TRIGONOMETRIC POLYNOMIALS WITH MANY REAL ZEROS AND A LITTLEWOOD-TYPE PROBLEM

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Abstract. We examine the size of a real trigonometric polynomial of degree at most n having at least k zeros in $K := \mathbb{R} \pmod{2\pi}$ (counting multiplicities). This result is then used to give a new proof of a theorem of Littlewood concerning flatness of unimodular trigonometric polynomials. Our proof is shorter and simpler than Littlewood's. Moreover our constant is explicit in contrast to Littlewood's approach, which is indirect.

1. Introduction

The set of all polynomials of degree n with coefficients ± 1 will be denoted by $\mathcal{L}_n$. Specifically

$$\mathcal{L}_n := \left\{ p : p(z) = \sum_{j=0}^{n} a_j z^j \,, \;\; a_j \in \{-1, 1\} \right\}.$$

Let D denote the closed unit disk of the complex plane. Let ∂D denote the unit circle of the complex plane. Littlewood made the following conjecture about $\mathcal{L}_n$ in the fifties.

Conjecture 1.1 (Littlewood). *There are at least infinitely many values of $n \in \mathbb{N}$ for which there are polynomials $p_n \in \mathcal{L}_n$ so that*

$$C_1(n+1)^{1/2} \le |p_n(z)| \le C_2(n+1)^{1/2}$$

for all $z \in \partial D$. Here the constants C_1 and C_2 are independent of n.

Since the $L_2(\partial D)$ norm of a polynomial from $\mathcal{L}_n$ is exactly $(2\pi)^{1/2}(n+1)^{1/2}$, the constants must satisfy $C_1 \le 1$ and $C_2 \ge 1$. See Problem 19 of [Li-68]. While there is much literature on this problem and its variants this is still open. See [Saf-90] and [Bor-98]. In fact, finding polynomials that satisfy just the lower bound in Conjecture 1.1 is still open. The Rudin–Shapiro polynomials satisfy the upper bound.

There is a related conjecture of Erdős [Er-62].

1991 *Mathematics Subject Classification*. Primary: 41A17.

Key words and phrases. real trigonometric polynomials, real zeros, unimodular trigonometric polynomials, flatness.

Research of the first author supported, in part, by NSERC of Canada. Research of the second author is supported, in part, by NSF under Grant No. DMS–9623156.

Conjecture 1.2 (Erdős). *There is a constant $\varepsilon > 0$ (independent of n) so that*

$$\max_{z \in \partial D} |p_n(z)| \geq (1+\varepsilon)(n+1)^{1/2}$$

for every $p_n \in \mathcal{L}_n$ and $n \in \mathbb{N}$. That is, the constant C_2 in Conjecture 1.1 must be bounded away from 1 (independently of n).

This conjecture is also open. Kahane [Kah-85], however, shows that if the polynomials are allowed to have complex coefficients of modulus 1 then Conjecture 1.1 holds and Conjecture 1.2 fails. That is, for every $\varepsilon > 0$ there are infinitely many values of $n \in \mathbb{N}$ for which there are polynomials p_n of degree n with complex coefficients of modulus 1 that satisfy

$$(1-\varepsilon)(n+1)^{1/2} \leq |p_n(z)| \leq (1+\varepsilon)(n+1)^{1/2}$$

for all $z \in \partial D$. Beck [Bec-91] extends Kahane's result (with two constants $C_1 > 0$ and $C_2 > 0$ instead of $1-\varepsilon$ and $1+\varepsilon$) for the class of polynomials of degree n whose coefficients are 400th roots of unity.

Our main result is a reproving of Conjecture 1.2 for real trigonometric polynomials. This is Corollary 2.4 of the next section. Littlewood gives a proof of this in [Li-61] and explores related issues in [Li-62], [Li-66a], and [Li-66b]. Our approach is via Theorem 2.1 which estimates the measure of the set where a real trigonometric polynomial of degree at most n with at least k zeros in $K := \mathbb{R} \pmod{2\pi}$ is small. There are two reasons for doing this. First the approach is, we believe, easier and secondly it leads to explicit constants.

2. New Results

Let $K := \mathbb{R} \pmod{2\pi}$. For the sake of brevity the uniform norm of a continuous function p on K will be denoted by $\|p\|_K := \|p\|_{L_\infty(K)}$. Let $\mathcal{T}_n$ denote the set of all real trigonometric polynomials of degree at most n, and let $\mathcal{T}_{n,k}$ denote the subset of those elements of $\mathcal{T}_n$ that have at least k zeros in K (counting multiplicities).

Theorem 2.1. *Suppose $p \in \mathcal{T}_n$ has at least k zeros in K (counting multiplicities). Let $\alpha \in (0,1)$. Then*

$$m\{t \in K : |p(t)| \leq \alpha \|p\|_K\} \geq \frac{\alpha}{e} \frac{k}{n},$$

where $m(A)$ denotes the one-dimensional Lebesgue measure of $A \subset K$.

Theorem 2.2. *We have*

$$2\pi \left(1 - \frac{c_2 k}{n}\right) \leq \sup_{p \in \mathcal{T}_{n,k}} \frac{\|p\|_{L_1(K)}}{\|p\|_{L_\infty(K)}} \leq 2\pi \left(1 - \frac{c_1 k}{n}\right)$$

for some absolute constants $0 < c_1 < c_2$.

Theorem 2.3. *Assume that $p \in \mathcal{T}_n$ satisfies*

$$\|p\|_{L_2(K)} \leq An^{1/2} \tag{2.1}$$

and

$$\|p'\|_{L_2(K)} \geq Bn^{3/2}\,. \tag{2.2}$$

Then there is a constant $\varepsilon > 0$ depending only on A and B such that

$$\|p\|_K^2 \geq (2\pi - \varepsilon)^{-1} \|p\|_{L_2(K)}^2\,. \tag{2.3}$$

Here

$$\varepsilon = \frac{\pi^3}{1024e} \frac{B^6}{A^6}$$

works.

Corollary 2.4. *Let $p \in \mathcal{T}_n$ be of the form*

$$p(t) = \sum_{k=1}^{n} a_k \cos(kt - \gamma_k)\,, \qquad a_k = \pm 1\,, \quad \gamma_k \in \mathbb{R}\,, \quad k = 1, 2, \dots, n\,.$$

Then there is a constant $\varepsilon > 0$ such that

$$\|p\|_K^2 \geq (2\pi - \varepsilon)^{-1} \|p\|_{L_2(K)}^2\,.$$

Here

$$\varepsilon := \frac{\pi^3}{1024e} \frac{1}{27}$$

works.

3. Proofs

To prove Theorem 2.1 we need the lemma below that is proved in [BE-95, E.11 of Section 5.1 on pages 236–237].

Lemma 3.1. *Let $p \in \mathcal{T}_n$, $t_0 \in K$, and $r > 0$. Then p has at most $enr|p(t_0)|^{-1}\|p\|_K$ zeros in the interval $[t_0 - r, t_0 + r]$.*

Proof of Theorem 2.1. Suppose $p \in \mathcal{T}_n$ has at least k zeros in K, and let $\alpha \in (0,1)$. Then

$$\{t \in K : |p(t)| \leq \alpha \|p\|_K\}$$

can be written as the union of pairwise disjoint intervals I_j, $j = 1, 2, \dots, m$. Each of the intervals I_j contains a point $y_j \in I_j$ such that

$$|p(y_j)| = \alpha \|p\|_K\,.$$

Also, each zero of p from K is contained in one of the intervals I_j. Let μ_j denote the number of zeros of p in I_j. Since $p \in \mathcal{T}_n$ has at least k zeros in K, we have $\sum_{j=1}^m \mu_j \geq k$. Note also that Lemma 3.1 implies that

$$\mu_j \leq en|I_j|(\alpha\|p\|_K)^{-1}\|p\|_K = \frac{en}{\alpha}|I_j|\,.$$

Therefore

$$k \leq \sum_{j=1}^m \mu_j \leq \frac{en}{\alpha}\sum_{j=1}^m |I_j| \leq \frac{en}{\alpha} m(\{t \in K : |p(t)| \leq \alpha\|p\|_K\})\,,$$

and the result follows. □

Proof of Theorem 2.2. The upper bound of the theorem follows from Theorem 2.1 applied with $\alpha = 1/2$. The lower bound follows by considering

$$p(t) := D_m(0)^2 - D_m(kt)^2 \in \mathcal{T}_{n,k} \qquad \text{with} \qquad m = \left\lfloor \frac{n}{2(k+1)} \right\rfloor\,,$$

where

$$D_m(t) = \frac{1}{2} + \sum_{j=1}^m \cos jt$$

is the Dirichlet kernel of degree m. □

Proof of Theorem 2.3. First note that by Bernstein's inequality for real trigonometric polynomials in $L_2(K)$, we have $B \leq A$. Assume that $p \in \mathcal{T}_n$ satisfies (2.1) and (2.2) but (2.3) does not hold with $\varepsilon = \pi$. Then

$$M := \|p\|_K \leq (2\pi - \pi)^{-1/2}\|p\|_{L_2(K)} \leq \pi^{-1/2}An^{1/2}\,. \tag{3.1}$$

Combining this with Bernstein's inequality we obtain

$$\|p'\|_K \leq n\|p\|_K \leq \pi^{-1/2}An^{3/2}\,. \tag{3.2}$$

Using (2.2), we obtain

$$\begin{aligned} B^2n^3 &\leq \|p'\|^2_{L_2(K)} = \int_K |p'(t)|^2\,dt \\ &\leq \|p'\|_K \int_K |p'(t)|\,dt \leq \pi^{-1/2}An^{3/2}\|p'\|_{L_1(K)}\,, \end{aligned}$$

that is

$$\|p'\|_{L_1(K)} \geq \pi^{1/2}\frac{B^2}{A}n^{3/2}\,. \tag{3.3}$$

Associated with $p \in \mathcal{T}_n$, $M = \|p\|_K$, and $\gamma \in [0,1]$, let

$$A_\gamma = A_\gamma(p) = \{t \in K : |p(t)| \leq (1-\gamma)M\} \tag{3.4}$$

and

$$B_\gamma = B_\gamma(p) = \{t \in K : |p(t)| > (1-\gamma)M\}\,. \tag{3.5}$$

Since every horizontal line $y = c$ intersects the graph of $p \in \mathcal{T}_n$ in at most $2n$ points with x coordinates in K, we have

$$\int_{B_\gamma} |p'(t)|\,dt \le 4n\gamma M \le 4n\gamma\pi^{-1/2}An^{1/2} \le \frac{\pi^{1/2}}{2}\frac{B^2}{A}n^{3/2} \tag{3.6}$$

if

$$4\gamma\pi^{-1/2}A \le \frac{\pi^{1/2}}{2}\frac{B^2}{A} \qquad \text{that is, if} \qquad \gamma \le \frac{\pi}{8}\frac{B^2}{A^2}\,.$$

Now (3.3)–(3.6) give

$$\int_{A_\gamma} |p'(t)|\,dt \ge \frac{\pi^{1/2}}{2}\frac{B^2}{A}n^{3/2} \qquad \text{with} \qquad \gamma = \frac{\pi}{8}\frac{B^2}{A^2}\,.$$

From this, with the help of (3.1) we can deduce that there is a

$$\delta \in (-(1-\gamma)M, (1-\gamma)M)$$

such that $p - \delta$ has at least

$$\frac{\frac{\pi^{1/2}}{2}\frac{B^2}{A}n^{3/2}}{2(1-\gamma)M} \ge \frac{\pi^{1/2}}{4}\frac{B^2}{A}\frac{n^{3/2}}{M} \ge \frac{\pi^{1/2}}{4}\frac{B^2}{A}\frac{n^{3/2}}{\pi^{-1/2}An^{1/2}} = \frac{\pi}{4}\frac{B^2}{A^2}n$$

zeros in K. Therefore Theorem 2.1 yields that

$$\begin{aligned} m\left\{t \in K : |p(t)| \le \left(1 - \frac{\gamma}{2}\right)\|p\|_K\right\} &\ge m\left\{t \in K : |p(t) - \delta| \le \frac{\gamma}{2}\|p\|_K\right\} \\ &\ge m\left\{t \in K : |p(t) - \delta| \le \frac{\gamma}{4}\|p-\delta\|_K\right\} \\ &\ge \frac{1}{e}\frac{\gamma}{4}\frac{\pi}{4}\frac{B^2}{A^2}\frac{n}{n} \ge \frac{\pi^2}{128e}\frac{B^4}{A^4}\,. \end{aligned}$$

Therefore

$$\begin{aligned} 2\pi\|p\|_K^2 - \|p\|_{L_2(K)}^2 &= \int_K (\|p\|_K^2 - |p(t)|^2)\,dt \ge \frac{\pi^2}{128e}\frac{B^4}{A^4}\frac{\gamma}{2}\|p\|_K^2 \\ &= \frac{\pi^3}{1024e}\frac{B^6}{A^6}\|p\|_K^2\,. \end{aligned}$$

We now conclude that

$$\|p\|_{L_2(K)}^2 \le \left(2\pi - \frac{\pi^3}{1024e}\frac{B^6}{A^6}\right)\|p\|_K^2\,,$$

and the result follows. □

Proof of Corollary 2.4. Let $p \in \mathcal{T}_n$ be of the given form. We have

$$\|p\|^2_{L_2(K)} = \pi \sum_{k=1}^{n} a_k^2 = \pi n\,,$$

that is

$$\|p\|_{L_2(K)} = \pi^{1/2} n^{1/2}\,.$$

Also

$$\|p'\|^2_{L_2(K)} = \pi \sum_{k=1}^{n} k^2 a_k^2 = \pi \frac{n(n+1)(2n+1)}{6} \geq \frac{\pi}{3} n^3\,,$$

that is

$$\|p'\|_{L_2(K)} \geq \left(\frac{\pi}{3}\right)^{1/2} n^{3/2}\,.$$

Now the result follows from Theorem 2.3 with $A := \pi^{1/2}$ and $B := (\pi/3)^{1/2}$. $\square$

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