MARKOV-NIKOLSKII TYPE INEQUALITY FOR ABSOLUTELY MONOTONE POLYNOMIALS OF ORDER k

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Dedicated to the memory of Professor George G. Lorentz on the occasion of his 100th birthday.

Abstract. A function Q is called absolutely monotone of order k on an interval I if $Q(x) \geq 0$, $Q'(x) \geq 0, \ldots, Q^{(k)}(x) \geq 0$, for all $x \in I$. An essentially sharp (up to a multiplicative absolute constant) Markov inequality for absolutely monotone polynomials of order k in $L_p[-1,1]$, $p > 0$, is established. One may guess that the right Markov factor is cn^2/k and, indeed, this turns out to be the case. Moreover, similarly sharp results hold in the case of higher derivatives and Markov-Nikolskii type inequalities. There is a remarkable connection between the right Markov inequality for absolutely monotone polynomials of order k in the supremum norm, and essentially sharp bounds for the largest and smallest zeros of Jacobi polynomials. This is discussed in the last section of this paper.

1. Introduction

Let $\mathcal{P}_n$ denote the collection of all real algebraic polynomials of degree at most n. If f is a function defined on a measurable set A, then let

$$\|f\|_A := \|f\|_{L_\infty A} := \|f\|_{L_\infty(A)} := \sup_{x \in A} \{|f(x)|\} .$$

Let

$$\|f\|_{L_p A} := \|f\|_{L_p(A)} := \left(\int_A |f(x)|^p \, dx \right)^{1/p}, \qquad p > 0 ,$$

whenever the Lebesgue integral exists. A function Q is called absolutely monotone of order k on an interval I if $Q(x) \geq 0$, $Q'(x) \geq 0, \ldots, Q^{(k)}(x) \geq 0$, for all $x \in I$. Let $0 \leq k \leq n$. Observe that $Q \in \mathcal{P}_n$ is absolutely monotone of order k on $[-1,1]$ if and only if it is of the form

$$Q(x) := R(x) + \int_{-1}^{x} \int_{-1}^{x_k} \cdots \int_{-1}^{x_3} \int_{-1}^{x_2} S(x_1) \, dx_1 \, dx_2 \, dx_3 \cdots dx_k$$

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with an $S \in \mathcal{P}_{n-k}$ nonnegative on $[-1,1]$ and a polynomial $R \in \mathcal{P}_{k-1}$ of the form

$$R(x) = \sum_{j=0}^{k-1} a_j (x+1)^j \,, \qquad a_j \geq 0 \,, \quad j = 0, 1, \ldots, k-1 \,. \tag{1.1}$$

Also, if $0 \leq m \leq k-1$, then

$$Q^{(m)}(x) = R^{(m)}(x) + \int_{-1}^{x} \int_{-1}^{x_{k-m}} \cdots \int_{-1}^{x_3} \int_{-1}^{x_2} S(x_1)\, dx_1\, dx_2\, dx_3 \cdots dx_{k-m} \,.$$

A straightforward application of Fubini's Theorem gives that if $y \in [-1,1]$, then

$$Q(y) = R(y) + \frac{1}{(k-1)!} \int_{-1}^{y} S(x)(y-x)^{k-1}\, dx \,. \tag{1.2}$$

Similarly, if $0 \leq m \leq k-1$, and $y \in [-1,1]$, then

$$Q^{(m)}(y) = R^{(m)}(y) + \frac{1}{(k-1-m)!} \int_{-1}^{y} S(x)(y-x)^{k-1-m}\, dx \,. \tag{1.3}$$

Markov's inequality [1, p. 233] asserts that

$$\|Q'\|_{[-1,1]} \leq n^2 \|Q\|_{[-1,1]} \tag{1.4}$$

for every $Q \in \mathcal{P}_n$. The Markov inequality in $L_p[-1,1]$ states that

$$\|Q'\|_{L_p[-1,1]} \leq c^{1+1/p} n^2 \|Q\|_{L_p[-1,1]} \tag{1.5}$$

holds for every $Q \in \mathcal{P}_n$ and $p > 0$. See [1, p. 402], for example. The essentially sharp Nikolskii-type inequality

$$\|Q\|_{L_p[-1,1]} \leq (c(2+qn))^{2/q-2/p} \|Q\|_{L_q[-1,1]} \tag{1.6}$$

for every $Q \in \mathcal{P}_n$ and $0 < q < p \leq \infty$ is proved in [1, p. 395] with $c := e^2(2\pi)^{-1}$.

It has been observed by Bernstein that Markov's inequality for monotone polynomials is not essentially better than that for all polynomials. He proved that

$$\sup_Q \frac{\|Q'\|_{[-1,1]}}{\|Q\|_{[-1,1]}} = \begin{cases} \frac{1}{4}(n+1)^2 \,, & \text{if } n \text{ is odd} \\ \frac{1}{4}n(n+2) \,, & \text{if } n \text{ is even} \,, \end{cases}$$

where the supremum is taken for all polynomials $0 \neq Q \in \mathcal{P}_n$ of degree at most n that are monotone on $[-1,1]$. See [5, p. 607], for instance.

In August, 2008, András Kroó asked me in an e-mail if I knew an analog of the above result of Bernstein for convex polynomials on $[-1,1]$. In a few days, with different methods, we both discovered independently that Markov's inequality for convex polynomials is not essentially better than that for all polynomials. A few weeks later Kroó informed me in an e-mail that with József Szabados he proved the essentially sharp cn^2/k Markov factor for absolutely monotone polynomials $Q \in \mathcal{P}_n$ of order k on $[-1,1]$ in the uniform norm. Meanwhile I had some work in progress about Markov inequality for absolutely monotone polynomials of order k on $[-1,1]$ in $L_p[-1,1]$ norm for $p > 0$. Kroó, Szabados, and I agreed that they would publish their results about Markov inequality for absolutely monotone polynomials of order k on $[-1,1]$ in C norm, while I would try to work out the right result in $L_p[-1,1]$ for $p > 0$. The results of Kroó and Szabados appeared in [3]. In this paper we prove the "right" Markov-Nikolskii type inequality for absolutely monotone polynomials of order k on $[-1,1]$.

2. New Result

Our main result in this paper is the following.

Theorem 2.1. *Let $k, m, n \in \mathbb{N}$, $1 \le k \le n$, $0 \le m \le k/2$. Let $0 < q \le p \le \infty$. There are absolute constants $c_1 > 0$ and $c_2 > 0$ and constants $c_{p,q} > 0$ and $c'_{p,q} > 0$ depending only on p and q such that*

$$c_{p,q}\left(\left(\frac{q}{q+1}\right)^2 \frac{c_1 n^2}{k}\right)^{m+1/q-1/p} \le \sup_Q \frac{\|Q^{(m)}\|_{L_p[-1,1]}}{\|Q\|_{L_q[-1,1]}} \le c'_{p,q}\left(\frac{c_2 n^2}{k}\right)^{m+1/q-1/p},$$

where the supremum is taken for all not identically zero absolutely monotone polynomials $Q \in \mathcal{P}_n$ of order k on $[-1,1]$.

We prove the above theorem in Section 4. There is an interesting relationship between the above result and essentially sharp lower and upper bounds for the smallest and largest zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}$. This will be explored in Section 5.

3. Lemmas

Lemma 3.1. *Let $k, m, n \in \mathbb{N}$, $1 \le k \le n$, $0 \le m \le k/2$. There is an absolute constant $c_3 \ge 1$ such that*

$$\int_{-1}^{1} S(x)(1-x)^{k-1-m}\,dx \le \left(\frac{c_3 n^2}{k^2}\right)^m \int_{-1}^{1} S(x)(1-x)^{k-1}\,dx$$

holds for every polynomial $S \in \mathcal{P}_{n-k}$ nonnegative on $[-1,1]$.

Proof. When $m = 0$ the lemma is obvious, so we may assume that $m \ge 1$. We base the proof on Bernstein's inequality [1, p. 232] stating that

$$\|T'\|_{[-\pi,\pi]} \le n\|T\|_{[-\pi,\pi]}$$

for all real trigonometric polynomials of degree at most n. Let $S \in \mathcal{P}_{n-k}$ be nonnegative on $[-1,1]$. Observe that for $a \in (-1,1)$ we have the obvious inequality

$$\int_{-1}^{a} S(x)(1-x)^{k-1-m}\,dx \le \left(\frac{1}{1-a}\right)^m \int_{-1}^{a} S(x)(1-x)^{k-1}\,dx\,. \tag{3.1}$$

Let

$$P(x) := \int_{-1}^{x} S(y)(1-y)^{k-1-m}\,dy\,.$$

Then $P \in \mathcal{P}_{n-m} \subset \mathcal{P}_n$,

$$P(1) = \int_{-1}^{1} S(y)(1-y)^{k-1-m}\,dy\,,$$

and $P^{(j)}(1) = 0$ for each $j = 1, 2, \ldots, k-1-m$. We set $T(t) := P(1) - P(\cos t)$. Observe that T is a trigonometric polynomial of degree at most n

$$T^{(j)}(0) = 0, \qquad j = 0, 1, \ldots, 2(k-m) - 1\,.$$

The Taylor expansion of T centered at 0,

$$T(y) = \sum_{j=0}^{\infty} \frac{T^{(j)}(0)}{j!}\, y^j = \sum_{j=2(k-m)}^{\infty} \frac{T^{(j)}(0)}{j!}\, y^j\,,$$

converges for every $y \in \mathbb{R}$ by the Root Test, since the Bernstein inequality

$$|T^{(j)}(0)| \le n^j \|T\|_{[-\pi,\pi]}$$

holds. Hence, using also $j! \ge (j/e)^j$, $k - m \ge k/2$, and the fact that $0 \le T(t) \le P(1)$ for every $t \in [-\pi, \pi]$, we obtain

$$\begin{aligned}|P(1) - P(\cos y)| = T(y) \le & \sum_{j=2(k-m)}^{\infty} \left(\frac{n^j P(1)}{j!}\right) y^j = \sum_{j=k}^{\infty} \left(\frac{n^j P(1)}{(j/e)^j}\right) y^j \\ \le & \sum_{j=k}^{\infty} \left(\frac{eny}{j}\right)^j P(1) \le \sum_{j=k}^{\infty} \left(\frac{eny}{k}\right)^j P(1) \\ \le & \frac{1}{2} P(1)\,, \qquad 0 \le y \le \frac{k}{4en}\,.\end{aligned}$$

Thus

$$P(1) \le 2P(\cos y) \qquad \text{with} \quad y := \frac{k}{4en}\,.$$

Combining this with (3.1) and the inequality $\cos y \le 1 - y^2/4$ we obtain

$$\begin{aligned}&\int_{-1}^{1} S(x)(1-x)^{k-1-m}\, dx = P(1) \le 2P(\cos y) \\ =& 2\int_{-1}^{\cos y} S(x)(1-x)^{k-1-m}\, dx \le 2\left(\frac{1}{1-\cos y}\right)^m \int_{-1}^{\cos y} S(x)(1-x)^{k-1}\, dx \\ \le& 2\left(\frac{cn^2}{k^2}\right)^m \int_{-1}^{1} S(x)(1-x)^{k-1}\, dx\end{aligned}$$

with $c = 64e^2$, and the lemma is proved. □

Lemma 3.2. *Let $k, m, N \in \mathbb{N}$, $k \ge 2$, $5k \le N$, $q > 0$. There are an absolute constant $c_4 > 0$ and not identically zero polynomials $S \in \mathcal{P}_{N-k}$ that are nonnegative on $[-1, 1]$ such that*

$$\frac{\displaystyle\int_{-1}^{1} S(x)(1-x)^{k-m}\, dx}{\displaystyle\int_{-1}^{1} S(x)(1-x)^{k}\, dx} \ge \left(\frac{c_4 N^2}{k^2}\right)^m$$

for every $0 \le m \le k$*.*

Proof. The lemma is obvious when $m = 0$, so we may assume that $m \ge 1$. Let $k \ge 2$, $5k \le N$,

$$\mu := \left\lfloor \frac{N-k}{4k} \right\rfloor \ge 1\,.$$

We define $U \in \mathcal{P}_\mu$ by $U(\cos t) = D_\mu(t)$, where

$$D_\mu(t) = \frac{1}{2} + \sum_{j=1}^{\mu} \cos(jt) = \frac{\sin((2\mu+1)t/2)}{2\sin(t/2)}\,,$$

and let $S := U^{4k} \in \mathcal{P}_{N-k}$. Clearly

$$S(1) = (\mu + \tfrac{1}{2})^{4k} \ge \mu^{4k}\,,$$

and

$$S(x) \le \frac{c_5^k}{(1-x)^{2k}}\,, \qquad x \in [-1,1]\,, \tag{3.2}$$

with an absolute constant $c_5 > 0$. Let

$$B := \left[1 - \frac{1}{2\mu^2}, 1 - \frac{1}{4\mu^2}\right]. \tag{3.3}$$

Using the Mean Value Theorem and Markov's inequality we obtain that there is a $\xi \in (x,1)$ such that

$$\begin{aligned} |\mu + \tfrac{1}{2} - U(x)| &= |U(1) - U(x)| = |U'(\xi)|(1-x) \le \mu^2 |U(1)|(1-x) \\ &\le \mu^2(\mu + \tfrac{1}{2})(1-x) \le \tfrac{1}{2}(\mu + \tfrac{1}{2})\,, \qquad x \in \left[1 - \frac{1}{2\mu^2}, 1\right], \end{aligned}$$

hence

$$U(x) \ge \tfrac{1}{2}(\mu + \tfrac{1}{2}) \ge \mu/2\,, \qquad x \in B \subset \left[1 - \tfrac{1}{2\mu^2}\,, 1\right].$$

Therefore, recalling (3.3), we have

$$\int_B S(x)(1-x)^k\,dx \ge \frac{1}{4\mu^2}\left(\frac{\mu}{2}\right)^{4k}\left(\frac{1}{4\mu^2}\right)^k \ge \frac{1}{4} 64^{-k} \mu^{2k-2}\,. \tag{3.4}$$

Also, it follows from (3.2) that

$$\begin{aligned} \int_{-1}^{1-c/\mu^2} S(x)(1-x)^k\,dx &\le \int_{-1}^{1-c/\mu^2} c_5^k (1-x)^{-k}\,dx \\ &\le c_5 \left(\frac{c_5\mu^2}{c}\right)^{k-1} \le 256 c_5 \frac{1}{4} 64^{-k} \mu^{2k-2} \end{aligned} \tag{3.5}$$

if $c := 64c_5$ and $1 - c/\mu^2 \geq -1$. When $1 - c/\mu^2 \geq -1$, from (3.4) and (3.5) we obtain

$$\int_{-1}^{1} S(x)(1-x)^k \, dx \leq (256c_5 + 1) \int_{1-c/\mu^2}^{1} S(x)(1-x)^k \, dx \,,$$

hence

$$\begin{aligned} \int_{-1}^{1} S(x)(1-x)^{k-m} \, dx &\geq \int_{1-c/\mu^2}^{1} S(x)(1-x)^{k-m} \, dx \\ &\geq \left(\frac{\mu^2}{c}\right)^m \int_{1-c/\mu^2}^{1} S(x)(1-x)^k \, dx \\ &\geq \left(\frac{\mu^2}{c}\right)^m \frac{1}{256c_5+1} \int_{-1}^{1} S(x)(1-x)^k \, dx \,. \end{aligned}$$

If $1 - c/\mu^2 < -1$, that is $\mu^2 < c/2$, then

$$\int_{-1}^{1} S(x)(1-x)^{k-m} \, dx \geq 2^{-m} \int_{-1}^{1} S(x)(1-x)^k \, dx \geq \left(\frac{\mu^2}{c}\right)^m \int_{-1}^{1} S(x)(1-x)^k \, dx \,,$$

which finishes the proof. $\square$

Our next lemma follows from (1.2), (1.3), Lemma 3.2, and the inequalities

$$\left(\frac{k}{e}\right)^m \leq \frac{k!}{(k-m)!} \leq k^m \,.$$

Lemma 3.3. *Let $k, m, N \in \mathbb{N}$, $k \geq 2$, $5k \leq N$. There are an absolute constant $c_4 > 0$ and not identically zero absolutely monotone polynomials $Q \in \mathcal{P}_N$ of order $k+1$ on $[-1,1]$ with a zero at -1 with multiplicity at least $k+1$, such that*

$$\frac{Q^{(m+1)}(1)}{Q(1)} = \frac{Q^{(m+1)}(1)}{\|Q'\|_{L_1[-1,1]}} \geq \left(\frac{c_4 N^2}{ek}\right)^{m+1}$$

for every $0 \leq m+1 \leq k$.

Observe that if $Q \in \mathcal{P}_N$ is an absolutely monotone polynomial of order $k+1$ on $[-1,1]$, then $P := Q'$ is an absolutely monotone polynomial of order k on $[-1,1]$. Hence Lemma 3.3 implies the following lemma.

Lemma 3.4. *Let $k, m, N \in \mathbb{N}$, $k \geq 1$, $5k \leq N$. There are an absolute constant $c_4 > 0$ and not identically zero absolutely monotone polynomials $P \in \mathcal{P}_N$ of order $k+1$ on $[-1,1]$ with a zero at -1 with multiplicity at least k such that*

$$\frac{P^{(m)}(1)}{\|P\|_{L_1[-1,1]}} \geq \left(\frac{c_4 N^2}{ek}\right)^{m+1}$$

for every $0 \leq m+1 \leq k$.

4. Proof of Theorem 2.1

Proof of Theorem 1.1. First we prove the upper bound of the theorem. Let $0 \le m \le k/2$, $1 \le k \le n$. We may assume that $k \ge 2$; the case $k = 1$, $m = 0$ follows from (1.6). Suppose $Q \in \mathcal{P}_n$ is an absolutely monotone polynomial of order k on $[-1, 1]$. If $Q(x) = 0$ for an $x \in [0, 1]$, then $Q \equiv 0$ and $Q^{(m)} \equiv 0$, hence

$$|Q^{(m)}(x)| \le \left(\frac{4c_3 n^2}{k} \right)^m Q(x) \tag{4.1}$$

trivially holds. If Q is not identically zero then scaling Lemma 3.1 linearly from the interval $[-1, 1]$ to $[-1, x]$ (note that $\frac{1}{2}(x+1) \ge \frac{1}{2}$ for $x \in [0, 1]$) and using (1.2), (1.3), and (1.1), we obtain

(4.2)

$$\begin{aligned}\frac{|Q^{(m)}(x)|}{|Q(x)|} &= \frac{Q^{(m)}(x)}{Q(x)} \le \frac{R^{(m)}(x)}{R(x)} + \frac{\frac{1}{(k-1-m)!} \int_{-1}^{x} S(t)(x-t)^{k-1-m}\, dt}{\frac{1}{(k-1)!} \int_{-1}^{x} S(t)(x-t)^{k-1}\, dt} \\ &\le k^m + k^m \left(\frac{2c_3 n^2}{k^2} \right)^m \le k^m + \left(\frac{2c_3 n^2}{k} \right)^m \le \left(\frac{4c_3 n^2}{k} \right)^m, \qquad x \in [0, 1].\end{aligned}$$

(Observe that if $R(x) = 0$ for an $x \in [0, 1]$, then $R \equiv 0$ and $R^{(m)} \equiv 0$, and $R^{(m)}(x)/R(x)$ in (4.2) can be interpreted as 0.) Hence (4.1) and (4.2) give

$$\int_0^1 |Q^{(m)}(x)|^p\, dx \le \left(\frac{4c_3 n^2}{k} \right)^{mp} \int_0^1 |Q(x)|^p\, dx. \tag{4.3}$$

Since $m \le k - 1$, $Q^{(m)}(x) \ge 0$ is increasing on $[-1, 1]$, hence we also have

$$\int_{-1}^0 |Q^{(m)}(x)|^p\, dx \le \int_0^1 |Q^{(m)}(x)|^p\, dx \le \left(\frac{4c_3 n^2}{k} \right)^{mp} \int_0^1 |Q(x)|^p\, dx. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$\int_{-1}^1 |Q^{(m)}(x)|^p\, dx \le 2 \left(\frac{4c_3 n^2}{k} \right)^{mp} \int_0^1 |Q(x)|^p\, dx,$$

that is,

$$\|Q^{(m)}\|_{L_p[-1,1]} \le 2^{1/p} \left(\frac{4c_3 n^2}{k} \right)^m \|Q\|_{L_p[-1,1]}. \tag{4.5}$$

Now observe that (4.1) and (4.2) with $m = 1$ gives

$$0 \le Q'(x) \le \frac{4c_3 n^2}{k} Q(x) \le \frac{4c_3 n^2}{k} Q(1), \qquad x \in [0, 1].$$

Combining this with the Mean Value Theorem gives that there is a $\xi \in (y,1)$ such that

$$|Q(1) - Q(x)| = (1-x)|Q'(\xi)| \le (1-x)\frac{4c_3n^2}{k}\,Q(1) \le \frac{1}{2}\,Q(1)$$

for every

$$x \in I := \left[1 - \frac{k}{8c_3n^2},\, 1\right],$$

that is,

$$\frac{1}{2}Q(1) \le Q(x)\,, \qquad x \in I\,.$$

Therefore, if $0 < q$, then

$$\begin{aligned}\|Q\|^q_{[-1,1]} \le & \frac{8c_3n^2}{k}\,2^q \int_I \left(\frac{1}{2}\|Q\|_{[-1,1]}\right)^q\,dx \le \frac{8c_3n^2}{k}\,2^q \int_I |Q(x)|^q\,dx \\ & \le \frac{8c_3n^2}{k}\,2^q \int_{-1}^1 |Q(x)|^q\,dx\,,\end{aligned}$$

and hence, if $0 < q \le p < \infty$, then

$$\begin{aligned}\|Q\|^p_{L_p[-1,1]} &= \int_{-1}^1 |Q(x)|^p\,dx \le \int_{-1}^1 \|Q\|^{p-q}_{[-1,1]}|Q(x)|^q\,dx \\ &\le 2^{p-q}\left(\frac{8c_3n^2}{k}\right)^{(p-q)/q} \|Q\|^{p-q}_{L_q[-1,1]}\|Q\|^q_{L_q[-1,1]} \\ &\le 2^{p-q}\left(\frac{8c_3n^2}{k}\right)^{(p-q)/q} \|Q\|^p_{L_q[-1,1]}\,.\end{aligned}$$

Therefore

$$\|Q\|_{L_p[-1,1]} \le 2^{1-q/p}\left(\frac{8c_3n^2}{k}\right)^{1/q-1/p} \|Q\|_{L_q[-1,1]}\,,$$

which, together with (4.5), finishes the proof of the upper bound of the theorem.

Now we prove the lower bound of the theorem. Since the case $k=1$ follows from the case $k=2$ we may assume that $k \ge 2$. First let $n \in \mathbb{N}$, $\nu := \lfloor 1/q \rfloor + 1$, $N := \lfloor n/\nu \rfloor$, $k \ge 2$, $5k \le N$, and $0 \le m+1 \le k/2$. By Lemma 3.4 there is a not identically 0 absolutely monotone polynomial $P \in \mathcal{P}_N$ of order k on $[-1,1]$ for which

$$\frac{P^{(m)}(1)}{\|P\|_{L_1[-1,1]}} \ge \left(\frac{c_4N^2}{ek}\right)^{m+1} \tag{4.6}$$

whenever $0 \le m+1 \le \frac{1}{2}(k+1)$. Using (4.6) and the already proved upper bound of the theorem with $m=0$, $q=1$, $p=\infty$, and $c' = c'_{\infty,1}$, we obtain

$$P(1) \le c'\frac{c_2N^2}{k}\|P\|_{L_1[-1,1]}\,. \tag{4.7}$$

and

$$\frac{P^{(m)}(1)}{P(1)} = \frac{P^{(m)}(1)}{\|P\|_{L_1[-1,1]}} \frac{\|P\|_{L_1[-1,1]}}{P(1)} \geq \left(\frac{c_4N^2}{ek}\right)^{m+1} \left(c' \frac{c_2N^2}{k}\right)^{-1} \geq \frac{c_4}{ec_2c'} \left(\frac{c_4N^2}{ek}\right)^{m} . \tag{4.8}$$

Let $R := P^\nu$. Then $R \in \mathcal{P}_n$ is an absolutely monotone polynomial of order k on $[-1,1]$, hence using (4.6), (4.7), and (4.8), we obtain that

$$\begin{aligned} \frac{R^{(m)}(1)}{\|R\|_{L_q[-1,1]}} &\geq \frac{(P(1))^{\nu-1}P^{(m)}(1)}{\left(\int_{-1}^{1}(P(x))^{\nu q}\,dx\right)^{1/q}} \geq \frac{(P(1))^{\nu-1}P^{(m)}(1)}{\left(\|P\|_{L_1[-1,1]}(P(1))^{\nu q-1}\right)^{1/q}} \\ &\geq \frac{(P(1))^{\nu-1}P^{(m)}(1)}{\left(\|P\|_{L_1[-1,1]}\left(c'\frac{c_2N^2}{k}\right)^{\nu q-1}\left(\|P\|_{L_1[-1,1]}\right)^{\nu q-1}\right)^{1/q}} \\ &\geq \left(c'\frac{c_2N^2}{k}\right)^{1/q-\nu}\left(\frac{P(1)}{\|P\|_{L_1[-1,1]}}\right)^{\nu}\left(\frac{P^{(m)}(1)}{P(1)}\right) \\ &\geq \left(c'\frac{c_2N^2}{k}\right)^{1/q-\nu}\left(\frac{c_4N^2}{ek}\right)^{\nu}\frac{c_4}{ec_2c'}\left(\frac{c_4N^2}{ek}\right)^{m} \\ &\geq c_{p,q}\left(\frac{c_6N^2}{k}\right)^{m+1/q} \end{aligned} \tag{4.9}$$

with an absolute constant $c_6 > 0$ and a constant $c_{p,q} > 0$ depending only on p and q. Since $R^{(m)} \in \mathcal{P}_{n-m}$ is an absolutely monotone polynomial of order $k-m \geq k/2$, the already proved upper bound of the theorem with proper substitutions imply

$$\frac{\|R^{(m)}\|_{L_p[-1,1]}}{\|R^{(m)}\|_{L_\infty[-1,1]}} \geq (c'_{p,q})^{-1}\left(\frac{2c_2n^2}{k}\right)^{-1/p} . \tag{4.10}$$

Now (4.9) and (4.10) give the lower bound of the theorem.

In the remaining cases, when $n \in \mathbb{N}$, $\nu := \lfloor 1/q \rfloor + 1$, $N := \lfloor n/\nu \rfloor$, $k \geq 2$, $N \leq 5k \leq 5n$, and $0 \leq m+1 \leq k/2$, the polynomials $Q(x) = (1-x)^n$ yield the lower bound of the theorem. □

5. Bounds for the Smallest and Largest zeros of Jacobi polynomials

In this section we point out a remarkable connection between the right Markov inequality for absolutely monotone polynomials of order k in the supremum norm, and essentially sharp bounds for the largest and smallest zeros of Jacobi polynomials. We hope that even the close experts of orthogonal polynomials would find some novelty in the discussion here.

A version of the following result is due to Chebyshev, see Theorem 7.72.1 on p. 188 in [6], who handled the slightly more technical case when $2n-2$ in the lemma below is replaced by $2n-1$ as well.

Lemma 5.1. *Associated with a weight function w on $[-1,1]$ let (P_n) be the sequence of orthonormal polynomials $P_n \in \mathcal{P}_n$ on $[-1,1]$ with respect to w. Denote the zeros of P_n by*

$$(-1 <) x_{nn} < x_{(n-1)n} < \cdots < x_{2n} < x_{1n} (< 1)\,.$$

Then

$$\inf_S \frac{\displaystyle\int_{-1}^1 S(x)(1-x)w(x)\,dx}{\displaystyle\int_{-1}^1 S(x)w(x)\,dx} = 1 - x_{1n}\,.$$

Equivalently,

$$\sup_S \frac{\displaystyle\int_{-1}^1 S(x)w(x)\,dx}{\displaystyle\int_{-1}^1 S(x)(1-x)w(x)\,dx} = \frac{1}{1-x_{1n}}\,,$$

where the infimum and supremum are taken for all $0 \neq S \in \mathcal{P}_{2n-2}$ nonnegative on $[-1,1]$.

Although Lemma 5.1 must be a well-known result for a reader familiar with the basics about orthogonal polynomials, we present a short proof of it here, which is different from that in [6, pp. 186–189]. We base the proof on the Gauss-Jacobi quadrature formula [6, pp. 47–48] and hope it would help the reader to see it reasonably clearly what is behind this good-looking result.

Proof. We prove the first statement of the lemma only, the second one is obviously equivalent to it. Let $0 \neq S \in \mathcal{P}_{2n-2}$ and $Q(x) := S(x)(x_{1n} - x)$. Since $Q \in \mathcal{P}_{2n-1}$, the well-known Gauss-Jacobi quadrature formula gives that

$$\begin{aligned}
\int_{-1}^1 S(x)(1-x)w(x)\,dx &= \int_{-1}^1 S(x)((1-x_{1n}) + (x_{1n}-x))w(x)\,dx \\
&= (1-x_{1n}) \int_{-1}^1 S(x)w(x)\,dx + \int_{-1}^1 S(x)(x_{1n}-x)w(x)\,dx \\
&= (1-x_{1n}) \int_{-1}^1 S(x)w(x)\,dx + \sum_{j=1}^n \lambda_j S(x_{jn})(x_{1n}-x_{jn}) \\
&\geq (1-x_{1n}) \int_{-1}^1 S(x)w(x)\,dx\,.
\end{aligned}$$

Here each $\lambda_j \geq 0$, hence each term $\lambda_j S(x_{jn})(x_{1n} - x_{jn})$ is non-negative. Hence the first statement of the lemma is already proved with the $\geq$ sign.

To prove the first statement of the lemma with the $\leq$ sign let

$$S(x) := \frac{P_n(x)^2}{(x-x_{1n})^2}\,.$$

Clearly $S \in \mathcal{P}_{2n-2}$ is non-negative on the real number line and

$$\begin{aligned}
&\int_{-1}^{1} S(x)(1-x)w(x)\,dx \\
&= \int_{-1}^{1} \frac{P_n(x)^2}{(x-x_{1n})^2}\,(1-x)w(x)\,dx = \int_{-1}^{1} \frac{P_n(x)^2}{(x-x_{1n})^2}\,((1-x_{1n})+(x_{1n}-x))w(x)\,dx \\
&= (1-x_{1n})\int_{-1}^{1} \frac{P_n(x)^2}{(x-x_{1n})^2}\,w(x)\,dx - \int_{-1}^{1} \frac{P_n(x)^2}{x-x_{1n}}\,w(x)\,dx \\
&= (1-x_{1n})\int_{-1}^{1} \frac{P_n(x)^2}{(x-x_{1n})^2}\,w(x)\,dx = (1-x_{1n})\int_{-1}^{1} S(x)\,w(x)\,dx\,.
\end{aligned}$$

Here we used the fact that the polynomial $P_n(x)/(x-x_{1n})$ is of degree $n-1$ and hence it is orthogonal to P_n with respect to the weight w. This is just the first statement of the lemma with the $\leq$ sign. □

Let $(P_n^{(\alpha,\beta)})$ be the sequence of orthonormal (Jacobi) polynomials of degree n associated with the weight $(1-x)^\alpha(1+x)^\beta$ on $[-1,1]$.

Corollary 5.2. *Let x_{1n} be the largest zero of the Jacobi polynomial $P_n^{(k+1,0)}$, $2 \leq k \leq n-1$. Then there are absolute constants $c_7 > 0$ and $c_8 > 0$ such that*

$$1 - c_7\left(\frac{k}{2n+k}\right)^2 \leq x_{1n} \leq 1 - c_8\left(\frac{k}{2n+k}\right)^2.$$

Proof. First we prove the upper bound of the corollary. Combining Lemma 5.1 and Lemma 3.1 applied with $m=1$, $2n+k-1$ in place of n, and k replaced by $k+1$, we obtain that

$$\frac{1}{1-x_{1n}} = \sup_S \frac{\displaystyle\int_{-1}^{1} S(x)(1-x)^{k-1}\,dx}{\displaystyle\int_{-1}^{1} S(x)(1-x)^{k}\,dx} \leq \frac{c_3(2n+k-1)^2}{(k+1)^2}\,,$$

where the supremum is taken for all not identically 0 polynomials $S \in \mathcal{P}_{2n-2}$ nonnegative on $[-1,1]$. This gives the upper bound of the corollary.

Now we prove the lower bound of the corollary. When $k \geq (n-1)/2$ the lower bound of the corollary follows from the well-known fact that $-1 < x_{1n} < 1$. So we may assume that $2 \leq k \leq (n-1)/2$. Combining Lemma 5.1 and Lemma 3.2 applied with $m=1$, $2 \leq k \leq (n-1)/2$, and $N := 2n+k-2$ (observe that $5k \leq N$ is satisfied), we obtain that

$$\frac{1}{1-x_{1n}} = \sup_S \frac{\displaystyle\int_{-1}^{1} S(x)(1-x)^{k-1}\,dx}{\displaystyle\int_{-1}^{1} S(x)(1-x)^{k}\,dx} \geq \frac{c_4(2n+k-2)^2}{k^2}\,, \tag{5.1}$$

where the supremum is taken for all not identically zero polynomials $S \in \mathcal{P}_{2n-2}$ that are nonnegative on $[-1,1]$. This gives the lower bound of the corollary. $\square$

There is much literature on bounds for the zeros of Jacobi polynomials, see e.g., Sections 6.2 and 6.21, pp. 116–123 in [6], but most are useful only when α and β are between $-1/2$ and $1/2$. The reader may wish to check [4], for example, and some of the other references in [2]. For large n, the extreme zeros behave like $-1 + j_\beta^2/(2n^2)$ and $1 - j_\alpha^2/(2n^2)$, where j_κ denotes the smallest positive zero of the Bessel function J_κ, see Section 8.1, p. 192 in [6].

The next theorem [2, Theorem 13] gives reasonably satisfactory lower and upper estimates for the zeros of the Jacobi polynomials.

Theorem 5.3. *Given $n = 1, 2, \ldots,$ the zeros*

$$(-1<)x_{nn} < x_{(n-1)n} < \cdots < x_{2n} < x_{1n} (< 1)$$

of the Jacobi polynomial of degree n with respect to a Jacobi weight $w(x) = (1-x)^\alpha (1+x)^\beta$ with $\alpha \geq -1/2$ and $\beta \geq -1/2$ satisfy

$$-1 + \frac{2\beta^2}{N^2} \leq x_{nn} \qquad \text{and} \qquad x_{1n} \leq 1 - \frac{2\alpha^2}{N^2}\,,$$

where $N := 2n + \alpha + \beta + 1$.

In [2] we did not prove the sharpness of the above theorem (up to an absolute constant). Combining Lemma 5.1 applied to the Jacobi weight $w(x) = (1-x)^\alpha(1+x)^\beta$ on $[-1,1]$ and appropriate (quite straightforward) modifications of Lemmas 3.1 and 3.2, we obtain the following result similarly to the proof of Corollary 5.2.

Theorem 5.4. *Given $n = 1, 2, \ldots,$ the zeros*

$$(-1<)x_{nn} < x_{(n-1)n} < \cdots < x_{2n} < x_{1n} (< 1)$$

of the Jacobi polynomial of degree n with respect to a Jacobi weight $w(x) = (1-x)^\alpha (1+x)^\beta$ with $\alpha \geq 1$ and $\beta \geq 1$ satisfy

$$-1 + \frac{c_9\beta^2}{N^2} \leq x_{nn} \leq -1 + \frac{c_{10}\beta^2}{N^2} \qquad \text{and} \qquad 1 - \frac{c_{10}'\alpha^2}{N^2} \leq x_{1n} \leq 1 - \frac{c_9'\alpha^2}{N^2}\,,$$

where $N = 2n + \alpha + \beta + 1$ and $c_9 > 0$, $c_{10} > 0$, $c_9' > 0$ and $c_{10}' > 0$ are appropriate absolute constants.

The details of the proof of the above theorem may be the subject matter of another note in the not too distant future.

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