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NOTES ON LACUNARY MÜNTZ POLYNOMIALS

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ABSTRACT

We prove that a Müntz system has Chebyshev polynomials on $[0,1]$ with uniformly bounded coefficients if and only if it is lacunary. A sharp Bernstein-type inequality for lacunary Müntz systems is established as well. As an application we show that a lacunary Müntz system fails to be dense in $C(A)$ in the uniform norm for every $A \subset [0,1]$ with positive outer Lebesgue measure. A bounded Remez-type inequality is conjectured for non-dense Müntz systems on $[0,1]$ which would solve Newman's problem concerning the density of products of Müntz systems.

1. Introduction and Notations

Denseness and approximation questions in Markov systems are intimately and essentially tied to the behavior of the associated Chebyshev polynomials; see, for example, [1, 2]. Our intention, in this paper, is to show that lacunary Müntz systems are completely characterized by the property that their associated Chebyshev polynomials on $[0,1]$ have uniformly bounded coefficients. This is the content of Theorems 2.1 and 2.2. This allows us to give an (essentially) sharp Bernstein-type inequality (Theorem 3.1) for these systems, and from this we can rederive a version of a Müntz-type theorem in [2] (Theorem 4.1). This theorem

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tells us that, under the assumption of lacunarity, a Müntz system fails to be dense in $C(A)$, where $A \subset [0,\infty)$ is any set with positive Lebesgue outer measure. We conjecture that this extension of the Müntz-Szász theorem holds in any non-dense Müntz system. In Section 5 a bounded Remez-type inequality is conjectured for non-dense Müntz systems on $[0,1]$ which would solve Newman's problem concerning the density of products of Müntz systems.

Let $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$, $0 \le \lambda_0 < \lambda_1 < \cdots$. The set of all Müntz polynomials of the form $p(x) = \Sigma_{j=0}^{n} a_j x^{\lambda_j}$ with real coefficients a_j will be denoted by $H_n(\Lambda)$. Let $H(\Lambda) = \bigcup_{n=0}^{\infty} H_n(\Lambda)$. The n-th Chebyshev polynomial T_n of $H(\Lambda)$ on $[0,1]$ is defined by the properties

(1) $T_n \in H_n(\Lambda)$,
(2) T_n equioscillates $n+1$ times on $[0,1]$,
(3) $\max_{0\le x\le 1} |T_n(x)| = 1$,
(4) $T_n(1) = 1$.

To be precise, property (2) means that $T_n(x)$ achieves the values

$$\pm \max_{0\le x\le 1} |T_n(x)| = \pm 1$$

$n+1$ times on $[0,1]$ with alternating signs. It is well-known that such a T_n $(n = 0,1,\ldots)$ exists and it is unique.

2. Chebyshev Polynomials of $H(\Lambda)$ on $[0,1]$ with Uniformly Bounded Coefficients

Let

$$T_n(x) = \sum_{j=0}^{n} a_{j,n} x^{\lambda_j}$$

be the n-th Chebyshev polynomial of $H(\Lambda)$ on $[0,1]$. We characterize the Müntz systems $H(\Lambda)$ for which $|a_{j,n}| \le K(\Lambda)$ for every $j = 0,1,\ldots,n$; $n = 0,1,\ldots$, where $K(\Lambda)$ is a constant depending only on Λ.

THEOREM 2.1: *There is a constant $c_1(\Lambda)$ depending only on Λ such that $|a_{j,n}| \le c_1(\Lambda)$ for every $j = 0,1,\ldots,n$; $n = 0,1,\ldots$ if and only if there is a constant $c_2(\Lambda) > 1$ depending only on Λ such that $\lambda_{i+1}/\lambda_i \ge c_2(\Lambda)$ for every $i = 1,2,\ldots$.*

In fact, in one of the directions we will prove more. Namely we have

THEOREM 2.2: *If $\lambda_{i+1}/\lambda_i \geq c_2(\Lambda) > 1$ for every $i = 1, 2, \ldots$ with some constant $c_2(\Lambda)$ depending only on Λ, then there is a constant $c_1(\Lambda)$ depending only on Λ such that*

$$|b_{j,n}| \leq c_1(\Lambda) \max_{0 \leq x \leq 1} |p(x)| \qquad (j = 0, 1, \ldots, n; n = 0, 1, \ldots)$$

for every

$$p(x) = \sum_{j=0}^{n} b_{j,n} x^{\lambda_j} \in H_n(\Lambda).$$

3. A Bernstein-type Inequality for Lacunary Müntz Systems

The following pretty inequalities were proved by D. Newman [9]. We have

$$\frac{2}{3} \sum_{i=0}^{n} \lambda_i \leq \sup_{p \in H_n(\Lambda)} \frac{|p'(1)|}{\max_{0 \leq x \leq 1} |p(x)|} \leq 11 \sum_{i=0}^{n} \lambda_i. \tag{3.1}$$

In the case that $\Sigma_{i=1}^{\infty} \lambda_i^{-1} < \infty$, $\lambda_0 = 0$, $\lambda_1 \geq 1$ and $\inf \{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0$, Lemma 2 of [1] gives

$$\max_{0 \leq x \leq y} |p'(x)| \leq c(\Lambda, y) \max_{0 \leq x \leq 1} |p(x)| \tag{3.2}$$

for every $p \in H(\Lambda)$ and $0 \leq y < 1$, where $c(\Lambda, y)$ is a constant depending only on Λ and y. In our next theorem we prove that if Λ is lacunary, $\lambda_0 = 0$ and $\lambda_1 \geq 1$, then in (3.2) $c(\Lambda, y)$ can be replaced by $c_3(\Lambda)/(1-y)$, where $c_3(\Lambda)$ is a constant depending only on Λ.

THEOREM 3.1: *If $\lambda_0 = 0$, $\lambda_1 \geq 1$ and $\lambda_{i+1}/\lambda_i \geq c_2(\Lambda) > 1$ for every $i = 1, 2, \ldots$ with some constant $c_2(\Lambda)$ depending only on Λ, then there is a constant $c_3(\Lambda)$ depending only on Λ such that*

$$|p'(y)| \leq \frac{c_3(\Lambda)}{1-y} \max_{0 \leq x \leq 1} |p(x)|$$

for every $p \in H(\Lambda)$ and $0 \leq y < 1$.

4. A Müntz-type Theorem for Lacunary Müntz Systems on Every $A \subset [0,1]$ with Positive Outer Lebesgue Measure

A beautiful theorem of Müntz of Szász [3, 7] states that a Müntz system $H(\Lambda)$ with $\lambda_0 = 0$ and $\lambda_i \nearrow \infty$ is dense in $C([0,1])$ in the uniform norm if and only if $\Sigma_{i=1}^{\infty}\lambda_i^{-1} = \infty$. In 1943 Clarkson and Erdös [4] showed that if $\inf\{\lambda_{i+1} - \lambda_i : i \in \mathbb{N}\} > 0$, then for an arbitrary $[a,b] \subset (0,\infty)$ a Müntz system $H(\Lambda)$ with $\lambda_i \nearrow \infty$ is dense in $C([a,b])$ in the uniform norm if and only if $\Sigma_{i=1}^{\infty}\lambda_i^{-1} = \infty$. The following conjecture seems to be hard.

CONJECTURE 4.1: *Suppose that $A \subset [0,1]$ is a closed set with positive Lebesgue measure. Then a Müntz system $H(\Lambda)$ with $\lambda_0 = 0$ and $\lambda_i \nearrow \infty$ is dense in $C(A)$ in the uniform norm if and only if $\Sigma_{i=1}^{\infty}\lambda_i^{-1} = \infty$.*

We prove an application of Theorem 3.1 for lacunary Müntz systems.

THEOREM 4.2: *If $\lambda_0 = 0$ and $\lambda_{i+1}/\lambda_i \geq c_2(\Lambda) > 1$ for every $i = 1,2,\dots$ with some constant $c_2(\Lambda)$ depending only on Λ, then $H(\Lambda)$ fails to be dense in $C(A)$ in the uniform norm for any $A \subset [0,1]$ with positive Lebesgue outer measure.*

5. Remez-type Inequalities for Müntz Systems

In [2] we pointed out that Conjecture 4.1 would trivially follow from the following Remez-type inequality.

CONJECTURE 5.1: *Let $\Sigma_{i=1}^{\infty}\lambda_i^{-1} < \infty$. For every $0 < s < 1$ there is a constant $c(s,\Lambda)$ depending only on s and Λ such that $|p(0)| \leq c(s,\Lambda)$ for every $p \in H(\Lambda)$ with $m(\{x \in [0,1] : |p(x)| \leq 1\}) \geq s$, where $m(\cdot)$ denotes the Lebesgue measure.*

By the already mentioned Müntz-Szász theorem, such a bounded Remez-type inequality cannot hold when $\Sigma_{i=1}^{\infty}\lambda_i^{-1} = \infty$. A discussion of Remez-type inequalities for algebraic and trigonometric polynomials can be found in [5] and [6]. In [2] we proved Conjecture 5.1 in the case when $\lambda_j > \lambda^j$, $j = 1,2,\dots$ with some $\lambda > 1$, and obtained Theorem 4.2 as a consequence of it. Our proof of Theorem 4.2 in Section 6 will be essentially shorter and it may open other ways to attack Conjecture 4.1.

There was another motivation to establish Conjecture 5.1. It would solve Newman's problem [10, P(10.5), p. 50] concerning the density of the classes

$$H^k(\Lambda) = \left\{ p = \prod_{j=1}^{k} p_j : \ p_j \in H(\Lambda),\ j = 1,2,\dots,k \right\}$$

in $C([0,1])$ in the uniform norm, when $\lambda_j = j^2$, $j = 1,2,\ldots$. Namely, if Conjecture 5.1 were true, then $H^k(\Lambda)$ would fail to be dense in $C([0,1])$ in the uniform norm for every $k \in \mathbb{N}$, whenever $\Sigma_{i=1}^{\infty}\lambda_i^{-1} < \infty$. Indeed, Conjecture 5.1 implies that

$$(5.1) \qquad m(\{x \in [0,1] : |q(x)| \geq \alpha^{-1}|q(0)|\}) \geq 1 - (2k)^{-1}$$

for every $q \in H(\Lambda)$ with $\alpha = c((2k)^{-1}, \Lambda) + 1$. Hence

$$(5.2) \qquad m(\{x \in [0,1] : p(x)| \geq \alpha^{-k}|p(0)|\}) \geq \frac{1}{2}$$

for every $p \in H^k(\Lambda)$ (if $p = p_1p_2\cdots p_k$ with $p_j \in H(\Lambda)$, $j = 1,2,\ldots,k$, then $|p(x)| \geq \alpha^{-k}|p(0)|$ holds for every $x \in [0,1]$ satisfying $|p_j(x)| \geq \alpha^{-1}|p_j(0)|$ for each $j = 1,2,\ldots,k$). Now let $f \in C([0,1])$ be such that $f(x) = 0$ if $1/4 \leq x \leq 1$, and $f(0) = 1$. If there were a $p \in H^k(\Lambda)$ such that

$$(5.3) \qquad \max_{0 \leq x \leq 1} |p(x) - f(x)| \leq \frac{1}{2}\alpha^{-k},$$

then it would contradict (5.2). Similarly, Conjecture 5.1 would imply that if $\Sigma_{i=1}^{\infty}\lambda_i^{-1} < \infty$, and $A \subset [0,1]$ is of positive measure, then $H^k(\Lambda)$ is not dense in $C(A)$ for every $k \in \mathbb{N}$.

6. Proofs

To prove Theorem 2.2 we need the following result of Hardy and Littlewood [8].

THEOREM A: *Assume that $\gamma_0 = 0$, $\gamma_{i+1}/\gamma_i \geq \eta > 1$ for every $i = 1,2,\ldots$, $f(x) = \Sigma_{i=0}^{\infty}b_ix^{\gamma_i}$ is convergent in $[0,1)$ and $\lim_{x\to 1-} f(x) = A$ exists. Then $A = \Sigma_{i=0}^{\infty}b_i$.*

Proof of Theorem 2.2: Without loss of generality we may assume that $\lambda_0 = 0$. Assume indirectly that there are

$$(6.1) \qquad P_k(x) = \sum_{j=0}^{n_k} b_{j,n_k}x^{\lambda_j}$$

such that

$$(6.2) \qquad \max_{0 \leq x \leq 1} |P_k(x)| = 1, \quad k = 1,2\ldots$$

and

$$\max_{0\le j\le n_k} |b_{j,n_k}| \ge k^4, \quad k = 1, 2, \ldots. \tag{6.3}$$

Choose a sequence $\{\alpha_k\}_{k=1}^{\infty}$ of positive integers such that

$$\alpha_1 = 1 \quad \text{and} \quad \alpha_{k+1} \ge 2\alpha_k \lambda_{n_k} \quad \text{for } k = 1, 2, \ldots. \tag{6.4}$$

Now let

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} P_k(x^{\alpha_k}). \tag{6.5}$$

Note that the above sum converges uniformly on $[0, 1]$ because of (6.2). Therefore f is continuous on $[0, 1]$. For the sake of brevity let

$$N_0 = 0 \quad \text{and} \quad N_k = \sum_{i=1}^{k} n_i, \quad k = 1, 2, \ldots. \tag{6.6}$$

Further let

$$\gamma_0 = 0, \qquad b_0 = \sum_{k=1}^{\infty} \frac{1}{k^2} b_{0,n_k} \tag{6.7}$$

and

$$\gamma_{N_{k-1}+j} = \alpha_k \lambda_j, \quad j = 1, 2, \ldots, n_k \quad \text{and} \quad k = 1, 2, \ldots \tag{6.8}$$

Observe that the sum in (6.7) converges, since by (6.2) $|b_{0,n_k}| = |P_k(0)| \le 1$. Also, from $\lambda_{i+1}/\lambda_i \ge c_2(\Lambda) > 1$, $i = 1, 2, \ldots$, and (6.4) we can deduce that $\gamma_{i+1}/\gamma_i \ge \eta > 1$, $i = 1, 2, \ldots$ with $\eta = \min\{c_2(\Lambda), 2\}$. Let $\Lambda' = \{\gamma_i\}_{i=1}^{\infty}$. Then by (6.5) $f \in C([0, 1])$ can be approximated by Müntz polynomials from $H(\Lambda')$ with arbitrary accuracy. Hence by Theorem 3 of [4] f is of the form

$$f(x) = \sum_{i=0}^{\infty} b_i x^{\gamma_i}, \tag{6.9}$$

where the sum converges in $[0, 1)$. Since f is continuous on $[0, 1]$, Theorem A implies that

$$\sum_{i=0}^{\infty} b_i = A \tag{6.10}$$

exists. By Theorem 3 of [4] each b_{j,n_k} ($j = 1,2,\ldots,n_k$; $k = 1,2,\ldots$) is equal to one of the coefficients b_i ($i = 1,2,\ldots$). Since $|b_{0,n_k}| = |P_k(0)| \le 1$ for every $k = 1,2,\ldots$, from (6.3) and (6.5) we deduce that for every $k \in \mathbb{N}$ there is an $i \in \mathbb{N}$ such that $|b_i| \ge k^2$. This contradicts (6.10), thus the theorem is proved. ∎

Proof of Theorem 2.1: We show that if $\liminf_{i\to\infty} \lambda_i/\lambda_{i-1} = 1$, then there is no $c_1(\Lambda)$ such that $|a_{j,n}| \le c_1(\Lambda)$ for every $j = 0,1,\ldots,n$; $n = 0,1,\ldots$. To see this, for an arbitrary $\varepsilon > 0$ we select an $n \in \mathbb{N}$ such that $\lambda_{n-1}/\lambda_n > 1-\varepsilon$. Observe that $P_n(x) = x^{\lambda_n} - x^{\lambda_{n-1}}$ achieves its maximum modulus on $[0,1]$ at

$$x = \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{1/(\lambda_n-\lambda_{n-1})},$$

hence

$$\max_{0\le x\le 1} |P_n(x)| \le \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{\frac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}} \left(1-\frac{\lambda_{n-1}}{\lambda_n}\right) \le 1-\frac{\lambda_{n-1}}{\lambda_n} < \varepsilon,$$

which shows that the leading coefficient of the n-th Chebyshev polynomial T_n of $H(\Lambda)$ on $[0,1]$ is at least $1/\varepsilon$, otherwise

$$\frac{1}{a_{n,n}}T_n - P_n \in H_{n-1}(\Lambda)$$

would have at least n zeros in $(0,1)$, a contradiction. ∎

Proof of Theorem 3.1: Let $p = H(\Lambda)$ be of the form

$$p(x) = b_{0,n} + \sum_{j=1}^{n} b_{j,n}x^{\lambda_j} \tag{6.11)}$$

such that

$$\max_{0\le x\le 1} |p(x)| \le 1. \tag{6.12}$$

From $\lambda_{i+1}/\lambda_i \ge c_2(\Lambda) > 1$ ($i = 1,2,\ldots$), $\lambda_1 \ge 1$, Theorem 2.2 and (6.12), we obtain

$$\begin{aligned} |p'(y)| &= \Big|\sum_{j=1}^{n} b_{j,n}\lambda_j y^{\lambda_j-1}\Big| \le \sum_{j=1}^{n} |b_{j,n}|\lambda_j y^{\lambda_j-1} \\ &\le c_1(\Lambda)\sum_{j=1}^{n} \lambda_j y^{\lambda_j-1} \le c_3(\Lambda)\sum_{j=0}^{\infty} y^j = \frac{c_3(\Lambda)}{1-y}, \end{aligned} \tag{6.13}$$

which yields the theorem. ∎

To prove Theorem 4.1 we need two lemmas.

LEMMA 6.1: *If $\lambda_0 = 0$ and $\lambda_{i+1}/\lambda_i \geq c_2(\Lambda) > 1$ for every $i = 1, 2, \ldots$, then for every $0 < y < 1$ there is an integer $k(y, \Lambda) > 0$ depending only on y and Λ such that the n-th Chebyshev polynomial T_n of $H(\Lambda)$ on $[0,1]$ has at most $k(y,\Lambda)$ zeros in $[0, y]$ $(n = 1, 2, \ldots)$.*

The proof of Lemma 6.1 can be found in [1, Theorem 3].

LEMMA 6.2: *Let $\lambda_0 = 0$, $\lambda_1 \geq 1$ and $\lambda_{i+1}/\lambda_i \geq c_2(\Lambda) > 1$ for every $i = 1, 2, \ldots$. Denote the extreme points of the n-th Chebyshev polynomial T_n of $H(\Lambda)$ on $[0,1]$ by $1 = y_{0,n} > y_{1,n} > \ldots > y_{n,n} = 0$. Then there is a constant $c = c_4(\Lambda) > 0$ depending only on Λ such that*

$$(6.14)\quad T_n(x) \geq \frac{1}{2} \quad \text{if} \quad y_{j,n} \leq x \leq y_{j,n} + c(1 - y_{j,n}), \qquad 1 \leq j \leq n,\ j \text{ is even}$$

and

$$(6.15)\quad T_n(x) \leq -\frac{1}{2} \quad \text{if} \quad y_{j,n} \leq x \leq y_{j,n} + c(1 - y_{j,n}), \qquad 1 \leq j \leq n,\ j \text{ is odd.}$$

Proof of Lemma 6.2: The proof is a straightforward combination of the equioscillation of the Chebyshev polynomials T_n, the Mean Value Theorem and Theorem 3.1. ∎

Proof of Theorem 4.2: Without loss of generality we may assume that $\lambda_1 \geq 1$; the case $\lambda_1 > 0$ can be obtained from this by the scaling $x \to x^{1/\lambda_1}$. Denote the Lebesgue outer measure of a set $A \subset [0,1]$ by $m(A)$. If $m(A) > 0$ and $A \subset [0,1]$, then by the Lebesgue Density Theorem there is an $0 < a \in A$ such that the left hand side density of A at a is 1. By the scaling $x \to x/a$, without loss of generality we may assume that $a = 1$, that is

$$(6.16)\qquad \lim_{x \to 0+} \frac{m((1-x, 1) \cap A)}{x} = 1.$$

Hence, there is a $0 < y < 1$ such that

$$(6.17)\qquad \frac{m((1-x) \cap A)}{x} \geq \max\{1 - c/2, 3/4\} \quad \text{for every } 0 < x \leq y,$$

where $c = c_4(\Lambda) > 0$ is the same as in Lemma 6.2. By Lemma 6.1 there is an integer $k = k(y/2, \Lambda) > 0$ such that for the extreme points $1 = y_{0,n} > y_{1,n} > \cdots > y_{n,n} = 0$ of the n-th Chebyshev polynomial T_n of $H(\Lambda)$ on $[0,1]$ we have

$$(6.18)\qquad y_{j,n} > 1 - y/2 \quad \text{if} \quad 0 \leq j \leq n - k.$$

Since $m((1-y, 1-y/2) \cap A) > 0$ (see (6.17)), there are $k+3$ distinct points $a_1 < a_2 < \cdots < a_{k+3}$ in $(1-y, 1-y/2) \cap A$. Now let g be a continuous function on $[0,1]$ (and hence on A) such that

$$g(x) = 0 \quad \text{if} \quad 1 - y/2 \le x \le 1 \tag{6.19}$$

and

$$g(a_j) = 2(-1)^j \quad \text{for every } j = 1, 2, \ldots, k+3. \tag{6.20}$$

Assume that there is a $p \in H(\Lambda)$ such that

$$\max_{x \in A} |p(x) - g(x)| \le \tfrac{1}{4}. \tag{6.21}$$

We will show that $p - T_n \in H_n(\Lambda)$ has at least $n+1$ different zeros in $(0,1)$, a contradiction. Indeed, it follows from Lemma 6.2 and (6.17) that there are $n-k$ distinct points $z_{1,n} > z_{2,n} > \cdots > z_{n-k,n}$ from A such that

$$T(z_{j,n}) \ge \tfrac{1}{2} \quad \text{if} \quad 1 \le j \le n-k \text{ and } j \text{ is even} \tag{6.22}$$

and

$$T(z_{j,n}) \le -\tfrac{1}{2} \quad \text{if} \quad 1 \le j \le n-k \text{ and } j \text{ is odd.} \tag{6.23}$$

Now (6.19) – (6.23) and $\max_{0 \le x \le 1} |T_n(x)| = 1$ imply

$$(p - T_n)(z_{j,n}) < 0 \quad \text{if} \quad 1 \le j \le n-k \text{ and } j \text{ is even,} \tag{6.24}$$

$$(p - T_n)(z_{j,n}) > 0 \quad \text{if} \quad 1 \le j \le n-k \text{ and } j \text{ is odd,} \tag{6.25}$$

$$(p - T_n)(a_j) < 0 \quad \text{if} \quad 1 \le j \le k+3 \text{ and } j \text{ is odd,} \tag{6.26}$$

and

$$(p - T_n)(z_{j,n}) > 0 \quad \text{if} \quad 1 \le j \le k+3 \text{ and } j \text{ is even.} \tag{6.27}$$

From (6.24) – (6.27) we can deduce that $p - T_n$ has at least $k+2$ zeros in $(a_1, a_{k+3}) \subset (1-y, 1-y/2)$. Thus $p - T_n \in H_n$ has at least $n+1$ different zeros in $(0,1)$, but $p \not\equiv T_n$, since $\max_{0 \le x \le 1} |p(x)| \ge 7/4$ and $\max_{0 \le x \le 1} |T_n(x)| = 1$. This is a contradiction which finishes the proof. ∎

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