A SIMPLE PROOF OF "FAVARD'S THEOREM" ON THE UNIT CIRCLE

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Abstract. A very short constructive proof is given for the unit circle analogue of the "Favard Theorem" on the orthogonality of a system of polynomials satisfying a Szegő type recurrence relation.

In what follows we will adopt the following notation. D is the open unit disk, that is $D = \{z \in \boldsymbol{C} : |z| < 1\}$, $T = \partial D$ is the unit circle. For a given polynomial Π_k of degree k its reverse Π_k^* is defined by $\Pi_k^*(z) = z^k \overline{\Pi_k(1/\overline{z})}$. Let $\{\varphi_n(d\mu)_{n=0}^{\infty}\}$ be the orthonormal polynomials corresponding to a given finite positive Borel measure μ on T with infinite support, that is

$$\varphi_n(d\mu, z) = \kappa_n(d\mu)z^n + \cdots + \varphi_n(d\mu, 0), \qquad \kappa_n > 0, \tag{1}$$

and

$$\frac{1}{2\pi}\int_T \varphi_n(d\mu, e^{i\theta})\overline{\varphi_m(d\mu, e^{i\theta})}d\mu(\theta) = \delta_{nm}, \qquad n, m \geq 0. \tag{2}$$

Let $\Phi_n(d\mu) = \varphi_n(d\mu)/\kappa_n(d\mu)$ denote the monic orthogonal polynomials. Then they satisfy the the Szegő recursion

$$\Phi_n(d\mu, z) = z\Phi_{n-1}(d\mu, z) + \Phi_n(d\mu, 0)\Phi_{n-1}^*(d\mu, z), \qquad n = 1, 2, \ldots, \tag{3}$$

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where $\Phi_0(d\mu, z) = 1$. It is well known that

$$|\Phi_n(d\mu, 0)| < 1, \qquad n = 1, 2, \ldots, \tag{4}$$

and

$$\kappa_n^2(d\mu) = \sum_0^n |\varphi_k(d\mu, 0)|^2 = \prod_1^n \frac{1}{1 - |\Phi_k(d\mu, 0)|^2}. \tag{5}$$

The so called "Favard Theorem" on the real line is about the orthogonality of a system of polynomials which satisfies a three-term recurrence with appropriate coefficients, and its following cousin on the unit circle is also well known (cf. [3, Theorem 8.1, p. 156] and [4, Theorem 8.3, p. 140]).

Theorem. *Assume $\{\epsilon_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $|\epsilon_n| < 1$ for $n = 1, 2, \ldots$. Let $\{\Phi_n\}_{n=0}^{\infty}$ satisfy the Szegő recursion*

$$\Phi_n(z) = z\Phi_{n-1}(z) + \epsilon_n \Phi_{n-1}^*(z), \qquad \Phi_0(z) = 1, \tag{6}$$

and let φ_n be defined by

$$\varphi_n(z) = \kappa_n \Phi_n(z) \quad \textit{where} \quad \kappa_0 = 1 \quad \textit{and} \quad \kappa_n = 1/\prod_1^n \sqrt{1 - |\epsilon_k|^2}, \quad n = 1, 2, \ldots. \tag{7}$$

Then there exists a unique finite positive Borel measure μ on T with infinite support such that we have $\varphi_n = \varphi_n(d\mu)$, that is $\{\varphi_n\}_{n=0}^{\infty}$ is orthonormal with respect to μ.

The purpose of this paper is to give a very short constructive proof of the above "Favard Theorem". The basic idea of our proof can be traced back to a series of papers by A. Máté, P. Nevai and V. Totik where weak and strong convergence properties of $\frac{1}{|\varphi_n|^2}$ have been shown to play a crucial role in a variety of problems related to the extension of Szegő's theory of orthogonal polynomials on the unit circle (cf. [5] and the references therein).

After the completion of the manuscript we realized that our method had been used earlier by P. Delsarte, Y. V. Genin and Y. G. Kamp in [1, Theorem 1.5, p. 155] who consider the matrix-valued case. Their beautiful IEEE paper must have avoided the attention of many mathematicians, and none of the people we contacted have been aware of the extensive use of weak convergence properties of (matrix-valued) orthogonal polynomials in [1].

Lemma 1. *Suppose $\{\epsilon_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that $|\epsilon_n| < 1$ for $n = 1, 2, \ldots$. Let φ_n and Φ_n be constructed by formulas (6) and (7). Then (i) all the zeros of φ_n are in the unit disk D, (ii) $\kappa_n^2 = \kappa_{n-1}^2/(1 - |\Phi_n(0)|^2) = \kappa_{n-1}^2 + |\varphi_n(0)|^2$, and (iii) $\kappa_n \varphi_n(z) = \kappa_{n-1} z \varphi_{n-1}(z) + \varphi_n(0)\varphi_n^*(z)$.*

Lemma 2. *Let $\{\varphi_n\}_{n=0}^{\infty}$ be constructed by (6) and (7) with $|\epsilon_n| < 1$. Then*

$$\frac{1}{2\pi} \int_T \varphi_k(e^{i\theta}) \overline{\varphi_j(e^{i\theta})} \frac{d\theta}{|\varphi_n(e^{i\theta})|^2} = \delta_{jk}, \qquad 0 \le j \le k \le n < \infty.$$

After these two lemmas, we are ready for the

Proof of the Theorem. First we prove the existence of the measure μ. Since the functions

$$\mu_n(\tau) := \int_0^\tau d\mu_n(\theta) := \int_0^\tau \frac{d\theta}{|\varphi_n(e^{i\theta})|^2}$$

are all increasing and uniformly bounded (cf. Lemma 2 applied with $j = k = 0$), by Helly's selection and convergence theorems there exist a subsequence $\{n_k\}$ and an increasing function μ on $[0, 2\pi)$, such that

$$\lim_{k\to\infty} \mu_{n_k}(\theta) = \mu(\theta)$$

and

$$\lim_{k\to\infty} \frac{1}{2\pi} \int_T f(e^{i\theta}) d\mu_{n_k}(\theta) = \int_0^{2\pi} f(e^{i\theta}) d\mu(\theta)$$

for every $f \in C(T)$. Hence,

$$\frac{1}{2\pi} \int_T \varphi_k(e^{i\theta}) \overline{\varphi_j(e^{i\theta})} d\mu(\theta) = \lim_{n\to\infty} \frac{1}{2\pi} \int_T \varphi_k(e^{i\theta}) \overline{\varphi_j(e^{i\theta})} \frac{d\theta}{|\varphi_n(e^{i\theta})|^2} = \delta_{kj}.$$

The uniqueness directly follows from the unique representation of bounded linear functionals on $C(T)$ (F. Riesz). $\Box$

For the sake of completeness we also include the proof of the two lemmas.

Proof of Lemma 1. (i) We will use induction to prove that all zeros of $\Phi_n^*(z)$ are in $\{z : |z| > 1\}$. If $n = 1$ then $\Phi_1(z) = z + \epsilon_1$ so $\Phi_1^*(z) = \overline{\epsilon}_1 z + 1$ and $|-1/\overline{\epsilon}_1| > 1$ because $|\epsilon_1| < 1$. Hence (i) is true for $n = 1$. Suppose (i) holds for $n = k$ then $|\Phi_k(z)/\Phi_k^*(z)| \le 1$ because $\Phi_k^*(z) \ne 0$ on the closed unit disk D∪T and $|\Phi_k(z)/\Phi_k^*(z)| = 1$ on T. Hence, using $|\epsilon_k| < 1$ and the reverse of (6) we obtain

$$\Phi_{k+1}^*(z) = \Phi_k^*(z)(1 + \overline{\epsilon_k} z \Phi_k(z)/\Phi_k^*(z)) \ne 0, \qquad |z| \le 1.$$

(ii) By definition, $\kappa_n^2 = \kappa_{n-1}^2/(1 - |\epsilon_n|^2)$. In view of (6) $\Phi_n(0) = \epsilon_n$ so that

$$\kappa_n^2 = \kappa_{n-1}^2 + \kappa_n^2 |\Phi_n(0)|^2 = \kappa_{n-1}^2 + |\varphi_n(0)|^2.$$

(iii) By (6) we have $\Phi_n(0) = \epsilon_n$ and

$$\Phi_n(z) = z\Phi_{n-1}(z) + \Phi_n(0)\Phi_{n-1}^*(z).$$

Taking the reverse of Φ_n in the above formula we obtain

$$\Phi_n^*(z) = \Phi_{n-1}^*(z) + \overline{\Phi_n(0)} z \Phi_{n-1}(z).$$

Eliminating Φ_{n-1}^* from above two formulas we get

$$\Phi_n(z) = z\Phi_{n-1}(z)[1-|\Phi_n(0)|^2] + \Phi_n(0)\Phi_n^*(z).$$

Noting that $1-|\Phi_n(0)|^2 = \kappa_{n-1}^2/\kappa_n^2$ and $\kappa_n\Phi_n = \varphi_n$, we finally obtain $\kappa_n\varphi_n(z) = \kappa_{n-1}z\varphi_{n-1}(z) + \varphi_n(0)\varphi_n^*(z)$. $\square$

Proof of Lemma 2. This is based on the proof of [1, Theorem 5.2.1, p. 198-199], and it uses backward induction for $k = n,\ n-1,\ \ldots, 1, 0$. For $k = n$ we have

$$\frac{1}{2\pi}\int_T |\varphi_n(e^{i\theta})|^2 \frac{d\theta}{|\varphi_n(e^{i\theta})|^2} = 1.$$

For $0 \le j \le n-1$ we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_T \varphi_n(e^{i\theta})\overline{\varphi_j(e^{i\theta})}\frac{d\theta}{|\varphi_n(e^{i\theta})|^2} &= \frac{1}{2\pi}\int_T \frac{\overline{\varphi_j(e^{i\theta})}}{\overline{\varphi_n(e^{i\theta})}}d\theta \\ = \frac{1}{2\pi}\int_T \frac{e^{(n-j)i\theta}\varphi_j^*(e^{i\theta})}{\varphi_n^*(e^{i\theta})}d\theta &= \left.\frac{e^{(n-j)i\theta}\varphi_j^*(e^{i\theta})}{\varphi_n^*(e^{i\theta})}\right|_{z=0} = 0,\end{aligned}$$

since by (i) in Lemma 1 $\varphi_n^*(z) \neq 0 \quad (|z| \le 1)$ and so $z^{n-j}\varphi_j^*(z)/\varphi_n^*(z)$ is analytic in $|z| \le 1$ and it vanishes at 0. Now assuming that the lemma holds for some $k \le n$, that is

$$\frac{1}{2\pi}\int_T \varphi_k(e^{i\theta})\overline{\varphi_j(e^{i\theta})}\frac{d\theta}{|\varphi_n(e^{i\theta})|^2} = \delta_{jk}, \qquad 0 \le j \le k,$$

we will to prove the lemma for $k-1$. From (iii) of Lemma 1 and from the above orthogonality we obtain

$$\begin{aligned}&\frac{\kappa_k^2}{2\pi}\int_T |\varphi_k(e^{i\theta})|^2\frac{d\theta}{|\varphi_n(e^{i\theta})|^2} \\ &= \frac{\kappa_{k-1}^2}{2\pi}\int_T |\varphi_{k-1}(e^{i\theta})|^2\frac{d\theta}{|\varphi_n(e^{i\theta})|^2} + \frac{|\varphi_k(0)|^2}{2\pi}\int_T |\varphi_k^*(e^{i\theta})|^2\frac{d\theta}{|\varphi_n(e^{i\theta})|^2}.\end{aligned}$$

Noting that $|\varphi_k^*(z)| = |\varphi_k(z)|$ on T and applying (ii) of Lemma 1, we get

$$\frac{1}{2\pi}\int_T |\varphi_{k-1}(e^{i\theta})|^2\frac{d\theta}{|\varphi_n(e^{i\theta})|^2} = \frac{\kappa_k^2 - |\varphi_k(0)|^2}{\kappa_{k-1}^2} = 1.$$

On the other hand, for $0 \le j \le k-2$ we have

$$\begin{aligned}&\frac{\kappa_{k-1}}{2\pi}\int_T \varphi_{k-1}(e^{i\theta})\overline{\varphi_j(e^{i\theta})}\frac{d\theta}{|\varphi_n(e^{i\theta})|^2} \\ &= \frac{\kappa_k}{2\pi}\int_T \varphi_k(e^{i\theta})\overline{e^{i\theta}\varphi_j(e^{i\theta})}\frac{d\theta}{|\varphi_n(e^{i\theta})|^2} - \frac{\varphi_k(0)}{2\pi}\int_T \varphi_k^*(e^{i\theta})\overline{e^{i\theta}\varphi_j(e^{i\theta})}\frac{d\theta}{|\varphi_n(e^{i\theta})|^2} \\ &= -\frac{\varphi_k(0)}{2\pi}\int_T \overline{\varphi_k(e^{i\theta})}\varphi_j^*(e^{i\theta})e^{(k-j-1)i\theta}\frac{d\theta}{|\varphi_n(e^{i\theta})|^2} = 0.\end{aligned}$$

This completes this proof. □

It is possible to prove Lemma 2 without induction as well. Such a proof, however, requires the intruduction of other solutions of the matrix version of the recurrence equation (3), and, consequently, the proof becomes somewhat longer.

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Finally, we hope that the readers of this paper will move on to study [1] by P. Delsarte, Y. V. Genin and Y. G. Kamp which is a rich source of ideas that have not fully been exploited yet by the community of experts on orthogonal polynomials.

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