FLATNESS OF CONJUGATE RECIPROCAL UNIMODULAR POLYNOMIALS

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Abstract. A polynomial is called unimodular if each of its coefficients is a complex number of modulus 1. A polynomial P of the form $P(z) = \sum_{j=0}^{n} a_j z^j$ is called conjugate reciprocal if $a_{n-j} = \overline{a}_j$, $a_j \in \mathbb{C}$ for each $j = 0, 1, \dots, n$. Let ∂D be the unit circle of the complex plane. We prove that there is an absolute constant $\varepsilon > 0$ such that

$$\max_{z \in \partial D} |f(z)| \geq (1+\varepsilon)\sqrt{4/3}\, m^{1/2}$$

for every conjugate reciprocal unimodular polynomial f of degree m and for all sufficiently large m. We also prove that there is an absolute constant $\varepsilon > 0$ such that

$$M_q(f') \leq \exp(\varepsilon(q-2)/q) \left(\frac{m(m+1)(2m+1)}{6} \right)^{1/2}, \qquad 1 \leq q < 2\,,$$

and

$$M_q(f') \geq \exp(\varepsilon(q-2)/q) \left(\frac{m(m+1)(2m+1)}{6} \right)^{1/2}, \qquad 2 < q\,,$$

for every conjugate reciprocal unimodular polynomial f of degree m and for all sufficiently large m, where

$$M_q(f')) := \left(\frac{1}{2\pi} \int_0^{2\pi} |f'(e^{it})|^q \, dt \right)^{1/q}, \qquad q > 0\,.$$

1. Introduction

Let $\mathcal{T}_n$ be the set of all real trigonometric polynomials of degree at most n. Let $\mathcal{P}_n^c$ be the set of all algebraic polynomials of degree at most n with complex coefficients. Throughout this paper it will be comfortable for us to denote an appropriate period $[a, a+2\pi)$ by K.

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Let ∂D be the unit circle of the complex plane. Let

$$\mathcal{A}_n := \left\{ Q : Q(t) = \sum_{j=1}^{n} \cos(jt + \gamma_j)\,, \quad \gamma_j \in \mathbb{R} \right\}$$

and

$$\mathcal{B}_{n+1/2} := \left\{ Q : Q(t) = \sum_{j=0}^{n} \cos\left(\frac{2j+1}{2}\, t + \gamma_j\right), \quad \gamma_j \in \mathbb{R} \right\}.$$

We use the notation

$$\|Q\|_p := \left(\frac{1}{2\pi} \int_K |Q(t)|^p \, dt \right)^{1/p}, \qquad p > 0\,,$$

and

$$\|Q\|_\infty := \max_{t \in K} |Q(t)|\,.$$

The Bernstein–Szegő inequality (see p. 232 in [7], for instance) gives that

$$|Q'(t)|^2 + n^2 |Q(t)|^2 \le n^2 \|Q\|_\infty^2\,, \qquad Q \in \mathcal{T}_n\,, \quad t \in \mathbb{R}\,.$$

Integrating the left hand side on the period and using Parseval's formula we obtain

$$\frac{n(n+1)(2n+1)}{12} + \frac{n^3}{2} \le n^2 \|Q\|_\infty^2\,, \qquad Q \in \mathcal{A}_n\,,$$

and hence

$$\|Q\|_\infty \ge \sqrt{4/3}\,\sqrt{n/2}\,, \qquad Q \in \mathcal{A}_n\,. \tag{1.1}$$

One of the highlights of this paper is to improve (1.1) by showing that there is an absolute constant $\varepsilon > 0$ such that

$$\|Q\|_\infty \ge (1+\varepsilon)\sqrt{4/3}\,\sqrt{n/2}\,, \qquad Q \in \mathcal{A}_n\,.$$

for all sufficiently large n. Let

$$\mathcal{K}_m := \left\{ P : P(z) = \sum_{j=0}^{n} a_j z^j\,, \quad a_j \in \mathbb{C}\,, \ |a_j| = 1\,, \ j = 0, 1, \dots, m \right\}$$

be the set of all unimodular polynomials of degree m. Associated with an algebraic polynomial P of the form

$$P(z) = \sum_{j=0}^{m} a_j z^j\,, \qquad a_j \in \mathbb{C}\,, \quad a_m \ne 0\,,$$

let

$$\overline{P}(z) := \sum_{j=0}^{m} \overline{a}_j z^j \qquad \text{and} \qquad P^*(z) := z^m \overline{P}(1/z)\,.$$

The polynomial P of degree m is called conjugate reciprocal if $P^* = P$. The classes $\mathcal{A}_n$, $\mathcal{B}_{n+1/2}$, and $\mathcal{K}_m$ and flatness properties of their elements were studied by many authors, see [1–40], for instance. Let

$$M_q(f) := \|f(e^{it})\|_q = \left(\frac{1}{2\pi}\int_K |f(e^{it})|^q\,dt\right)^{1/q}, \qquad q \in (0,\infty)\,,$$

and

$$M_\infty(f) := \sup_{t\in K} |f(e^{it})|\,.$$

There is a beautiful short argument to see that

$$M_\infty(f) \geq \sqrt{4/3}\, m^{1/2} \tag{1.2}$$

for every conjugate reciprocal unimodular polynomial $f \in \mathcal{K}_m$. Namely, Parseval's formula gives

$$M_\infty(f') \geq M_2(f') = \left(\frac{m(m+1)(2m+1)}{6}\right)^{1/2}, \qquad f \in \mathcal{K}_m\,.$$

Combining this with Malik's extension of Lax's Bernstein-type inequality

$$M_\infty(f') \leq \frac{m}{2}\, M_\infty(f)$$

valid for all conjugate reciprocal algebraic polynomials $f \in \mathcal{P}_m^c$ (see p. 438 in [7], for instance), we obtain

$$M_\infty(f) \geq \frac{2}{m}\left(\frac{m(m+1)(2m+1)}{6}\right)^{1/2} \geq \sqrt{4/3}\, m^{1/2}$$

for all conjugate reciprocal unimodular polynomials $f \in \mathcal{K}_m$. One of the highlights of this paper is to improve (1.2) by showing that there is an absolute constant $\varepsilon > 0$ such that

$$M_\infty(f) \geq (1+\varepsilon)\sqrt{4/3}\, m^{1/2}\,,$$

for every conjugate reciprocal unimodular polynomial $f \in \mathcal{K}_m$ and for all sufficiently large m. We also prove that there is an absolute constant $\varepsilon > 0$ such that

$$M_q(f') \leq \exp(\varepsilon(q-2)/q)\left(\frac{m(m+1)(2m+1)}{6}\right)^{1/2}, \qquad 1 \leq q < 2\,,$$

and

$$M_q(f') \geq \exp(\varepsilon(q-2)/q)\left(\frac{m(m+1)(2m+1)}{6}\right)^{1/2}, \qquad 2 < q\,,$$

for every conjugate reciprocal unimodular polynomial of degree m and for all sufficiently large m. See Theorem 2.7.

2. New Results

Theorem 2.1. *Let $Q \in \mathcal{A}_n$ and $P = (Q')^2 + n^2Q^2$. There is an absolute constant $\delta > 0$ such that*

$$\|P\|_{1/2} \leq (1-\delta)\|P\|_1$$

for all sufficiently large n.

Theorem 2.1*. *Let $Q \in \mathcal{B}_{n+1/2}$ and $P = (Q')^2 + (n+1/2)^2Q^2$. There is an absolute constant $\delta > 0$ such that*

$$\|P\|_{1/2} \leq (1-\delta)\|P\|_1\,.$$

for all sufficiently large n.

Theorem 2.2. *Let $Q \in \mathcal{A}_n$ and $P = (Q')^2 + n^2Q^2$. There is an absolute constant $\delta > 0$ such that*

$$\|P\|_\infty \geq (1+\delta)\|P\|_1$$

for all sufficiently large n.

Theorem 2.2*. *Let $Q \in \mathcal{B}_{n+1/2}$ and $P = (Q')^2 + (n+1/2)^2Q^2$. There is an absolute constant $\delta > 0$ such that*

$$\|P\|_\infty \geq (1+\delta)\|P\|_1$$

for all sufficiently large n.

Theorem 2.3. *There is an absolute constant $\delta > 0$ such that*

$$\|Q\|_\infty \geq (1+\delta)\sqrt{4/3}\,\sqrt{n/2}$$

for every $Q \in \mathcal{A}_n$ and for all sufficiently large n.

Theorem 2.3*. *There is an absolute constant $\delta > 0$ such that*

$$\|Q\|_\infty \geq (1+\delta)\sqrt{4/3}\,\sqrt{n/2}$$

for every $Q \in \mathcal{B}_{n+1/2}$ and for all sufficiently large n.

Theorem 2.4. *There is an absolute constant $\varepsilon > 0$ such that*

$$M_1(f') \leq (1-\varepsilon)\sqrt{1/3}\,m^{3/2}$$

for every conjugate reciprocal unimodular polynomial $f \in \mathcal{K}_m$ and for all sufficiently large m.

Theorem 2.5. *There is an absolute constant $\varepsilon > 0$ such that*

$$M_\infty(f') \geq (1+\varepsilon)\sqrt{1/3}\,m^{3/2}$$

for every conjugate reciprocal unimodular polynomial $f \in \mathcal{K}_m$ and for all sufficiently large m.

Theorem 2.6. *There is an absolute constant* $\varepsilon > 0$ *such that*

$$M_\infty(f) \geq (1+\varepsilon)\sqrt{4/3}\, m^{1/2}$$

for every conjugate reciprocal unimodular polynomial $f \in \mathcal{K}_m$ *and for all sufficiently large* m*.*

Theorem 2.7. *There is an absolute constant* $\varepsilon > 0$ *such that*

$$M_q(f') \leq \exp(\varepsilon(q-2)/q) \left(\frac{m(m+1)(2m+1)}{6}\right)^{1/2}, \qquad 1 \leq q < 2\,,$$

and

$$M_q(f') \geq \exp(\varepsilon(q-2)/q) \left(\frac{m(m+1)(2m+1)}{6}\right)^{1/2}, \qquad 2 < q\,,$$

for every conjugate reciprocal unimodular polynomial $f \in \mathcal{K}_m$ *and for all sufficiently large* m*.*

The above results were well known before without the absolute constants $\delta > 0$ and $\varepsilon > 0$,respectively.

Remark 2.1. *The factor* $(1+\delta)$ *in Theorem 2.2 cannot be replaced by* $(1+\delta)3/2$*.*

Remark 2.2. *The factor* $(1+\delta)\sqrt{4/3}$ *in Theorem 2.3 cannot be replaced by* $(1+\delta)\sqrt{2}$*.*

Remark 2.3. *The factor* $(1+\varepsilon)\sqrt{1/3}$ *in Theorem 2.5 cannot be replaced by* $(1+\varepsilon)\sqrt{1/2}$*.*

Remark 2.4. *The factor* $(1+\varepsilon)\sqrt{4/3}$ *in Theorem 2.6 cannot be replaced by* $(1+\varepsilon)\sqrt{2}$*.*

A polynomial $f \in \mathcal{P}_m^c$ of degree m is called skew-reciprocal if $f^*(z) = f(-z)$. A polynomial $f \in \mathcal{P}_m^c$ of degree m is called plain-reciprocal if $f^* = \overline{f}$, that is, $f(z) = z^m f(1/z)$ for all $z \in \mathbb{C} \setminus \{0\}$. Observe that Corollary 2.8 in [28] may be formulated as follows.

Remark 2.5. *There is an absolute constant* $\varepsilon > 0$ *such that*

$$\max_{z \in \partial D} |f'(z)| - \min_{z \in \partial D} |f'(z)| \geq \varepsilon m^{3/2}$$

for all conjugate reciprocal, plain-reciprocal, and skew-reciprocal unimodular polynomials $f \in \mathcal{K}_m$ *and for all sufficiently large* m*.*

Observe that for conjugate reciprocal unimodular polynomials Theorem 2.5 is stronger than Remark 2.5

Problem 2.1. *Is there an absolute constant* $\varepsilon > 0$ *such that*

$$M_\infty(f') \geq (1+\varepsilon)\sqrt{1/3}\, m^{3/2}$$

holds for all plain-reciprocal and skew-reciprocal unimodular polynomials $f \in \mathcal{K}_m$ *and for all sufficiently large* m*?*

Problem 2.2. *Is there an absolute constant $\varepsilon > 0$ such that*

$$M_\infty(f') \geq (1+\varepsilon)\sqrt{1/3}\, m^{3/2}$$

or at least

$$\max_{z\in\partial D} |f'(z)| - \min_{z\in\partial D} |f'(z)| \geq \varepsilon m^{3/2}$$

holds for all unimodular polynomials $f \in \mathcal{K}_m$ and for all sufficiently large m?

Our method to prove Theorem 2.5 does not seem to work for all unimodular polynomials $f \in \mathcal{K}_m$. In an e-mail communication several years ago B. Saffari speculated that the answer to Problem 2.2 is no. However we do not know the answer even to Problem 2.1.

Let $\mathcal{L}_m$ be the collection of all polynomials of degree m with each of their coefficients in $\{-1, 1\}$. The elements of $\mathcal{L}_m$ are called Littlewood polynomials of degree m.

Problem 2.3. *Is there an absolute constant $\varepsilon > 0$ such that*

$$M_\infty(f') \geq (1+\varepsilon)\sqrt{1/3}\, m^{3/2}$$

or at least

$$\max_{z\in\partial D} |f'(z)| - \min_{z\in\partial D} |f'(z)| \geq \varepsilon m^{3/2}$$

holds for all Littlewood polynomials $f \in \mathcal{L}_m$ and for all sufficiently large m?

The following problem due to Erdős [29] is open for a long time.

Problem 2.4. *Is there an absolute constant $\varepsilon > 0$ such that*

$$M_\infty(f) \geq (1+\varepsilon) m^{1/2}$$

or at least

$$\max_{z\in\partial D} |f(z)| - \min_{z\in\partial D} |f(z)| \geq \varepsilon m^{1/2}$$

holds for all Littlewood polynomials $f \in \mathcal{L}_m$ and for all sufficiently large m?

The same problem may be raised only for all skew-reciprocal Littlewood polynomials $f \in \mathcal{L}_m$, and as far as we know, it is also open.

3. Lemmas

Let $m(A)$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. The following lemma is due to Littlewood, see Theorem 1 in [34].

Lemma 3.1. *Let $R \in \mathcal{T}_n$ be of the form*

$$R(t) = R_n(t) = \sum_{j=1}^{n} a_j \cos(jt + \gamma_j)\,, \qquad a_j\,, \gamma_j \in \mathbb{R}\,, \quad j = 1, 2, \ldots, n\,.$$

Let $s_m := \sum_{j=1}^m a_j^2$, $m = 1, 2, \dots, n$, *and let* $\mu := \|R\|_2$, *that is,* $\mu^2 = s_n$. *Suppose*

$$\|R\|_1 \geq c\mu \,,$$

where $c > 0$ *is a constant (necessarily not greater than* 1*). Suppose also that the coefficients of* R *satisfy*

$$s_{[n/h]}/s_n = \mu^{-2} \sum_{1 \leq j \leq n/h} a_j^2 \leq 2^{-9} c^6$$

for some constant $h > 0$. *Let* $V = 2^{-5} c^3$. *Then there exists a constant* $B > 0$ *depending only on* c *and* h *such that*

$$m(\{t \in K : v_1 \mu \leq |R(t)| \leq v_2 \mu\}) \geq B(v_2 - v_1)^2$$

for every v_1 *and* v_2 *such that* $-V \leq v_1 < v_2 \leq V$.

Lemma 3.2. *Associated with* $Q \in \mathcal{T}_n$ *we define the sets*

$$E_\delta := \{t \in K : |Q'(t)| < \delta n^{3/2}\}$$

and

$$F_\delta := \{t \in K : |Q''(t)| \leq \delta^{1/2} n^{5/2}\} \,.$$

We have

$$m(E_\delta \setminus F_\delta) \leq 8\delta^{1/2} \,.$$

Proof of Lemma 3.2. Observe that $E_\delta \setminus F_\delta$ is the union of at most $4n$ pairwise disjoint open subintervals of the period. Let these intervals be (x_j, y_j), $j = 1, 2, \dots, \mu$, where $\mu \leq 4n$. By the Mean Value Theorem we can deduce that there are $\xi_j \in (x_j, y_j)$ such that

$$2\delta n^{3/2} \geq |Q'(y_j) - Q'(x_j)| = |Q''(\xi_j)|(y_j - x_j) \geq \delta^{1/2} n^{5/2} (y_j - x_j) \,,$$

and hence

$$y_j - x_j \leq 2\delta^{1/2} n^{-1} \,, \qquad j = 1, 2, \dots, \mu \,.$$

Hence

$$m(E_\delta \setminus F_\delta) = \sum_{j=1}^{\mu} (y_j - x_j) \leq \mu (2\delta^{1/2} n^{-1}) \leq 8\delta^{1/2} \,.$$

□

Lemma 3.3. *Let* $Q \in \mathcal{A}_n$, $P := (Q')^2 + n^2 Q^2$, $\delta \in (0, 1)$, *and*

$$G_\delta := \{t \in K : \big| |P(t)|^{1/2} - \|P\|_1^{1/2} \big| \leq \delta^{1/4} \|P\|_1^{1/2}\} \,.$$

Suppose

$$\|P\|_{1/2} \geq (1 - \delta) \|P\|_1 \,. \tag{3.1}$$

Then

$$m(K \setminus G_\delta) \le 4\pi\delta^{1/2}\,.$$

Proof of Lemma 3.3. Using (3.1) we have

$$\|P\|_{1/2}^{1/2} \ge (1-\delta)^{1/2}\|P\|_1^{1/2} \ge (1-\delta)\|P\|_1^{1/2}\,.$$

Hence

$$\begin{aligned}\int_K \big||P(t)|^{1/2} - \|P\|_1^{1/2}\big|^2\,dt &= 4\pi\,\|P\|_1^{1/2}\big(\|P\|_1^{1/2} - \|P\|_{1/2}^{1/2}\big)\\ &\le 4\pi\|P\|_1^{1/2}\delta\|P\|_1^{1/2} = 4\pi\delta\|P\|_1\,.\end{aligned}$$

Letting $a := m(K \setminus G_\delta)$ we have

$$a\delta^{1/2}\|P\|_1 \le 4\pi\delta\|P\|_1\,,$$

and the lemma follows. □

Lemma 3.4. *Let $Q \in \mathcal{A}_n$, $P := (Q')^2 + n^2Q^2$, and $\|P\|_{1/2} \ge \frac{31}{32}\|P\|_1$. Then $R := n^2Q^2 + Q''$ satisfies the assumptions of Lemma 3.1 with $c := 1/32$ and $h = 2^9 32^6$ for all sufficiently large n.*

Proof of Lemma 3.4. Let $n \ge 3$. Observe that

$$R(t) = \sum_{j=1}^n a_j\cos(jt+\gamma_j)\,, \qquad a_j := n^2 - j^2, \quad \gamma_j \in \mathbb{R}\,, \quad j = 1,2,\dots,n\,.$$

Then

$$s_n = \mu^2 = \|R\|_2^2 = \frac{1}{2}\sum_{j=1}^n (n^2-j^2)^2\,,$$

hence

$$\frac{n^5}{6} \le s_n = \mu^2 \le \frac{n^5}{2}\,.$$

Using Parseval's formula, we have

$$\|P\|_1 = \frac{1}{2}\sum_{j=1}^n (j^2+n^2) \ge \frac{2n^3}{3}\,,$$

(3.2)
$$\|Q''\|_1 \le \|Q''\|_2 = \left(\frac{1}{2}\sum_{j=1}^n j^4\right)^{1/2} \le (1+o(1))\,\frac{n^{5/2}}{\sqrt{10}}\,,$$

and

$$\|Q'\|_2 = \left(\frac{1}{2}\sum_{j=1}^{n} j^2\right)^{1/2} \leq (1+o(1))\,\frac{n^{3/2}}{\sqrt{6}}\,,$$

and hence

$$\begin{aligned}
\|nQ\|_1 &= \frac{1}{2\pi}\int_K |P(t) - Q'(t)^2|^{1/2}\,dt \\
&\geq \frac{1}{2\pi}\int_K |P(t)|^{1/2}\,dt - \frac{1}{2\pi}\int_K |Q'(t)|\,dt \\
&= \|P\|_{1/2}^{1/2} - \|Q'\|_1 \geq \left(\frac{31}{32}\right)^{1/2} \|P\|_1^{1/2} - \|Q'\|_2 \\
&\geq \frac{31}{32}\sqrt{\frac{2}{3}}\,n^{3/2} - (1+o(1))\sqrt{\frac{1}{6}}\,n^{3/2}\,.
\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3) we conclude

$$\begin{aligned}
\|R\|_1 &= \|n^2Q + Q''\|_1 \geq \|n^2Q\|_1 - \|Q''\|_1 \\
&\geq \frac{31}{32}\sqrt{\frac{2}{3}}\,n^{5/2} - (1+o(1))\sqrt{\frac{1}{6}}\,n^{5/2} - (1+o(1))\,\frac{n^{5/2}}{\sqrt{10}} \\
&\geq \frac{1}{32}\,n^{5/2}
\end{aligned}$$

for all sufficiently large n. Also, $s_{[n/h]} \leq (n/h)n^4 = n^5/h$, hence $s_{[n/h]}/s_n \leq 1/h$. Therefore the assumptions of Lemma 3.1 are satisfied with $c := 1/32$ and $h = 2^9 32^6$. $\square$

4. Proof of the Theorems

Proof of Theorem 2.1. Let $Q \in \mathcal{A}_n$, $P = (Q')^2 + n^2Q^2$, and $R := n^2Q^2 + Q''$. Let $\delta \in (0,1)$. Suppose $\|P\|_{1/2} \geq (1-\delta)\|P\|_1$. Lemma 3.4 states that if $0 < \delta \leq 1/32$ then R satisfies the assumptions of Lemma 3.1 with $c = 1/32$ and $h = 2^9 32^6$. Now let

$$E_\delta := \{t \in K : |Q'(t)| < \delta n^{3/2}\}\,,$$

$$F_\delta := \{t \in K : |Q''(t)| \leq \delta n^{5/2}\}\,,$$

$$G_\delta := \{t \in K : |P(t)^{1/2} - \|P\|_1^{1/2}| \leq \delta^{1/4}\|P\|_1^{1/2}\}\,,$$

and

$$H_\gamma := \{t \in K : \gamma n^{5/2} \leq |R(t)| < 2\gamma n^{5/2}\}\,.$$

Recall that by Parseval's formula we have

$$\|P\|_1 = \frac{n^3}{2} + \frac{n(n+1)(2n+1)}{12}\,. \tag{4.1}$$

Hence, if $t \in G_\delta \cap E_\delta \cap F_\delta$ and the absolute constant $\delta > 0$ is sufficiently small, then

$$\begin{aligned}|R(t)| \geq & n^2|Q(t)| - |Q''(t)| = n(P(t) - Q'(t)^2)^{1/2} - |Q''(t)| \\ \geq & n\left((1-\delta^{1/4})^2\left(\frac{n^3}{2} + \frac{n(n+1)(2n+1)}{12}\right) - \delta^2 n^3\right)^{1/2} - \delta n^{5/2} \\ \geq & \frac{1}{2}\, n^{5/2}\,,\end{aligned}$$

that is,

$$|R(t)| \geq \frac{1}{2}\, n^{5/2}\,, \qquad t \in G_\delta \cap E_\delta \cap F_\delta\,. \tag{4.2}$$

By Lemma 3.2 we have

$$m(E_\delta \setminus F_\delta) \leq 8\delta^{1/2}\,. \tag{4.3}$$

By Lemma 3.3 we have

$$m(K \setminus G_\delta) \leq 4\pi\delta^{1/2}\,. \tag{4.4}$$

Observe that if $0 < \gamma < 1/4$ then (4.2) implies that $H_\gamma \subset K \setminus (G_\delta \cap E_\delta \cap F_\delta)$, hence

$$H_\gamma \cap E_\delta \subset (E_\delta \setminus G_\delta) \cup (E_\delta \setminus F_\delta)\,.$$

Therefore, by (4.3) and (4.4) we can deduce that

$$\begin{aligned}m(H_\gamma \cap E_\delta) \leq & m(E_\delta \setminus G_\delta) + m(E_\delta \setminus F_\delta) \\ \leq & 4\pi\delta^{1/2} + 8\delta^{1/2}\,.\end{aligned} \tag{4.5}$$

By Lemmas 3.1 and 3.4 there are absolute constants $0 < \gamma < 1/4$ and $B > 0$ such that

$$m(H_\gamma) \geq B\gamma^2. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$m(H_\gamma \setminus E_\delta) \geq \frac{1}{2}\, B\gamma^2 \tag{4.7}$$

for all sufficiently small absolute constants $\delta > 0$. Observe that

$$|2Q'(t)R(t)| \geq 2\delta n^{3/2} \gamma n^{5/2} = 2\gamma\delta n^4\,, \qquad t \in H_\gamma \setminus E_\delta\,, \tag{4.8}$$

and

$$P'(t) = 2Q'(t)R(t)\,. \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we obtain

$$m(\{t \in K : |P'(t)| \geq 2\gamma\delta n^4\}) \geq \frac{1}{2}\, B\gamma^2\,,$$

and hence

$$\int_K |P'(t)|\, dt \geq \frac{1}{2} B\gamma^2 (2\gamma\delta n^4) = B\gamma^3\delta n^4\,. \tag{4.10}$$

Now let $\widetilde{P} := P - 2\pi\|P\|_1 \in \mathcal{T}_{2n}$. Then (4.10) can be rewritten as

$$\int_K |\widetilde{P}'(t)|\, dt \geq B\gamma^3\delta n^4\,,$$

and by Bernstein's inequality in L_1 (see p. 390 of [7], for instance), we have

$$2\pi\|\widetilde{P}\|_1 = \int_K |\widetilde{P}(t)|\, dt \geq \frac{1}{2}\, B\gamma^3\delta n^3\,. \tag{4.11}$$

Observe that

(4.12)

$$\begin{aligned}
2\pi\|\widetilde{P}\|_1 &= \int_K |\widetilde{P}(t)|\, dt \\
&= \int_K \big|P(t) - \|P\|_1\big|\, dt \leq \int_K \big|\big(P(t)^{1/2} - \|P\|_1^{1/2}\big)\big(P(t)^{1/2} + \|P\|_1^{1/2}\big)\big|\, dt \\
&\leq \left(\int_K \big|(P(t)^{1/2} - \|P\|_1^{1/2}\big|^2\, dt\right)^{1/2} \left(\int_K \big|(P(t)^{1/2} + \|P\|_1^{1/2}\big|^2\, dt\right)^{1/2} \\
&= 2\pi\big(2\|P\|_1^{1/2}(\|P\|_1^{1/2} - \|P\|_{1/2}^{1/2})\big)^{1/2}\big(2\|P\|_1^{1/2}(\|P\|_1^{1/2} + \|P\|_{1/2}^{1/2})\big)^{1/2} \\
&\leq 4\pi\|P\|_1^{1/2}\big(\|P\|_1 - \|P\|_{1/2}\big)^{1/2} = 4\pi n^{3/2}\big(\|P\|_1 - \|P\|_{1/2}\big)^{1/2}\,.
\end{aligned}$$

Combining (4.11), (4.12), and (4.1), we conclude

$$\begin{aligned}
\|P\|_1 - \|P\|_{1/2} &\geq \left(\frac{2\pi\|\widetilde{P}\|_1}{4\pi n^{3/2}}\right)^2 \geq \left(\frac{B\gamma^3\delta n^{3/2}}{8\pi}\right)^2 \geq \delta^* n^3 \\
&\geq \delta^*\|P\|_1
\end{aligned}$$

with an absolute constant $\delta^* > 0$. □

Proof of Theorem 2.2. By Theorem 2.1 there is an absolute constant $\delta > 0$ such that

$$\begin{aligned}
\|P\|_1 = \frac{1}{2\pi}\int_K |P(t)|\, dt &= \frac{1}{2\pi}\int_K |P(t)|^{1/2}|P(t)|^{1/2}\, dt \leq \frac{1}{2\pi}\int_K |P(t)|^{1/2}\|P\|_\infty^{1/2}\, dt \\
&\leq \|P\|_{1/2}^{1/2}\|P\|_\infty^{1/2} \leq (1-\delta)^{1/2}\|P\|_1^{1/2}\|P\|_\infty^{1/2}\,.
\end{aligned}$$

Hence

$$\|P\|_1^{1/2} \le (1-\delta)^{1/2}\|P\|_\infty^{1/2}\,,$$

and the result follows. □

Proof of Theorem 2.3. The Bernstein–Szegő inequality (see p. 232 of [7], for instance) yields

$$Q'(t)^2 + n^2Q(t)^2 \le n^2\|Q\|_\infty^2, \qquad t \in \mathbb{R}\,,\;\; Q \in \mathcal{A}_n \subset \mathcal{T}_n\,,$$

hence if $P = (Q')^2 + n^2Q^2$, then

$$\|P\|_\infty \le n^2\|Q\|_\infty\,.$$

Hence, using Theorem 2.1 and Parseval's formula we can deduce that

$$\begin{aligned}\|Q\|_\infty^2 \ge& n^{-2}\|P\|_\infty \ge n^{-2}(1+\delta)\|P\|_1 = n^{-2}(1+\delta)\big(\|Q'\|_2^2 + n^2\|Q\|_2^2\big)\\ =& n^{-2}(1+\delta)\left(\frac{n(n+1)(2n+1)}{12} + \frac{n^3}{2}\right) \ge (1+\delta)(4/3)(n/2)\end{aligned}$$

with an absolute constant $\delta > 0$ and the theorem follows. □

The proofs of Theorems 2.1*, 2.2*, and 2.3* are similar to those of Theorems 2.1, 2.2, and 2.3 respectively. The modifications required in the proofs of Theorems 2.1*, 2.2*, and 2.3* are straightforward for the experts and we omit the details.

Proof of Theorem 2.4. First assume that $m = 2n$ is even and $f \in \mathcal{K}_m$ is a conjugate reciprocal unimodular polynomial. Let $f(z) = \sum_{j=0}^m a_jz^j$, where $a_j \in \mathbb{C}$ and $|a_j| = 1$ for each $j = 0, 1, \ldots, m$. As f is conjugate reciprocal, we have

$$a_{m-j} = \overline{a}_j\,, \qquad j = 0, 1, \ldots, m\,,$$

and $a_n \in \{-1, 1\}$, in particular. Let $Q \in \mathcal{A}_n$ be defined by $2Q(t) := e^{-int}f(e^{it}) - a_n$. Then

$$ie^{it}f'(e^{it}) = e^{int}(2Q'(t) + in(2Q(t) + a_n))\,,$$

hence the triangle inequality implies that

$$\begin{aligned}|f'(e^{it})| \le&\; 2|e^{int}(Q'(t) + inQ(t))| + |e^{int}ina_n| = 2|Q'(t) + inQ(t)| + n\\ =&\; 2|P(t)|^{1/2} + n\,,\end{aligned}$$

where $P := (Q')^2 + n^2Q^2$ is the same as in Theorem 2.1, and the theorem follows from Theorem 2.1 as

$$\begin{aligned}M_1(f') \le&\; 2\|P\|_{1/2}^{1/2} + n \le 2(1-\delta)^{1/2}\|P\|_1^{1/2} + n\\ \le&\; 2(1-\delta)^{1/2}\left(\frac{n(n+1)(2n+1)}{12} + \frac{n^3}{2}\right)^{1/2} + n\\ \le&\; 2(1-\delta)^{1/2}\left(\frac{2(n+1)^3}{3}\right)^{1/2} + n\\ \le&\; (1-\delta)^{1/2}\sqrt{1/3}\,m^{3/2} + o(m^{3/2})\,.\end{aligned}$$

Now assume that $m = 2n+1$ is odd and $f \in \mathcal{K}_m$ is a conjugate reciprocal unimodular polynomial. Let $Q \in \mathcal{B}_{n+1/2}$ be defined by $2Q(t) := e^{-imt/2} f(e^{it})$. Then

$$ie^{it} f'(e^{it}) = 2e^{imt/2}(Q'(t) + (im/2)Q(t))$$

implies that

$$\begin{aligned} |f'(e^{it})| &= 2|e^{imt/2}(Q'(t) + (im/2)Q(t))| = 2|Q'(t) + (im/2)Q(t)| \\ &= 2|P(t)|^{1/2}, \end{aligned}$$

where $P := (Q')^2 + (n+1/2)^2 Q^2$ is the same as in Theorem 2.1*, and the theorem follows from Theorem 2.1* and Parseval's formula as

$$\begin{aligned} M_1(f') &\leq 2\|P\|_{1/2}^{1/2} \leq 2(1-\delta)^{1/2} \|P\|_1^{1/2} \\ &\leq 2(1-\delta)^{1/2} \left(\frac{(n+1)(n+2)(2n+3)}{12} + \frac{(n+1)^3}{2} \right)^{1/2} \\ &\leq 2(1-\delta)^{1/2} \left(\frac{2(n+1)^3}{3} \right)^{1/2} o(n^{3/2}) \\ &\leq (1-\delta)^{1/2} \sqrt{1/3}\, m^{3/2} + o(m^{3/2}). \end{aligned}$$

□

Proof of Theorem 2.5. Let $f \in \mathcal{K}_m$ be a conjugate reciprocal unimodular polynomial. By Theorem 2.4 there is an absolute constant $\varepsilon > 0$ such that

$$\begin{aligned} \frac{m(m+1)(2m+1)}{6} &= \big(M_2(f')\big)^2 = \frac{1}{2\pi} \int_K |f'(e^{it})|^2 \, dt = \frac{1}{2\pi} \int_K |f'(e^{it})| |f'(e^{it})| \, dt \\ &\leq \frac{1}{2\pi} \int_K |f'(e^{it})| \max_{\tau \in K} |f'(e^{i\tau})| \, dt \\ &\leq M_1(f') M_\infty(f') \\ &\leq (1-\varepsilon) \sqrt{1/3}\, m^{3/2} M_\infty(f'). \end{aligned}$$

Hence

$$\sqrt{1/3}\, m^{3/2} \leq (1-\varepsilon) M_\infty(f'),$$

and the theorem follows. □

Proof 1 of Theorem 2.6. First assume that $m = 2n$ is even and $f \in \mathcal{K}_m$ is a conjugate reciprocal unimodular polynomial. Let $f(z) = \sum_{j=0}^m a_j z^j$, where $a_j \in \mathbb{C}$ and $|a_j| = 1$ for each $j = 0, 1, \ldots, m$. As f is conjugate reciprocal, we have

$$a_{m-j} = \overline{a}_j, \qquad j = 0, 1, \ldots, m,$$

and $a_n \in \{-1, 1\}$, in particular. Let $Q \in \mathcal{A}_n$ be defined by $2Q(t) = e^{-int} f(e^{it}) - a_n$. Observe that

$$\left| \max_{z \in \partial D} |f(z)| - \|2Q\|_\infty \right| \leq 1 \,,$$

hence the theorem follows from Theorem 2.3. Now assume that $m = 2n+1$ is odd and $f \in \mathcal{K}_m$ is a conjugate reciprocal unimodular polynomial. Let $Q \in \mathcal{B}_{n+1/2}$ be defined by $2Q(t) := e^{-imt/2} f(e^{it})$. Observe that

$$\max_{z \in \partial D} |f(z)| = \|2Q\|_\infty \,,$$

hence the theorem follows from Theorem 2.3*. □

Proof 2 of Theorem 2.6. It is well known (see p. 438 of [7], for instance) that if f is a conjugate reciprocal unimodular polynomial of degree m then $\|f'\|_\infty = (m/2)\, \|f\|_\infty$. Hence the theorem follows from a combination of this and Theorem 2.5. □

Proof of Theorem 2.7. Let $f \in \mathcal{K}_m$ be conjugate reciprocal. Observe that Parseval's formula gives

$$M_2(f') = \left(\frac{m(m+1)(2m+1)}{6} \right)^{1/2} . \tag{4.13}$$

As we will see, both inequalities of the theorem follow from Theorem 2.4 and the following convexity property of the function $h(q) := q \log M_q(g)$ on $(0, \infty)$. Let g be a continuous function on ∂D and let

$$I_q(g) := M_q(g)^q = \frac{1}{2\pi} \int_K |g(e^{it})|^q \, dt \,.$$

Then $h(q) := \log I_q(g) = q \log M_q(g)$ is a convex function of q on $(0, \infty)$. This is a simple consequence of Hölder's inequality. For the sake of completeness, before we apply it, we present the short proof of this fact. We need to see that if $q < r < p$, then

$$I_r(g) \leq I_p(g)^{\frac{r-q}{p-q}} I_q(g)^{\frac{p-r}{p-q}} \,,$$

that is,

$$\left(\frac{1}{2\pi} \int_K |g(e^{it})|^r \, dt \right)^{p-q} \leq \left(\frac{1}{2\pi} \int_K |g(e^{it})|^p \, dt \right)^{r-q} \left(\frac{1}{2\pi} \int_K |g(e^{it})|^q \, dt \right)^{p-r} . \tag{4.14}$$

To see this let

$$\alpha := \frac{p-q}{r-q} \,, \qquad \beta := \frac{p-q}{p-r} \,, \qquad \gamma := \frac{p}{\alpha} \,, \qquad \delta := \frac{q}{\beta} \,,$$

hence $1/\alpha + 1/\beta = 1$ and $\gamma + \delta = r$. Let

$$F(t) := |g(e^{it})|^\gamma = |g(e^{it})|^{\frac{p(r-q)}{p-q}} \,,$$

and

$$G(t) := |g(e^{it})|^{\delta} = |g(e^{it})|^{\frac{q(p-r)}{p-q}}\,.$$

Then by Hölder's inequality we conclude

$$\int_K F(t)G(t)\,dt \le \left(\int_K F(t)^{\alpha}\,dt\right)^{1/\alpha} \left(\int_K G(t)^{\beta}\,dt\right)^{1/\beta}\,,$$

and (4.14) follows.

Let $q \in [1,2)$. Then, using the convexity property of the function $h(q) := q\log M_q(f')$ on $(0,\infty)$, we obtain

$$\frac{2\log M_2(f') - q\log M_q(f')}{2-q} \ge \frac{2\log M_2(f') - \log M_1(f')}{2-1}\,.$$

Combining this with Theorem 2.4 and (4.13) gives the theorem.

Now let $q \in (2,\infty)$. Then, using the convexity property of the function $h(q) := q\log M_q(f')$ on $(0,\infty)$, we obtain

$$\frac{q\log M_q(f') - 2\log M_2(f')}{q-2} \ge \frac{2\log M_2(f') - \log M_1(f')}{2-1}\,.$$

Combining this with Theorem 2.4 and (4.13) gives the theorem. □

Proof of Remark 2.1. Let (f_n) be an ultraflat sequence of unimodular polynomials $f_n \in \mathcal{K}_n$ satisfying $M_\infty(f_n) \le (1+\varepsilon_n)\sqrt{n}$ with a sequence (ε_n) of numbers $\varepsilon_n > 0$ converging to 0. It is shown in [32] that such a sequence (f_n) exists. Let $g_n(z) = zf_{n-1}(z)$. Let $Q_n \in \mathcal{A}_n$ be defined by $2Q_n(t) := \mathrm{Re}(g_n(e^{it}))$. Then the Bernstein–Szegő inequality (see p. 232 in [7], for instance) gives that $P_n := (Q_n')^2 + n^2Q_n^2$ satisfy

$$\|P_n\|_\infty \le n^2\|Q_n\|_\infty^2 \le (1+\varepsilon_n)^2 n^3,$$

while by Parseval's formula we have

$$\|P_n\|_1 = \frac{n^3}{2} + \frac{n(n+1)(2n+1)}{12} \ge \frac{2n^3}{3}\,.$$

□

Proof of Remark 2.2. Let $Q_n \in \mathcal{A}_n$ be the same as in the proof of Remark 2.1. Then

$$\|Q_n\|_\infty \le n^{-1}\|P_n\|_\infty^{1/2} \le (1+\varepsilon_n)n^{1/2}\,.$$

□

Proof of Remark 2.4. Let $f_n \in \mathcal{K}_n$ and $g_n(z) = zf_{n-1}(z)$ be the same as in the proof of Remark 2.1. For $m = 2n$ we define $h_m \in \mathcal{K}_m$ by

$$h_m(z) := z^n(g_n(z) + \overline{g}_n(1/z) + 1)\,.$$

We have

$$M_\infty(h_m) \le 2(1+\varepsilon_n)\sqrt{n} + 1 \le (1+\varepsilon_n)\sqrt{2}\,\sqrt{m} + 1\,.$$

□

Proof of Remark 2.3. For $m = 2n$ let $h_m \in \mathcal{K}_m$ be the same as in the proof of Remark 2.4. Then using the well-known Bernstein-type inequality for conjugate reciprocal polynomials (see p. 438 in [7], for instance), we have

$$M_\infty(h'_m) \le \frac{m}{2}\,(1+\varepsilon_n)\sqrt{2}\,\sqrt{m} \le \frac{1}{\sqrt{2}}\,(1+\varepsilon_n)m^{3/2}\,.$$

□

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