SHARP EXTENSIONS OF BERNSTEIN'S INEQUALITY TO RATIONAL SPACES

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Abstract. Sharp extensions of some classical polynomial inequalities of Bernstein are established for rational function spaces on the unit circle, on $K := \mathbb{R} \pmod{2\pi}$, on $[-1,1]$ and on $\mathbb{R}$. The key result is the establishment of the inequality

$$|f'(z_0)| \leq \max\left\{ \sum_{\substack{j=1 \\ |a_j|>1}} \frac{|a_j|^2-1}{|a_j-z_0|^2}, \quad \sum_{\substack{j=1 \\ |a_j|<1}} \frac{1-|a_j|^2}{|a_j-z_0|^2} \right\} \|f\|_{\partial D}$$

for every rational function $f = p_n/q_n$, where p_n is a polynomial of degree at most n with complex coefficients and

$$q_n(z) = \prod_{j=1}^{n} (z - a_j)$$

with $|a_j| \neq 1$ for each j, and for every $z_0 \in \partial D$, where $\partial D := \{z \in \mathbb{C} : |z| = 1\}$. The above inequality is sharp at every $z_0 \in \partial D$.

1. Introduction, Notation.

We denote by $\mathcal{P}_n^r$ and $\mathcal{P}_n^c$ the sets of all algebraic polynomials of degree at most n with real or complex coefficients, respectively. The sets of all trigonometric polynomials of degree at most n with real or complex coefficients, respectively, are denoted by $\mathcal{T}_n^r$ and $\mathcal{T}_n^c$. We will use the notation

$$\|f\|_A = \sup_{z \in A} |f(z)|$$

for continuous functions f defined on A. Let

$$D := \{z \in \mathbb{C} : |z| \leq 1\},$$
$$\partial D := \{z \in \mathbb{C} : |z| = 1\}$$

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and

$$K := \mathbb{R}\ (\mathrm{mod} 2\pi).$$

The classical inequalities of Bernstein [1] state that

$$\begin{aligned} |p'(z_0)| &\leq n\|p\|_{\partial D}, & p \in \mathcal{P}_n^c, \quad z_0 \in \partial D, \\ |t'(\theta_0)| &\leq n\|t\|_K, & t \in \mathcal{T}_n^c, \quad \theta_0 \in K, \\ |p'(x_0)| &\leq \frac{n}{\sqrt{1-x_0^2}}\|p\|_{[-1,1]}, & p \in \mathcal{P}_n^c, \quad x_0 \in (-1,1). \end{aligned}$$

Proofs of the above inequalities may be found in almost every book on approximation theory, see [4], [5], [6] or [8] for instance. An extensive study of Markov- and Bernstein-type inequalities is presented in [7].

In this paper we study the rational function spaces:

$$\mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \partial D) := \left\{ \frac{p_n(z)}{\prod_{j=1}^n (z-a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on ∂D with $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \partial D$;

$$\mathcal{T}_n^c(a_1, a_2, \cdots, a_{2n}; K) := \left\{ \frac{t_n(\theta)}{\prod_{j=1}^{2n} \sin((\theta-a_j)/2)} : t_n \in \mathcal{T}_n^c \right\}$$

on K with $\{a_1, a_2, \cdots, a_{2n}\} \subset \mathbb{C} \setminus \mathbb{R}$;

$$\mathcal{P}_n^c(a_1, a_2, \cdots, a_n; [-1,1]) := \left\{ \frac{p_n(x)}{\prod_{j=1}^n (x-a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on $[-1,1]$ with $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus [-1,1]$;

$$\mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \mathbb{R}) := \left\{ \frac{p_n(x)}{\prod_{j=1}^n (x-a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on $\mathbb{R}$ with $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$, and

$$\mathcal{P}_n^r(a_1, a_2, \cdots, a_n; \mathbb{R}) := \left\{ \frac{p_n(x)}{\prod_{j=1}^n |x-a_j|} : p_n \in \mathcal{P}_n^r \right\}$$

on $\mathbb{R}$ with $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$.

The spaces

$$\mathcal{T}_n^r(a_1, a_2, \cdots, a_{2n}; K) := \left\{ \frac{t_n(\theta)}{\prod_{j=1}^{2n} |\sin((\theta - a_j)/2)|} : t_n \in \mathcal{T}_n^r \right\}$$

on K with $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$ and

$$\mathcal{P}_n^r(a_1, a_2, \cdots, a_n; [-1,1]) := \left\{ \frac{p_n(x)}{\prod_{j=1}^{n} |x - a_j|} : p_n \in \mathcal{P}_n^r \right\}$$

on $[-1,1]$ with $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus [-1,1]$ have been studied in [2] and [3], and the sharp Bernstein-Szegő type inequalities

$$f'(\theta_0)^2 + \widetilde{B}_n(\theta_0)^2 \, f(\theta_0)^2 \le \widetilde{B}(\theta_0)^2 \|f\|_K^2, \qquad \theta_0 \in K$$

for every $f \in \mathcal{T}_n^r(a_1, a_2, \cdots, a_{2n}; K)$ with

$$(a_1, a_2, \cdots, a_{2n}) \subset \mathbb{C} \setminus \mathbb{R}, \qquad \mathrm{Im}(a_j) > 0, \quad j = 1, 2, \cdots, 2n$$

and

$$(1 - x_0^2) f'(x_0)^2 + B_n(x_0)^2 f(x_0)^2 \le B_n(x_0)^2 \|f\|_{[-1,1]}^2, \qquad x_0 \in (-1,1)$$

for every $f \in \mathcal{P}_n^r(a_1, a_2, \cdots, a_n; [-1,1]))$ with

$$\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus [-1,1]$$

have been proved, where

$$\widetilde{B}_n(\theta) := \frac{1}{2} \sum_{j=1}^{2n} \frac{1 - |e^{ia_j}|^2}{|e^{ia_j} - e^{i\theta}|^2}, \qquad \theta \in K$$

and

$$B_n(x) := \mathrm{Re} \left(\sum_{j=1}^{n} \frac{\sqrt{a_j^2 - 1}}{a_j - x} \right), \qquad x \in [-1,1]$$

with the choice of $\sqrt{a_j^2 - 1}$ is determined by

$$\left| a_j - \sqrt{a_j^2 - 1} \right| < 1.$$

These inequalities give sharp upper bound for $|f'(\theta)|$ and $|f'(x_0)|$ only at n points in K and $[-1,1]$, respectively. In this paper we establish Bernstein-type inequalities for the spaces

$$\mathcal{P}_n^c(a_1, a_2, \cdots, a_n, \partial D) \quad \text{and} \quad \mathcal{T}_n^c(a_1, a_2, \cdots, a_{2n}; K)$$

which are sharp at every $z \in \partial D$ and $\theta \in K$, respectively. An essentially sharp Bernstein-type inequality is also established for the space

$$\mathcal{P}_n^c(a_1, a_2, \cdots, a_n; [-1,1]).$$

A Bernstein-type inequality of Russak [7] is extended to the spaces

$$\mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \mathbb{R}),$$

and a Bernstein-Szegő type inequality is established for the spaces

$$\mathcal{P}_n^r(a_1, a_2, \cdots, a_n; \mathbb{R}).$$

For a polynomial

$$q_n(z) = c \prod_{j=1}^{n} (z - a_j), \qquad 0 \neq c \in \mathbb{C}, \qquad a_j \in \mathbb{C}$$

we define

$$q_n^*(z) = \overline{c} \prod_{j=1}^{n} (1 - \overline{a}_j z) = z^n \overline{q}_n(z^{-1}).$$

It is well-known, and simple to check, that

$$|q_n(z)| = |q_n^*(z)|, \qquad z \in \partial D.$$

We also define the Blaschke products

$$S_n(z) := \prod_{j=1}^{n} \frac{1 - \overline{a}_j z}{z - a_j}$$

associated with $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \partial D$, and

$$\widetilde{S}_n(z) := \prod_{j=1}^{n} \frac{z - \overline{a}_j}{z - a_j}$$

associated with $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$.

2. New Results.

Theorem 1. *Let* $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \partial D$. *Then*

$$|f'(z_0)| \leq \max\left\{ \sum_{\substack{j=1 \\ |a_j|>1}} \frac{|a_j|^2 - 1}{|a_j - z_0|^2}, \sum_{\substack{j=1 \\ |a_j|<1}} \frac{1 - |a_j|^2}{|a_j - z_0|^2} \right\} \|f\|_{\partial D}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \partial D)$ *and* $z_0 \in \partial D$. *If the first sum is not less than the second sum for a fixed* $z_0 \in \partial D$, *then equality holds for* $f = c\, S_n^+$, $c \in \mathbb{C}$, *where* S_n^+ *is the Blaschke product associated with those* a_j *for which* $|a_j| > 1$. *If the first sum is not greater than the second sum for a fixed* $z_0 \in \partial D$, *then equality holds for* $f = c\, S_n^-$, $c \in \mathbb{C}$, *where* S_n^- *is the Blaschke product associated with those* a_j *for which* $|a_j| < 1$.

Theorem 2. *Let* $\{a_1, a_2, \cdots, a_{2n}\} \subset \mathbb{C} \setminus \mathbb{R}$. *Then*

$$|f'(\theta_0)| \le \max\left\{ \sum_{\substack{j=1\\ \mathrm{Im}(a_j)<0}}^{2n} \frac{|e^{ia_j}|^2 - 1}{|e^{ia_j} - e^{i\theta_0}|^2}, \sum_{\substack{j=1\\ \mathrm{Im}(a_j)>0}}^{2n} \frac{1 - |e^{ia_j}|^2}{|e^{ia_j} - e^{i\theta_0}|^2} \right\} \|f\|_K$$

for every $f \in \mathcal{T}_n^c(a_1, a_2, \cdots, a_{2n}; K)$ *and* $\theta_0 \in K$. *If the first sum is not less than the second sum for a fixed* $\theta_0 \in K$, *then equality holds for* $f(\theta) = cS_{2n}^+(e^{i\theta})$, $c \in \mathbb{C}$. *If the first sum is not greater than the second sum for a fixed* $\theta_0 \in K$, *then equality holds for* $f(\theta) = cS_{2n}^-(e^{i\theta})$, $c \in \mathbb{C}$. S_{2n}^+ *and* S_{2n}^- *associated with* $\{e^{ia_1}, e^{ia_2}, \cdots, e^{ia_{2n}}\}$ *are defined as in Theorem 1.*

Theorem 3. *Let* $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C}/[-1, 1]$ *and*

$$c_j := a_j - \sqrt{a_j^2 - 1}, \qquad |c_j| < 1$$

with the choice of root in $\sqrt{a_j^2 - 1}$ *determined by* $|c_j| < 1$. *Then*

$$|f'(x_0)| \le \frac{1}{\sqrt{1 - x_0^2}} \max\left\{ \sum_{j=1}^{n} \frac{|c_j|^{-2} - 1}{|c_j^{-1} - z_0|^2}, \sum_{j=1}^{n} \frac{1 - |c_j|^2}{|c_j - z_0|^2} \right\} \|f\|_{[-1,1]}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \cdots, a_n; [-1, 1])$ *and* $x_0 \in (-1, 1)$, *where* z_0 *is defined by*

$$z_0 := x_0 + i\sqrt{1 - x_0^2}, \qquad x_0 \in (-1, 1).$$

Note that

$$B_n(x_0) = \mathrm{Re}\left(\sum_{j=1}^{n} \frac{\sqrt{a_j^2 - 1}}{a_j - x_0} \right) = \sum_{j=1}^{n} \frac{1 - |c_j|^2}{|c_j - z_0|^2}, \qquad x_0 \in (-1, 1).$$

Our next result extends an inequality established by Russak [7] to wider families of rational functions.

Theorem 4. *Let* $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}$. *Then*

$$|f'(x_0)| \le \max\left\{ \sum_{\substack{j=1\\ \mathrm{Im}(a_j)>0}}^{n} \frac{2|\mathrm{Im}(a_j)|}{|x_0 - a_j|^2}, \sum_{\substack{j=1\\ \mathrm{Im}(a_j)<0}}^{n} \frac{2|\mathrm{Im}(a_j)|}{|x_0 - a_j|^2} \right\} \|f\|_{\mathbb{R}}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \mathbb{R})$ *and* $x_0 \in \mathbb{R}$. *If the first sum is not less than the second sum for a fixed* $x_0 \in \mathbb{R}$, *then equality holds for* $f = c\tilde{S}_n^+$, $c \in \mathbb{C}$, *where* $\tilde{S}_n^+$ *is the Blaschke product associated with the poles* a_j *lying in the upper half-plane*

$$H^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

If the first sum is not greater than the second sum for a fixed $x_0 \in \mathbb{R}$, then equality holds for $f = c\tilde{S}_n^-$, $c \in \mathbb{C}$, where $\tilde{S}_n^-$ is the Blaschke product associated with the poles a_j lying in the lower half-plane

$$H^- := \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}.$$

Our last result is a Bernstein-Szegő type inequality for

$$\mathcal{P}_n^r(a_1, a_2, \cdots, a_{2n}; \mathbb{R})$$

which follows from the Bernstein-Szegő type inequality for

$$\mathcal{P}_n^r(a_1, a_2, \cdots, a_n; [-1, 1])$$

mentioned in the introduction.

Theorem 5. *Let*

$$\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}, \qquad \mathrm{Im}(a_j) > 0, \quad j = 1, 2, \cdots, n.$$

Then

$$f'(x_0)^2 + \widehat{B}_n(x_0)^2 f(x_0)^2 \leq \widehat{B}_n(x_0)^2 \|f\|_{\mathbb{R}}^2, \qquad x_0 \in \mathbb{R},$$

for every $f \in \mathcal{P}_n^r(a_1, a_2, \cdots, a_n; \mathbb{R})$, where

$$\widehat{B}_n(x) := \sum_{j=1}^n \frac{\mathrm{Im}(a_j)}{|x - a_j|^2}, \qquad x \in \mathbb{R}.$$

We remark that equality holds in Theorem 5 if and only if x_0 is a maximum point of f (i.e. $f(x_0) = \pm\|f\|_{\mathbb{R}}$) or f is a "Chebyshev polynomial" for the space $\mathcal{P}_n^r(a_1, a_2, \cdots, a_n; \mathbb{R})$ which can be explicitly expressed by using the results of [2] and [3].

Note that Bernstein's classical inequalities are contained in Theorem 1, 2, and 3 as limiting cases, by taking

$$\{a_1^{(k)}, a_2^{(k)}, \cdots, a_n^{(k)}\} \subset \mathbb{C} \setminus D$$

in Theorems 1 and 3 so that $\lim_{k\to\infty} |a_j^{(k)}| = \infty$ for each $j = 1, 2, \cdots, n$, and by taking

$$\{a_1^{(k)}, a_2^{(k)}, \cdots, a_{2n}^{(k)}\} \subset \mathbb{C} \setminus \mathbb{R}$$

in Theorem 2 so that $a_{n+j}^{(k)} = \overline{a}_j^{(k)}$ and $\lim_{k\to\infty} |\mathrm{Im}(a_j^{(k)})| = \infty$ for each $j = 1, 2, \cdots, n$. Further results can be obtained as limiting cases by fixing $a_1, a_2, \cdots, a_m$, $1 \leq m \leq n$, in Theorems 1 and 3, and by taking

$$\{a_1, a_2, \cdots, a_m, a_{m+1}^{(k)}, a_{m+2}^{(k)}, \cdots, a_n^{(k)}\} \subset \mathbb{C} \setminus D$$

so that $\lim_{k\to\infty} |a_j^{(k)}| = \infty$ for each $j = m+1, m+2, \cdots, n$. One may also fix the poles $a_1, a_2, \cdots, a_m$, $a_{n+1}, a_{n+2}, \cdots, a_{n+m}$, $1 \leq m \leq n$, in Theorem 2 and take

$$\{a_1, \cdots, a_m, a_{m+1}^{(k)}, \cdots, a_n^{(k)}, a_{n+1}, \cdots, a_{n+m}, a_{n+m+1}^{(k)}, \cdots, a_{2n}^{(k)}\} \subset \mathbb{C} \setminus \mathbb{R}$$

so that $a_{n+j}^{(k)} = \overline{a}_j^{(k)}$ and $\lim_{k\to\infty} |\mathrm{Im}(a_j^{(k)})| = \infty$ for each $j = m+1, m+2, \cdots, n$. Several interesting corollaries of the above three theorems can be obtained. We formulate only one of these.

Corollary 6. *Suppose* $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C}$ *and*

$$1 < R \le |a_j|, \qquad j = 1, 2, \cdots, n.$$

Then

$$|f'(z_0)| \le \frac{R+1}{R-1} n \|f\|_{\partial D}, \qquad z \in \partial D,$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \partial D)$. *For a fixed* $z_0 \in \partial D$ *equality holds if and only if*

$$a_1 = a_2 = \cdots = a_n = Rz_0$$

and $f = c\,S_n,\ c \in \mathbb{C}$, *where* S_n *is the Blaschke product associated with the poles* $a_j,\ j = 1, 2, \cdots, n$.

3. Proofs.

To prove Theorem 1 we need the following result (see [9, p. 38] for instance).

Interpolation Theorem. Let V be an $n+1$ dimensional subspace over $\mathbb{C}$ of $C(Q)$, the linear space of complex-valued continuous functions defined on a compact Hausdorff space Q, and let $L \not\equiv 0$ be a linear functional on V. Then there exists distinct points $x_1, x_2, \cdots, x_r$ in Q, where $1 \le r \le 2n+1$, and nonzero real numbers $c_1, c_2, \cdots, c_r$ so that

$$L(f) = \sum_{j=1}^{r} c_j f(x_j), \qquad f \in V$$

and

$$\|L\| := \max_{0 \ne f \in V} \frac{|L(f)|}{\|f\|_Q} = \sum_{j=1}^{r} |c_j|.$$

Proof of Theorem 1. For the reason of symmetry it is sufficient to prove the theorem when $z = 1$. Without loss of generality we may assume that

$$\text{Re}\left(\sum_{j=1}^{n} \frac{1}{1-a_j}\right) \ne \frac{n}{2} \tag{1}$$

the other cases follow from this by a limiting argument. Let $Q := \partial D$ (with the usual metric topology),

$$V := \mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \partial D)$$

and

$$L(f) := f'(1), \qquad f \in V.$$

We show in this situation that $n+1 \le r$ in the Interpolation Theorem. Suppose to the contrary that $r \le n$. By the Interpolation Theorem there are r distinct points $x_1, x_2, \cdots, x_r$ on ∂D so that

$$\frac{p_n'(1)q_n(1) - q_n'(1)p_n(1)}{q_n(1)^2} = \sum_{j=1}^{r} c_j \frac{p_n(x_j)}{q_n(x_j)}, \qquad p_n \in \mathcal{P}_n^c, \tag{2}$$

where

$$q_n(z) := \prod_{j=1}^{n}(z - a_j). \tag{3}$$

We claim that $x_j \neq 1$ for each $j = 1, 2, \cdots, r$. Indeed, if there is an index j so that $x_j = 1$, then the Interpolation Theorem implies that

$$p_n(z) := (z+1)^{n-r} \prod_{j=1}^{r}(z - x_j) \in \mathcal{P}_n^c$$

has a zero at 1 with multiplicity at least 2, a contradiction. Applying (2) to the above p_n, we obtain

$$p_n'(1)q_n(1) - q_n'(1)p_n(1) = 0,$$

and since $p_n(1) \neq 0$ and $q_n(1) \neq 0$, this is equivalent to

$$\frac{q_n'(1)}{q_n(1)} = \frac{p_n'(1)}{p_n(1)}$$

or in terms of the zeros of p_n and q_n

$$\sum_{j=1}^{n} \frac{1}{1-a_j} = \frac{n-r}{2} + \sum_{j=1}^{r} \frac{1}{1-x_j}. \tag{4}$$

Since $x_j \in \partial D$ and $x_j \neq 1$, $\ j = 1, 2, \cdots, r$, we have

$$\operatorname{Re}\left(\frac{1}{1-x_j}\right) = \frac{1}{2}, \qquad j = 1, 2, \cdots, r. \tag{5}$$

It follows from (4) and (5) that

$$\operatorname{Re}\left(\sum_{j=1}^{n} \frac{1}{1-a_j}\right) = \frac{n}{2}$$

which contradicts assumption (1). So $n + 1 \leq r$, indeed.

A simple compactness argument shows that there is a function $\tilde{f} \in V$ so that $\|\tilde{f}\|_{\partial D} = 1$ and $|L(\tilde{f})| = \|L\|$. The interpolation Theorem implies

$$|\tilde{f}(x_j)| = 1, \qquad j = 1, 2, \cdots, r.$$

Hence, if

$$\tilde{f} = \frac{\tilde{p}_n}{q_n}, \quad \tilde{p}_n \in \mathcal{P}_n^c, \quad q_n(z) = \prod_{j=1}^{n}(z - a_j),$$

then

$$h(z) = |\tilde{p}_n(z)|^2 - |q_n(z)|^2 \leq 0, \qquad z \in \partial D \tag{6}$$

and

$$h(x_j) = 0, \qquad j = 1, 2, \cdots, r. \tag{7}$$

Note that $t(\theta) := h(e^{i\theta}) \in \mathcal{T}_n^r$ vanishes at each θ_j, where $\theta_j \in [0, 2\pi)$ is defined by $x_j = e^{i\theta_j}$, $j = 1, 2, \cdots, r$. Because of (6), each of these zeros is of even multiplicity. Hence, $n + 1 \le r$ implies that $t \in \mathcal{T}_n$ has at least $2n + 2$ zeros with multiplicities, therefore $t(\theta) \equiv 0$. From this we can deduce that $h(z) = 0$ for every $z \in \partial D$, so

$$|\tilde{p}_n(z)| = |q_n(z)|, \qquad z \in \partial D. \tag{8}$$

We have

$$z^{-n}\tilde{p}_n(z)\tilde{p}_n^*(z) = |\tilde{p}_n(z)|^2 = |q_n(z)|^2 = z^{-n}q_n(z)q_n^*(z), \qquad z \in \partial D,$$

so by the Unicity Theorem of analytic functions

$$\tilde{p}_n\tilde{p}_n^* = q_nq_n^*.$$

¿From this it follows that there is a constant $0 \neq c \in \mathbb{C}$ so that

$$\tilde{f}(z) = \frac{\tilde{p}_n(z)}{q_n(z)} = c\prod_{j=1}^{m} \frac{z - 1/\overline{\alpha}_j}{z - \alpha_j}$$

with some $m \le n$ and

$$\alpha_j := a_{k_j}, \quad j = 1, 2, \cdots, m, \quad 1 \le k_1 < k_2 < \cdots < k_m \le n.$$

A straightforward calculation gives

$$|\tilde{f}'(1)| = \left|\frac{\tilde{f}'(1)}{\tilde{f}(1)}\right| = \left|\sum_{j=1}^{m}\left(\frac{1}{1 - 1/\overline{\alpha}_j} - \frac{1}{1 - \alpha_j}\right)\right|$$

$$= \left|\sum_{j=1}^{m} \frac{|\alpha_j|^2 - 1}{|\alpha_j - 1|^2}\right| \le \max\left\{\sum_{\substack{j=1\\|a_j|>1}} \frac{|a_j|^2 - 1}{|a_j - 1|^2}, \sum_{\substack{j=1\\|a_j|<1}} \frac{1 - |a_j|^2}{|a_j - 1|^2}\right\}$$

which finishes the proof. □

Proof of Theorem 2. Observe that if

$$h_n(\theta) := \prod_{j=1}^{2n} \sin((\theta - a_j)/2) \in \mathcal{T}_n^c$$

and $t_n \in \mathcal{T}_n^c$, then there are $p_{2n} \in \mathcal{P}_{2n}^c$ and $q_{2n} \in \mathcal{P}_{2n}^c$ so that

$$\frac{t_n(\theta)}{h_n(\theta)} = \frac{p_{2n}(e^{i\theta})e^{-in\theta}}{q_{2n}(e^{i\theta})e^{-in\theta}} = \frac{p_{2n}(e^{i\theta})}{q_{2n}(e^{i\theta})},$$

where

$$q_{2n}(z) = c\prod_{j=1}^{2n}(z - e^{ia_j})$$

with some $0 \neq c \in \mathbb{C}$. Therefore the theorem follows from Theorem 1. □

Proof of Theorem 3. The result follows from Theorem 1 by the substitution

$$x = \frac{1}{2}(z + z^{-1}).$$

□

Proof of Theorem 4. The function

$$x = i\frac{z+1}{z-1}$$

maps $\partial D \setminus \{1\} = \{z \in \mathbb{C} : |z| = 1, \quad z \neq 1\}$ onto the real line. A straightforward calculation shows that the inequality of the theorem follows from Theorem 1 by the above substitution. □

Proof of Theorem 5. By Corollary 3.3. of [2] we have

$$(1 - y_0^2)g'(y_0)^2 + B_n(y_0)^2 g(y_0)^2 \leq B_n(y_0)^2 \|g\|^2_{[-1,1]} \tag{9}$$

for every $g \in \mathcal{P}_n^r(b_1, b_2, \cdots, b_n; [-1,1])$ and $y_0 \in [-1,1]$, where

$$\{b_1, b_2, \cdots, b_n\} \subset \mathbb{C} \setminus [-1,1]$$

and

$$B_n(y_0) := \operatorname{Re}\left(\sum_{j=1}^{n} \frac{\sqrt{b_j^2 - 1}}{b_j - y_0}\right), \qquad y_0 \in [-1,1],$$

with the choice of root in $\sqrt{b_j^2 - 1}$ determined by

$$|b_j - \sqrt{b_j^2 - 1}| < 1.$$

Let $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \mathbb{R}, \quad x_0 \in \mathbb{R}$, and

$$f \in \mathcal{P}_n(a_1, a_2, \cdots, a_n; \mathbb{R})$$

be fixed. Let $a \in \mathbb{R}$ be chosen so that $|x_0| < a$, let $y_0 := x_0/a \in (-1,1)$, $b_j := a_j/a, \;\; j = 1, 2, \cdots, n$, and

$$g(x) := f(ax) \in \mathcal{P}_n^r(b_1, b_2, \cdots, b_n; [-1,1]).$$

Applying (9) with the above g and y_0, we obtain

$$(1 - y_0)^2 a^2 f'(x_0)^2 + B_n(y_0)^2 f(x_0)^2 \leq B_n(y_0)^2 \|f\|^2_{[-a,a]}$$

so

$$\frac{a^2 - x_0^2}{a^2} f'(x_0)^2 + (a^{-1}B_n(y_0))^2 f(x_0)^2 \le (a^{-1}B_n(y_0))^2 \|f\|_{\mathbb{R}}^2 \tag{10}$$

where

$$\begin{aligned} \lim_{a\to+\infty} a^{-1}B_n(y_0) &= \lim_{a\to+\infty} \mathrm{Re}\left(\sum_{j=1}^n \frac{\sqrt{b_j^2-1}}{a(b_j-y_0)}\right) \\ &= \lim_{a\to+\infty} \mathrm{Re}\left(\sum_{j=1}^n \frac{\sqrt{(a_j/a)^2-1}}{a_j-x_0}\right) \\ &= \lim_{a\to+\infty} \mathrm{Re}\left(\sum_{j=1}^n \frac{\sqrt{(a_j/a)^2-1}-a_j/a}{a_j-x_0}\right) \\ &= \mathrm{Re}\left(\sum_{j=1}^n \frac{i\ \mathrm{sign}\left(\mathrm{Im}\left(\sqrt{a_j^2-1}-a_j\right)\right)(\overline{a}_j-x_0)}{|\overline{a}_j-x_0|^2}\right) \\ &= \sum_{j=1}^n \frac{\mathrm{Im}(a_j)}{|a_j-x_0|^2} = \widehat{B}_n(x_0) \end{aligned} \tag{11}$$

(note that the map $a \to \sqrt{(a_j/a)^2-1} - a_j/a$ is a continuous map on $(0,\infty)$ taking only nonreal values, and

$$\mathrm{Im}\left(\sqrt{a_j^2-1}-a_j\right) < 0$$

follows from $\left|a_j - \sqrt{a_j^2-1}\right| < 1$ and $\mathrm{Im}(a_j) > 0$.) Therefore, taking the limit on (10) when $a \to +\infty$, we obtain the theorem by (11).

Proof of Corollary 6. The inequality follows from Theorem 1 since $R \le |a_j|$ and $|z_0| = 1$ imply

$$\frac{|a_j|^2-1}{|a_j-z_0|^2} \le \frac{R+1}{R-1}, \qquad j = 1, 2, \cdots, n.$$

Now assume that $\tilde{f} \ne 0$ satisfies

$$|\tilde{f}'(z_0)| = \frac{R+1}{R-1}n, \qquad \|\tilde{f}\|_{\partial D} = 1,$$

for some $z_0 \in \partial D$. Then we obtain from Theorem 1 that

$$\frac{|a_j|^2-1}{|a_j-z_0|^2} = \frac{R+1}{R-1}, \qquad j = 1, 2, \cdots, n,$$

therefore

$$a_j = Rz_0, \qquad j = 1, 2, \cdots, n.$$

Now observe that $1 < R \le |a_j|, \quad j = 1, 2, \cdots, n$, implies

$$\operatorname{Re}\left(\sum_{j=1}^{n} \frac{1}{1-a_j}\right) < \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2},$$

so the proof of Theorem 1 yields that $\tilde{f} = c\,S_n, \ \ |c| = 1$, where S_n is the Blaschke product associated with $\{a_1, a_2, \cdots, a_n\}$.

On the other hand, if $z_0 \in \partial D, \ \ a_1 = a_2 = \cdots = a_n = Rz_0, \ \ S_n$ is the Blaschke product associated with $\{a_1, a_2, \cdots, a_n\}$ and $f = c\,S_n, \ \ c \in \mathbb{C}$, then

$$|f'(z_0)| = \frac{R+1}{R-1}\|f\|_{\partial D} = c\frac{R+1}{R-1}$$

and the proof is finished. □

References

[1]. Bernstein, S. N., *Collected Works I*, Acad. Nuak. SSSR, Moscow, 1952.

[2]. Borwein, P., Erdélyi, T., & Zhang, J., *Chebyshev polynomials and Markov-Bernstein type inequalities for rational spaces*, J. London Math. Soc. (to appear).

[3]. Borwein, P. & Erdélyi, T., *Polynomials and Polynomial Inequalities*, Springer-Verlag (to appear).

[4]. Cheney, E. W., *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966.

[5]. DeVore, R. A. & Lorentz, G. G., *Constructive Approximation*, Springer-Verlag, New York, 1993.

[6]. Lorentz, G. G., *Approximation of Functions*, Holt Rinehart and Winston, New York, 1966.

[7]. Petrushev, P. P. & Popov, V. A., *Rational Approximations of Real Functions*, Cambridge University Press, 1987.

[8]. Rahman, Q. I. & Schmieisser, G., *Les Inegalités de Markoff et de Bernstein*, Les Presses de L'Université de Montreal, 1983.

[9]. Sharpiro, H. S., *Topics in Approximation Theory, Lecture Notes in Mathematics*, Springer-Verlag, New York, 1971.