ON THE UNIQUENESS OF THE SOLUTION TO SOME POLYNOMIAL EQUATIONS

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Problem. *Let $n > 1$ be an integer. Let $k_j > 1$ for each $j = 1, 2, \ldots, n$. Show that the equation*

$$\prod_{j=1}^{n} (1 - x^{k_j}) = 1 - x\,.$$

has exactly one solution in the interval $(0,1)$.

Proposed solution. Since $n > 1$, the derivative of the right hand side of the equation vanishes at 0 and 1, so the existence of at least one solution in $(0,1)$ is a consequence of the Intermediate Value Theorem. We have the equation A simple Application of the Intermediate Value Theorem gives that the equation always has at least one zero in the interval $(0,1)$. The non-trivial part of the solution is to show that the equation does not have more than one solution. After a substitution

$$u = -\log(1-x)\,, \qquad e^{-u} = 1 - x\,, \qquad x = 1 - e^{-u}\,,$$

we are lead to the equation

$$\begin{aligned} -u &= \log \prod_{j=1}^{n} (1 - (1 - e^{-u})^{k_j}) \\ &= \sum_{j=0}^{n} \log\,(1 - (1 - e^{-u})^{k_j}) \\ &= \sum_{j=0}^{n} g_j(u)\,, \end{aligned}$$

where

$$g_j(u) := \log\,(1 - (1 - e^{-u})^{k_j})\,.$$

Here

$$g_j'(u) = \frac{-k_j(1-e^{-u})^{k_j-1}e^{-u}}{1 - (1 - e^{-u})^{k_j}}\,.$$

We show that each g_j' is decreasing on $(0,\infty)$. Indeed, using the substitution $y = 1 - e^{-u}$, we have

$$g_j'(u) = \frac{-k_j y^{k_j - 1}(1-y)}{1 - y^{k_j}}\,.$$

Hence an elementary calculus shows that $g_j'(y)$ is a decreasing function of y on $(0,1]$, hence of u on $(0,\infty)$. Thus the right hand side of the equation $-u = \sum_{j=1}^n g_j(u)$ is concave down. Since $g_j(0) = 0$, the right hand side of the equation $-u = \sum_{j=1}^n g_j(u)$ also vanishes at 0, and the result follows. $\square$

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