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PROOF OF SAFFARI'S NEAR-ORTHOGONALITY CONJECTURE FOR ULTRAFLAT SEQUENCES OF UNIMODULAR POLYNOMIALS

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Abstract. Let $P_n(z) = \sum_{k=0}^n a_{k,n}z^k \in \mathbb{C}[z]$ be a sequence of unimodular polynomials ($|a_{k,n}| = 1$ for all k, n) which is ultraflat in the sense of Kahane, i.e.,

$$\lim_{n\to\infty} \max_{|z|=1} \left|(n+1)^{-1/2}|P_n(z)| - 1\right| = 0\,.$$

We prove the following conjecture of Saffari (1991): $\sum_{k=0}^n a_{k,n}a_{n-k,n} = o(n)$ as $n \to \infty$, that is, the polynomial $P_n(z)$ and its "conjugate reciprocal" $P_n^*(z) = \sum_{k=0}^n \overline{a}_{n-k,n}z^k$ become "nearly orthogonal" as $n \to \infty$. To this end we use results from [Er1] where (as well as in [Er3]) we studied the structure of ultraflat polynomials and proved several conjectures of Saffari.

Preuve de la conjecture de quasi-orthogonalité de Saffari pour les suites ultra-plates de polynômes unimodulaires

Résumé. Soit $P_n(z) = \sum_{k=0}^n a_{k,n}z^k \in \mathbb{C}[z]$ une suite de polynômes unimodulaires ($|a_{k,n}| = 1$ pour tout k, n) supposée ultra-plate au sens de Kahane, c.à.d.

$$\lim_{n\to\infty} \max_{|z|=1} \left|(n+1)^{-1/2}|P_n(z)| - 1\right| = 0\,.$$

Nous prouvons la conjecture suivante de Saffari (1991): $\sum_{k=0}^n a_{k,n}a_{n-k,n} = o(n)$ pour $n \to \infty$, c.à.d. que le polynôme $P_n(z)$ et son "reciproque conjugué" $P_n^*(z) = \sum_{k=0}^n \overline{a}_{n-k,n}z^k$ deviennent "quasi-orthogonaux" lorsque $n \to \infty$. Pour ce faire nous employons des résultats de [Er1] où (ainsi que dans [Er3]) nous avons étudié la structure des polynômes ultra-plats et avons prouvé plusieurs conjectures de Saffari.

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Version Française Abrégée

Une suite de polynômes $P_n(z) = \sum_{k=0}^{n} a_{k,n} z^k \in \mathbb{C}[z]$ à coefficients unimodulaires (appelés, pour abréger, "polynômes unimodulaires") est dite ultra-plate s'il existe une suite positive (ε_n) tendant vers zéro telle que, pout tout n, on ait

$$(1-\varepsilon_n)\sqrt{n+1} \leq |P_n(e^{it})| \leq (1+\varepsilon_n)\sqrt{n+1} \qquad (\text{pour tout } t \in \mathbb{R})\,.$$

Le problème de l'existence de telles suites ultra-plates fut soulevé en 1966 par Littlewood [Li1] qui, selon ses collègues et selon des écrits ultérieurs, tantôt conjecturait leur existence et tantôt partageait l'opinion générale (laquelle penchait pour la conjecture d'inexistence). Cependant, en 1980, Kahane [Ka] prouva finalement leur *existence* par une méthode probabiliste (non constructive).

En 1991 B. Saffari [Sa] étudia les polynômes ultra-plats (a priori quelconques, et pas seulement ceux obtenus par la méthode de Kahane [Ka]). Dans deux articles très récents [Er1] et [Er3] nous avons étudié la structure des polynômes ultra-plats (a priori quelconques) et prouvé plusieurs conjectures de Saffari [Sa]. Dans cette Note, nous prouvons le résultat suivant, également conjecturé par Saffari [Sa]:

Théorème. *Si la suite $P_n(z) = \sum_{k=0}^{n} a_{k,n} z^k \in \mathbb{C}[z]$ est ultra-plate, alors*

$$\sum_{k=0}^{n} a_{k,n} a_{n-k,n} = o(n)$$

ce qui signifie que $P_n(z)$ et $P_n^*(z) = \sum_{k=0}^{n} \overline{a}_{n-k,n} z^k$, le polynôme "réciproque-conjugué" de $P_n(z)$, deviennent "quasi-orthogonaux" pour $n \to \infty$.

La démonstration, donnée dans la version anglaise, est basée sur des techniques d'analyse réelle et sur des résultats que nous avons prouvés dans [Er3] par des techniques d'analyse complexe.

1. Introduction and the New Result

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let

$$\mathcal{K}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \mathbb{C}\,, \;\; |a_k| = 1 \right\}.$$

The class $\mathcal{K}_n$ is often called the collection of all (complex) unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k, \;\; a_k \in \{-1, 1\} \right\}.$$

The class $\mathcal{L}_n$ is often called the collection of all (real) unimodular polynomials of degree n. By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2 \, dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z\in\partial D} |P_n(z)| \leq \sqrt{n+1} \leq \max_{z\in\partial D} |P_n(z)|\,.$$

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's Flatness Problem). *How close can a unimodular polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying*

$$|P_n(z)| = \sqrt{n+1}\,, \qquad z \in \partial D? \tag{1.1}$$

Obviously (1.1) is impossible if $n \geq 1$. So one must look for less than (1.1), but then there are various ways of seeking such an "approximate situation". One way is the following. In his paper [Li1] Littlewood had suggested that, conceivably, there might exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that $(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definition.

Definition 1.2. *Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is ε-flat if*

$$(1-\varepsilon)\sqrt{n+1} \leq |P_n(z)| \leq (1+\varepsilon)\sqrt{n+1}\,, \qquad z \in \partial D\,.$$

Definition 1.3. *Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is (ε_{n_k})-ultraflat if each P_{n_k} is (ε_{n_k})-flat. We simply say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is ultraflat if it is (ε_{n_k})-ultraflat with a suitable sequence (ε_{n_k}) of positive numbers tending to 0.*

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely, in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \geq 1$,

$$\max_{z\in\partial D} |P_n(z)| \geq (1+\varepsilon)\sqrt{n+1}\,, \tag{1.2}$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ which is (ε_n)-ultraflat, where $\varepsilon_n = O\left(n^{-1/17}\sqrt{\log n}\right)$. (Kahane's paper contained though a slight error which was corrected in [QS2].) Thus the Erdős conjecture (1.2) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$ the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true, and consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$. An interesting result related to Kahane's breakthrough is given in [Be]. For an account of some of the work done till the mid 1960's, see Littlewood's book [Li2] and [QS2].

Let (ε_n) be a sequence of positive numbers tending to 0. Let the sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ be (ε_n)-ultraflat. We write

$$P_n(e^{it}) = R_n(t)e^{i\alpha_n(t)}\,, \qquad R_n(t) = |P_n(e^{it})|\,, \qquad t \in \mathbb{R}\,. \tag{1.3}$$

It is a simple exercise to show that α_n can be chosen so that it is differentiable on $\mathbb{R}$. This is going to be our understanding throughout the paper.

The structure of ultraflat sequences of unimodular polynomials is studied in [Er1] and [Er3] where several conjectures of Saffari are proved. Here, based on the results in [Er1], we prove yet another Saffari conjecture formulated in [Sa].

Theorem 1.4 (Saffari's Near-Orthogonality Conjecture). *Assume that (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Let*

$$P_n(z) := \sum_{k=0}^{n} a_{k,n} z^k\,.$$

Then

$$\sum_{k=0}^{n} a_{k,n} a_{n-k,n} = o(n)\,.$$

Here, as usual, $o(n)$ denotes a quantity for which $\lim_{n\to\infty} o(n)/n = 0$. The statement remains true if the ultraflat sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ is replaced by an ultraflat sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$, $0 < n_1 < n_2 < \dots$.

If Q_n is a polynomial of degree n of the form $Q_n(z) = \sum_{k=0}^{n} a_k z^k\,,\ a_k \in \mathbb{C}\,,$ then its conjugate reciprocal polynomial is defined by $Q_n^*(z) := z^n \overline{Q}_n(1/z) := \sum_{k=0}^{n} \overline{a}_{n-k} z^k$. In terms of the above definition Theorem 1.4 may be rewritten as

Corollary 1.5. *Assume that (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then*

$$\int_{\partial D} |P_n(z) - P_n^*(z)|^2\,|dz| = 2n + o(n)\,.$$

2. Proof of Theorem 1.4

To prove the theorem we need a few lemmas. The first two are from [Er1].

Lemma 2.1 (Uniform Distribution Theorem for the Angular Speed). *Suppose (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Then, with the notation (1.3), in the interval $[0, 2\pi]$, the distribution of the normalized angular speed $\alpha_n'(t)/n$ converges to the uniform distribution as $n \to \infty$. More precisely, we have*

$$\text{meas}(\{t \in [0,2\pi] : 0 \le \alpha_n'(t) \le nx\}) = 2\pi x + \gamma_n(x)$$

for every $x \in [0,1]$, where $\lim_{n\to\infty} \max_{x\in[0,1]} |\gamma_n(x)| = 0$.

Lemma 2.2 (Negligibility Theorem for Higher Derivatives). *Suppose* (P_n) *is an ultraflat sequence of unimodular polynomials* $P_n \in \mathcal{K}_n$. *Then, with the notation (1.3), for every integer* $r \geq 2$, *we have*

$$\max_{0 \leq t \leq 2\pi} |\alpha_n^{(r)}(t)| \leq \gamma_{n,r} n^r$$

with suitable constants $\gamma_{n,r} > 0$ *converging to* 0 *for every fixed* $r = 2, 3, \dots$.

Lemma 2.3. *Suppose* (P_n) *is an ultraflat sequence of unimodular polynomials* $P_n \in \mathcal{K}_n$. *Let*

$$P_n(z) := \sum_{k=0}^{n} a_{k,n} z^k \,.$$

Then, with the notation (1.3),

$$\sum_{k=0}^{n} a_{k,n} a_{n-k,n} - \frac{n}{2\pi} \int_0^{2\pi} \exp(i(2\alpha_n(t) - nt)) = o(n)\,.$$

Proof of Lemma 2.3. This follows easily by using the formula

$$\sum_{k=0}^{n} a_{k,n} a_{n-k,n} = \frac{1}{2\pi} \int_0^{2\pi} P_n(e^{it})^2 e^{-int}\, dt$$

and the ultraflatness inequalities

$$(1 - \varepsilon_n)\sqrt{n+1} \leq |P_n(e^{it})| \leq (1 + \varepsilon_n)\sqrt{n+1}\,, \qquad n = 1, 2, \dots\,,$$

(cf. Definitions 1.2 and 1.3), where (ε_n) is a sequence of positive numbers tending to 0.

Proof of Theorem 1.4. By Lemma 2.3 it is sufficient to prove that

$$\int_0^{2\pi} \exp(i\beta_n(t))\, dt = \eta_n\,, \qquad \text{with} \qquad \beta_n(t) := 2\alpha_n(t) - nt\,,$$

where (η_n) is a sequence tending to 0. To see this let $\varepsilon > 0$ be fixed. Let $K_n := \gamma_{n,2}^{-1/4}$, where $\gamma_{n,2}$ is defined in Lemma 2.2. We divide the interval $[0, 2\pi]$ into subintervals

$$I_m := [a_{m-1}, a_m] := \left[\frac{(m-1)K_n}{n}, \frac{mK_n}{n}\right], \qquad m = 1, 2, \dots, N-1 := \left\lfloor \frac{2\pi n}{K_n} \right\rfloor\,,$$

and

$$I_N := [a_{N-1}, a_N] := \left[\frac{(N-1)K_n}{n}, 2\pi\right]\,.$$

For the sake of brevity let

$$A_{m-1} := \beta_n(a_{m-1})\,, \qquad m = 1, 2, \dots, N\,,$$

and

$$B_{m-1} := \beta_n'(a_{m-1})\,, \qquad m = 1,2,\dots,N\,.$$

Then by Taylor's Theorem

$$|\beta_n(t) - (A_{m-1} + B_{m-1}(t-a_{m-1})| \le \gamma_{n,2} n^2 (K_n/n)^2 \le \gamma_{n,2}\gamma_{n,2}^{-1/2} \le \gamma_{n,2}^{1/2}$$

for every $t \in I_m$, where $\lim_{n\to\infty} \gamma_{n,2}^{1/2} = 0$ by Lemma 2.2. Hence

$$\int_{I_m} \exp(i\beta_n(t))\,dt = \int_{I_m} \exp(i(A_{m-1} + B_{m-1}(t-a_{m-1})))\,dt + \int_{I_m} \delta_n(t)\,dt\,,$$

with functions $\delta_n(t)$ satisfying

$$\lim_{n\to\infty} \max_{0\le t\le 2\pi} |\delta_n(t)| = 0\,.$$

Hence for $|B_{m-1}| \ge n\varepsilon$ we have

$$\left|\int_{I_m} \exp(i\beta_n(t))\,dt\right| \le \frac{2}{|B_{m-1}|} + \Delta_n \text{meas}(I_m) \le \frac{2}{n\varepsilon} + \Delta_n \text{meas}(I_m)\,,$$

where

$$\Delta_n := \max_{0\le t\le 2\pi} |\delta_n(t)| > 0 \qquad \text{with} \qquad \lim_{n\to\infty} \Delta_n = 0\,.$$

Therefore $\lim_{n\to\infty} K_n = \infty$ implies

$$\begin{aligned} \sum_m \left|\int_{I_m} \exp(i\beta_n(t))\,dt\right| &\le \frac{2}{n\varepsilon} N + 2\pi\Delta_n \\ &\le \frac{2}{n\varepsilon}\left(\frac{2\pi n}{K_n} + 1\right) + 2\pi\Delta_n \le \eta_n^*(\varepsilon)\,, \end{aligned} \tag{2.1}$$

where the summation is taken over all $m = 1,2,\dots,N$ for which $|B_{m-1}| \ge n\varepsilon$, and where $(\eta_n^*(\varepsilon))$ is a sequence tending to 0. Now let

$$E_{n,\varepsilon} := \bigcup_{m:\,|B_{m-1}|\le n\varepsilon} I_m\,.$$

For $|B_{m-1}| \le n\varepsilon$ we deduce by Lemma 2.2 that

$$\begin{aligned} |\beta_n'(t)| &\le |B_{m-1}| + \frac{K_n}{n} \max_{t\in I_m} |\beta_n''(t)| = \\ &= |B_{m-1}| + \frac{K_n}{n} \max_{t\in I_m} |2\alpha_n''(t)| = |B_{m-1}| + \frac{\gamma_{n,2}^{-1/4}}{n} 2\gamma_{n,2} n^2 \le 2n\varepsilon \end{aligned}$$

for every $t \in I_m$ and for every sufficiently large n (independent of m). So

$$E_{n,\varepsilon} \subset \{t \in [0,2\pi] : |\beta_n'(t)| \le 2n\varepsilon\} \subset \{t \in [0,2\pi] : |\alpha_n'(t) - n/2| \le n\varepsilon\}$$

for every $t \in I_m$ and every sufficiently large n. Hence we obtain by Lemma 2.1 that

$$\text{meas}(E_{n,\varepsilon}) \le 4\pi\varepsilon + \eta_n^{**}(\varepsilon)\,,$$

where $(\eta_n^{**}(\varepsilon))$ is a sequence tending to 0. Therefore

$$\sum_m \left|\int_{I_m} \exp(i\beta_n(t))\,dt\right| \le \text{meas}(E_{n,\varepsilon}) \le 4\pi\varepsilon + \eta_n^{**}(\varepsilon)\,, \tag{2.2}$$

where the summation is taken over all $m = 1,2,\dots,N$ for which $|B_{m-1}| \le n\varepsilon$. Since $\varepsilon > 0$ is arbitrary, the result follows from (2.1) and (2.2). □

3. Remarks

In [Sa] another "near orthogonality" relation has been conjectured. Namely it was suspected that if (P_{n_m}) is an ultraflat sequence of unimodular polynomials $P_{n_m} \in \mathcal{K}_{n_m}$ and

$$P_n(z) := \sum_{k=0}^{n} a_{k,n} z^k \,, \qquad n = n_m \,, \qquad m = 1, 2, \dots \,,$$

then

$$\sum_{k=0}^{n} a_{k,n} \overline{a}_{n-k,n} = o(n) \,, \qquad n = n_m \,, \qquad m = 1, 2, \dots \,,$$

where, as usual, $o(n_m)$ denotes a quantity for which $\lim_{n_m \to \infty} o(n_m)/n_m = 0$. However, it was Saffari himself, together with Queffelec [QS2], who showed that this could not be any farther away from being true. Namely they constructed an ultraflat sequence (P_{n_m}) of plain-reciprocal unimodular polynomials $P_{n_m} \in \mathcal{K}_{n_m}$ such that

$$P_n(z) := \sum_{k=0}^{n} a_{k,n} z^k \,, \qquad a_{k,n} = a_{n-k,n} \,, \qquad k = 0, 1, 2, \dots n \,,$$

and hence

$$\sum_{k=0}^{n} a_{k,n} \overline{a}_{n-k,n} = n + 1$$

for the values $n = n_m$, $m = 1, 2, \dots$.

References

[Be] J. Beck, *"Flat" polynomials on the unit circle – note on a problem of Littlewood*, Bull. London Math. Soc. (1991), 269–277.

[BE] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer-Verlag, New York, 1995.

[Er1] T. Erdélyi, *The phase problem of ultraflat unimodular polynomials: the resolution of the conjecture of Saffari*, Math. Annalen (to appear).

[Er2] T. Erdélyi, *On the zeros of polynomials with Littlewood-type coefficient constraints*, Michigan Math. J. **49** (2001), 97–111.

[Er3] T. Erdélyi, *How far is a sequence of ultraflat unimodular polynomials from being conjugate reciprocal*, Michigan Math. J. (to appear).

[Er] P. Erdős, *Some unsolved problems*, Michigan Math. J. **4** (1957), 291–300.

[Ka] J.P. Kahane, *Sur les polynomes a coefficient unimodulaires*, Bull. London Math. Soc. **12** (1980), 321–342.

[Kö] T. Körner, *On a polynomial of J.S. Byrnes*, Bull. London Math. Soc. **12** (1980), 219–224.

[Li1] J.E. Littlewood, *On polynomials* $\sum \pm z^m, \sum \exp(\alpha_m i) z^m, z = e^{i\theta}$, J. London Math. Soc. **41**, 367–376, yr 1966.

[Li2] J.E. Littlewood, *Some Problems in Real and Complex Analysis*, Heath Mathematical Monographs, Lexington, Massachusetts, 1968.

[QS1] H. Queffelec and B. Saffari, *Unimodular polynomials and Bernstein's inequalities*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995, 3), 313–318.

[QS2] H. Queffelec and B. Saffari, *On Bernstein's inequality and Kahane's ultraflat polynomials*, J. Fourier Anal. Appl. **2** (1996, 6), 519–582.

[Sa] B. Saffari, *The phase behavior of ultraflat unimodular polynomials*, in Probabilistic and Stochastic Methods in Analysis, with Applications (1992), Kluwer Academic Publishers, Printed in the Netherlands, 555–572.