas [1], pp. 92–93) with

$$x + r \cos \varphi \quad \text{and} \quad r \sin \varphi \qquad (x \in I, \ r > 0, \ 0 \leq \varphi \leq 2\pi)$$

in place of x and y, respectively.

We split the right hand side integral into three parts:

$$\frac{|y|}{\pi} \int_{-\infty}^{\infty} = \frac{|y|}{\pi} \left(\int_{|t| \leq 1 + \varepsilon/n} + \int_{1 + \varepsilon/n \leq |t| \leq 1 + \varepsilon^2} + \int_{|t| \geq 1 + \varepsilon^2} \right) := I_1 + I_2 + I_3,$$

and estimate these quantities separately. Since $p \in \mathcal{P}_n^c(\varepsilon)$, it is readily seen that p has no roots in the open circles with diameters $[-1,-1+2\varepsilon^2]$ and $[1-2\varepsilon^2,1]$, and hence by Lemma 4.1 of [5],

$$\log|p(t)| \le 8\frac{n(|t|-1)}{\varepsilon} + \log\|p\|_I \le 8\left(\frac{n(|t|-1)}{\varepsilon} - 1\right) \qquad (1 \le |t| \le 1+\varepsilon^2). \tag{4.3}$$

Thus $\max_{|t|\le 1+\varepsilon/n}|p(t)| \le 1$, i.e. $I_1 \le 0$.

Using again (4.3), as well as

$$|t|-|x|-r \ge \frac{|t|-|x|}{2} \qquad \left(0 < r \le \frac{1-|x|+\varepsilon/n}{2},\ |t| \ge 1+\varepsilon/n,\ x \in I\right) \tag{4.4}$$

we get

$$I_2 \le \frac{8nr}{\pi\varepsilon}\int_{1+\varepsilon/n\le|t|\le 1+\varepsilon^2} \frac{|t|-1}{(t-x-r\cos\varphi)^2+r^2\sin^2\varphi}\,dt \le$$

$$\le \frac{32nr}{\pi\varepsilon}\int_{1+\varepsilon/n\le|t|\le 1+\varepsilon^2} \frac{dt}{|t|-|x|} \le \frac{64nr}{\pi\varepsilon}\log\frac{1-|x|+\varepsilon^2}{1-|x|+\varepsilon/n} \le$$

$$\le \begin{cases} \frac{cnr\varepsilon}{1-x^2} & \text{if } x^2 \le 1-\varepsilon^2, \\ \frac{cnr}{\varepsilon}\log\frac{2\sqrt{2}\varepsilon}{\sqrt{1-x^2}+1/n} & \text{if } 1-\varepsilon^2 \le x^2 \le 1 \end{cases} \qquad \left(0 < r \le \frac{1-|x|+\varepsilon/n}{2}\right).$$

Finally, using the Chebyshev inequality

$$|p(t)| \le |T_n(t)|\|p\|_I \le \exp(cn\sqrt{|t|-1})\|p\|_I \qquad (|t| \ge 1)$$

and (4.4) again we obtain

$$I_3 \le cnr\int_{|t|\ge 1+\varepsilon^2}\frac{\sqrt{|t|-1}}{(|t|-|x|)^2}\,dt \le cnr\int_{|t|\ge 1+\varepsilon^2}\frac{dt}{(|t|-|x|)^{3/2}} \le$$

$$\le \frac{cnr}{(1-|x|+\varepsilon^2)^{1/2}} \le \frac{cnr}{\sqrt{1-x^2}+\varepsilon} \qquad \left(x \in I, 0 < r \le \frac{1-|x|+\varepsilon/n}{2}\right).$$

Since

$$\max\left(\frac{\varepsilon}{1-x^2}, \frac{1}{\sqrt{1-x^2}+\varepsilon}\right) \le \frac{1}{\sqrt{1-x^2}} \qquad (x^2 \le 1-\varepsilon^2)$$

and

$$\frac{\varepsilon}{\sqrt{1-x^2}+\varepsilon} \le c\log\frac{2\sqrt{2}\varepsilon}{\sqrt{1-x^2}+1/n} \qquad \left(1-\varepsilon^2 \le x^2 \le 1,\ 0<r\le\frac{1-|x|+\varepsilon/n}{2}\right),$$

collecting the above estimates we obtain

$$\log|p(x+r\cos\varphi+ir\sin\varphi)| \le \begin{cases} \frac{cnr}{\sqrt{1-x^2}} & \text{if } x^2\le 1-\varepsilon^2, \\ \frac{cnr}{\varepsilon}\log\frac{2\sqrt{2}\varepsilon}{\sqrt{1-x^2}+\frac{1}{n}} & \text{if } 1-\varepsilon^2\le x^2\le 1 \end{cases}$$

$$\left(0<r\le\frac{1-|x|+\varepsilon/n}{2}\right).$$

Hence choosing

$$r=\begin{cases} \frac{\sqrt{1-x^2}}{4n} & \text{if } x^2\le 1-\varepsilon^2, \\ \frac{\varepsilon\log 2}{4n\log\frac{2\sqrt{2}\varepsilon}{\sqrt{1-x^2}+1/n}} & \text{if } 1-\varepsilon^2\le x^2\le 1, \end{cases}$$

by Cauchy's integral formula we get

$$|p'(x)| \le \frac{1}{2\pi}\oint_{|z-x|=r}\frac{|p(z)|}{|z-x|^2}\,|dz| \le \frac{cr}{r^2}=\frac{c}{r} \qquad (x\in I),$$

which proves the upper estimate in (4.1).

As for the lower estimates in Theorem 4 in case $1/n\le\varepsilon\le\delta\le\frac{1}{4\sqrt{2}}2$, consider the polynomial

$$q_n(x)=T_{2n}\left(\frac{x}{\sqrt{1-\varepsilon^2}}\right)+T_{2n}\left(\frac{1}{\sqrt{1-\varepsilon^2}}\right)\in\mathcal{P}_{2n}.$$

Using the formula

$$T_n\left(\frac{z+z^{-1}}{2}\right)=\frac{z^n+z^{-n}}{2},$$

an easy calculation shows that the roots of this polynomial are

$$z_k=\cos t_k+i\varepsilon\sin t_k \qquad \left(t_k=\frac{(2k-1)\pi}{2n},\ k=1,\dots,2n\right),$$

and so $q_n\in\mathcal{P}_{2n}(\varepsilon)$.

Now let $p_n\in\mathcal{P}_n^c(\varepsilon)$ be that polynomial which is obtained from q_n by omitting the roots with negative imaginary parts and normalized such that $|p_n(x)|^2=|q_n(x)| \quad (x\in I)$. Since

$$T_{2n}\left(\frac{1}{\sqrt{1-\varepsilon^2}}\right)-1\le q_n(x)\le 2T_{2n}\left(\frac{1}{\sqrt{1-\varepsilon^2}}\right) \qquad (x\in I),$$

we have

$$|p_n(x)| \sim T_{2n}\left(\frac{1}{\sqrt{1-\varepsilon^2}}\right)^{1/2} \sim ||p_n||_I \qquad (x \in I).$$

Now

$$|p_n'(x)| \geq |p_n(x)| \cdot \operatorname{Im} \sum_{k=1}^{n} \frac{1}{x - z_k} \geq \tag{4.5}$$

$$\geq c||p_n||_I \sum_{k=1}^{n} \frac{\varepsilon \sin t_k}{(x-\cos t_k)^2 + \varepsilon^2 \sin^2 t_k} \qquad (x \in I),$$

and since

$$(x - \cos t_k)^2 + \varepsilon^2 \sin^2 t_k \leq c\{(1-x^2)^2 + \sin^4 t_k + \varepsilon^2 \sin^2 t_k\} \leq c\varepsilon^2 \sin^2 t_k$$

$$\left(1 + \frac{n}{3}\sqrt{1-x^2} \leq k \leq 2\sqrt{2}n\varepsilon,\ 1 - \varepsilon^2 \leq x^2 \leq 1\right)$$

we obtain

$$|p_n'(x)| \geq c||p_n||_I \sum_{1+\frac{n}{3}\sqrt{1-x^2} \leq k \leq 2\sqrt{2}n\varepsilon} \frac{1}{\varepsilon \sin t_k} \geq \frac{cn}{\varepsilon}||p_n||_I \log \frac{2\sqrt{2}n\varepsilon}{\frac{n}{3}(1-x^2)+1}$$

$$(1 - \varepsilon^2 \leq x^2 \leq 1).$$

This yields the second lower estimate in (4.1).

Next, we prove the first lower estimate in (4.1) (when $\delta = \varepsilon \leq \frac{1}{4\sqrt{2}}$). Let $x^2 \leq 1 - \varepsilon^2$, and apply the just proved lower estimate with $\varepsilon_0 = \sqrt{1-x^2} \geq \varepsilon$. We obtain a polynomial $p_n \in \mathcal{P}_{2n}(\varepsilon_0) \subset \mathcal{P}_{2n}(\varepsilon)$ such that

$$|p_n'(y)| \geq \frac{2\sqrt{2}n}{\varepsilon_0} \log \frac{cn\varepsilon_0}{n\sqrt{1-y^2}+1} \qquad (x^2 = 1 - \varepsilon_0^2 \leq y^2 \leq 1).$$

In particular,

$$|p_n'(x)| \geq \frac{cn}{\sqrt{1-x^2}} \log \frac{cn\sqrt{1-x^2}}{n\sqrt{1-x^2}+1} \geq \frac{cn}{\sqrt{1-x^2}} \qquad (x^2 \leq 1 - \varepsilon^2).$$

Finally, if $\frac{1}{4\sqrt{2}} \leq \varepsilon = \delta \leq 1$, then the lower estimates in (4.1) follow from the sharpness of (1.2).

5. Constrained trigonometric polynomials in L_p

Our main goal is to prove the following Bernstein-type inequality in $L_p := L_p(K)$ for all $f \in \mathcal{T}_n(r)$ and $p \in [1, \infty)$.

Theorem 5. *Let χ be a nonnegative, nondecreasing, convex function defined on $[0, \infty)$. Then, with the notation (2.1),*

$$\sup_{0 \neq t \in \mathcal{T}_n(r)} \frac{\left\| \chi \left(\sqrt{\frac{\Delta}{n}} t' \right) \right\|_{L_1}}{||\chi(ct)||_{L_1}} < \infty.$$

In particular, with $\chi(x) = x^p$,

$$\sup_{0 \neq t \in \mathcal{T}_n(r)} \frac{||t'||_{L_p}}{||t||_{L_p}} \leq c\sqrt{\frac{n}{\Delta}}$$

for every $p \in [1, \infty)$.

In the proof of Theorem 5 we need the following essentially sharp Nikolskii-type inequality for every $t \in \mathcal{T}_n(r)$. Both the upper and lower bounds of Lemma 2 will be needed.

Lemma 2. *Let $n \in \mathbf{N}$, $r \in (0, 1]$, and $p \in (0, \infty)$. Then*

$$\sup_{0 \neq t \in \mathcal{T}_n(r)} \frac{||t||_{L_\infty}}{||t||_{L_p}} \sim \sqrt{\frac{n}{\Delta}},$$

where the constants involved depend only on p.

Proof. First we prove the upper bound. Let $t \in \mathcal{T}_n(r)$. Let $\tau \in \mathbf{R}$ be a number where $t(\tau) = ||t||_{L_\infty}$. Let $I_\tau := \left[\tau - \frac{1}{2\lambda p} \sqrt{\frac{n}{\Delta}}, \tau + \frac{1}{2\lambda p} \sqrt{\frac{n}{\Delta}} \right]$, where $\lambda > 0$ is the constant corresponding to the upper estimate in Theorem 1. Combining the Mean Value Theorem and our Bernstein-type inequality in L_∞ for $t \in \mathcal{T}_n(r)$, we obtain that

$$||t||_{L_\infty} - t(\theta) = t(\tau) - t(\theta) = |(\theta - \tau)t'(\xi)| \leq \lambda \sqrt{\frac{n}{\Delta}} \frac{1}{2\lambda p} \sqrt{\frac{\Delta}{n}} ||t||_{L_\infty} = \frac{1}{2p} ||t||_{L_\infty}$$

for every $x \in I_\tau$ ($\xi \in I_\tau$ is a suitable number guaranteed by the Mean Value Theorem). Therefore

$$t(\theta)^p \geq \left(1 - \frac{1}{2p} \right)^p ||t||_{L_\infty}^p \geq c||t|||_{L_\infty}^p$$

for every $x \in I_\tau$. Hence, noting that $\lambda \geq 1$, we get

$$||t||_{L_p}^p \geq \int_{I_\tau} t(\theta)^p \, d\theta \geq \frac{2}{2p\lambda} \sqrt{\frac{n}{\Delta}} c||t||_{L_\infty}^p \geq \frac{c}{p\lambda} \sqrt{\frac{\Delta}{n}} ||t||_{L_\infty}^p,$$

and the upper bound of the lemma is proved.

Now we prove the lower bound. Let $\nu := 1/k$, where $k = 2m-1 \le n/2$ with a nonnegative integer m. We define

$$Q_k(z) := \left(\sum_{j=0}^{k} (1+\nu)^j z^j \right) \left(\sum_{j=0}^{k} (1+\nu)^j z^{-j} \right)$$

and

$$R_{n,\nu}(x) := \mu \left(Q_k(e^{ix}) \right)^{\lfloor n/k \rfloor},$$

where $\mu > 0$ is chosen so that

$$|R_{n,\nu}(0)| = 1. \tag{5.1}$$

Obviously $R_{n,\nu} \in \mathcal{T}_n(\nu/2)$. We have

$$|R_{n,\nu}(x)| = \frac{|R_{n,\nu}(x)|}{|R_{n,\nu}(0)|} \le \left(\frac{\left| \sum_{j=0}^{m-1} (1+\nu)^j e^{ijx} + \sum_{j=m}^{2m-1} (1+\nu)^j e^{ijx} \right|}{\sum_{j=0}^{k} (1+\nu)^j} \right)^{\lfloor n/k \rfloor} = \tag{5.2}$$

$$= \left(\frac{\left| (1 + (1+\nu)^m e^{imx} \right| \left| \sum_{j=0}^{m-1} (1+\nu)^j e^{ijx} \right|}{(1+(1+\nu)^m) \left(\sum_{j=0}^{m-1} (1+\nu)^j \right)} \right)^{\lfloor n/k \rfloor} \le \left| \frac{(1+(1+\nu)^m e^{imx}}{1+(1+\nu)^m} \right|^{2\lfloor n/(2k) \rfloor} =$$

$$= \left| \frac{(1^2 + (1+\nu)^{2m} + 2(1+\nu)^m \cos mx}{1 + (1+\nu)^{2m} + 2(1+\nu)^m} \right|^{\lfloor n/(2k) \rfloor} \le (1 - c(1-\cos mx))^{\lfloor n/(2k) \rfloor} \le$$

$$\le (1 - cm^2x^2)^{\lfloor n/(2k) \rfloor} \le \exp\left(-cm^2x^2 \lfloor n/(2k) \rfloor\right) \le \exp\left(-cn(k+1)x^2\right)$$

for every $x \in \mathbf{R}$ with $|x| \le \frac{6}{m} = \frac{12}{k+1}$. Also

$$|R_{n,\nu}(x)| = \frac{|R_{n,\nu}(x)|}{|R_{n,\nu}(0)|} \le \left(\frac{\left| \sum_{j=0}^{k} (1+\nu)^j e^{ijx} \right|}{\sum_{j=0}^{k} (1+\nu)^j} \right)^{\lfloor n/k \rfloor} = \tag{5.3}$$

$$= \left(\frac{\left| (1+\nu)^{k+1} e^{i(k+1)x} - 1 \right|}{|(1+\nu)e^{ix} - 1|} \frac{\nu}{(1+\nu)^{k+1} - 1} \right)^{\lfloor n/k \rfloor} \le$$

$$\leq \left(\frac{4+1}{e-1} \cdot \frac{\nu}{\frac{2}{\pi}x}\right)^{\lfloor n/k \rfloor} \leq \left(\frac{5\nu}{x}\right)^{\lfloor n/k \rfloor}.$$

Assume that $k+1 \geq \frac{12}{\pi}$; the case $1 \leq k \leq \frac{12}{\pi}$ is similar. Combining (5.2), (5.3), and $\nu := 1/k$, we have

$$\int_{-\pi}^{\pi} |R_{n,\nu}(x)|^p\,dx = \int_{-12/(k+1)}^{12/(k+1)} |R_{n,\nu}(x)|^p\,dx + \tag{5.4}$$

$$+\int_{-\pi}^{-12/(k+1)} |R_{n,\nu}(x)|^p\,dx + \int_{12/(k+1)}^{\pi} |R_{n,\nu}(x)|^p\,dx \leq$$

$$\leq 2\int_0^{12/(k+1)} \exp\left(-cn(k+1)px^2\right)dx + 2\int_{12/(k+1)}^{\pi} \left(\frac{5\nu}{x}\right)^{p\lfloor n/k \rfloor} \leq$$

$$\leq \frac{2}{\sqrt{cn(k+1)p}} \int_0^{12\sqrt{\frac{cnp}{k+1}}} e^{-u^2}\,du + 2(5\nu)^{p\lfloor n/k \rfloor} \left[\frac{x^{-p\lfloor n/k \rfloor+1}}{-p\lfloor n/k \rfloor+1}\right]_{\frac{12}{k+1}}^{\pi} \leq$$

$$\leq \frac{2}{\sqrt{cn(k+1)p}} + 2\left(\frac{5}{12} \cdot \frac{k+1}{k}\right)^{p\lfloor n/k \rfloor} \left(p\lfloor n/k \rfloor - 1\right)\frac{12}{k+1} \leq$$

$$\leq \frac{2}{\sqrt{cn(k+1)p}} + 2\exp\left(-c'pn/k\right)(pn/k)\frac{12}{k} \leq \frac{c''}{\sqrt{nkp}} \leq \frac{c''}{\sqrt{nk}}.$$

Now (5.1) and (5.4) together with $R_{n,\nu} \in \mathcal{T}_n(\nu/2)$ give the lower bound of the theorem. The elementary argument showing that the general case of $r \in (0,1]$ and $n \in \mathbf{N}$ can be reduced to the case of $r = \nu/2 = 1/(2k)$ with $k = 2m-1 \leq n/2$, $m, n \in \mathbf{N}$, is left to the reader.

Proof of Theorem 5. By Lemma 2 there is a trigonometric polynomial $t_{n,r} \in \mathcal{T}_n(r)$ such that

$$\frac{||t_{n,r}||_{L_\infty}}{||t_{n,r}||_{L_1}} \sim \sqrt{\frac{n}{\Delta}}.$$

Let $g \in \mathcal{T}_n(r)$. On applying the upper bound of Lemma 2 to

$$G := g t_{n,r} \in \mathcal{T}_{2n}(r), \tag{5.5}$$

we obtain

$$||g t_{n,r}||_{L_\infty} \leq c\sqrt{\frac{2n}{\Delta}}||g t_{n,r}||_{L_1(K)}. \tag{5.6}$$

If we apply the Bernstein-type inequality of Theorem 1 to (5.5) and use the Nikolskii-type inequality of (5.6), we can deduce that

$$|g'(\theta)t_{n,r}(\theta)+t'_{n,r}(\theta)g(\theta)| \le c\sqrt{\frac{2n}{\Delta}}||gt_{n,r}||_{L_\infty} \le c\sqrt{\frac{2n}{\Delta}}c\sqrt{\frac{2n}{\Delta}}||gt_{n,r}||_{L_1}$$

for every $\theta \in K$. By putting $\theta = 0$, and noticing that

$$t'_{n,r}(0) = 0 \qquad \text{and} \qquad c\sqrt{\frac{n}{\Delta}} \le \frac{|t_{n,r}(0)|}{||t_{n,r}||_{L_1}},$$

we get

$$|g'(0)| \le c\sqrt{\frac{2n}{\Delta}}\int_{-\pi}^{\pi} g(\theta)||t_{n,r}||_{L_1}^{-1}t_{n,r}(\theta)\,d\theta. \tag{5.7}$$

Now let $t \in \mathcal{T}_n(r)$ and $\tau \in K$ be fixed. On applying (5.7) to $g \in \mathcal{T}_n(r)$ defined by $g(\theta) := t(\theta+\tau)$, we conclude that

$$|t'(\tau)| \le c\sqrt{\frac{2n}{\Delta}}\int_{-\pi}^{\pi} g(\theta)||t_{n,r}||_{L_1}^{-1}t_{n,r}(\theta-\tau)\,d\theta.$$

Hence

$$\sqrt{\frac{\Delta}{n}}|t'(\tau)| \le c\int_{-\pi}^{\pi} g(\theta)||t_{n,r}||_{L_1}^{-1}t_{n,r}(\theta-\tau)\,d\theta. \tag{5.8}$$

Since

$$\int_{-\pi}^{\pi} ||t_{n,r}||_{L_1}^{-1}t_{n,r}(\theta-\tau)\,d\theta = 1, \tag{5.9}$$

Jensen's inequality and (5.8) imply that

$$\chi\left(\sqrt{\frac{n}{\Delta}}|t'(\tau)|\right) \le \int_{-\pi}^{\pi} \chi(cg(\theta))||t_{n,r}||_{L_1}^{-1}t_{n,r}(\theta-\tau)\,d\theta.$$

If we integrate both sides with respect to τ, Fubini's theorem and (5.9) (on interchanging the role of θ and τ) yield the inequality of the theorem. The following corollary is an obvious consequence of Theorem 5 applied with $p = 2$.

Corollary. *There is an absolute constant $c > 0$ such that every real trigonometric polynomial of the form*

$$Q(t) = a_0 + \sum_{k=1}^{n}(a_k\cos kt + b_k\sin kt)$$

has at least one zero in the strip $\{z \in \mathbf{C} \mid \operatorname{Im} z < cr\}$ *assuming*

$$r = r(Q) := n \frac{\sum_{k=1}^{n}(a_k^2 + b_k^2)}{\sum_{k=1}^{n} k^2(a_k^2 + b_k^2)} \leq 1.$$

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