MARKOV- AND BERNSTEIN-TYPE INEQUALITIES FOR POLYNOMIALS WITH RESTRICTED COEFFICIENTS

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Abstract. The Markov-type inequality

$$\|p'\|_{[0,1]} \leq cn \log(n+1) \|p\|_{[0,1]}$$

is proved for all polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$ with an absolute constant c. Here $\|\cdot\|_{[0,1]}$ denotes the supremum norm on $[0,1]$. The Bernstein-type inequality

$$|p'(y)| \leq \frac{c}{(1-y)^2} \|p\|_{[0,1]}\,, \qquad y \in [0,1)\,,$$

is shown for every polynomial p of the form

$$p(x) = \sum_{j=m}^{n} a_j x^j\,, \qquad |a_m| = 1\,, \quad |a_j| \leq 1\,, \quad a_j \in \mathbb{C}\,.$$

The inequality

$$|p'(y)| \leq \frac{c}{(1-y)} \log\left(\frac{2}{1-y}\right) \|p\|_{[0,1]}\,, \qquad y \in [0,1)\,,$$

is also proved for every analytic function p on the open unit disk D that satisfies the growth condition

$$|p(0)| = 1\,, \qquad |p(z)| \leq \frac{1}{1-|z|}\,, \quad z \in D\,.$$

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1. INTRODUCTION

In this paper n always denotes a nonnegative integer. We introduce the following classes of polynomials. Let

$$\mathcal{P}_n := \left\{ f : f(x) = \sum_{i=0}^{n} a_i x^i\,, \;\; a_i \in \mathbb{R} \right\}$$

denote the set of all algebraic polynomials of degree at most n with real coefficients.

Let

$$\mathcal{P}_n^c := \left\{ f : f(x) = \sum_{i=0}^{n} a_i x^i\,, \;\; a_i \in \mathbb{C} \right\}$$

denote the set of all algebraic polynomials of degree at most n with complex coefficients.

Let

$$\mathcal{F}_n := \left\{ f : f(x) = \sum_{i=0}^{n} a_i x^i\,, \;\; a_i \in \{-1, 0, 1\} \right\}$$

denote the set of polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$. So obviously

$$\mathcal{F}_n \subset \mathcal{P}_n \subset \mathcal{P}_n^c\,.$$

The following two inequalities are well known in approximation theory. See, for example, Duffin and Schaeffer [15], Bernstein [1], Cheney [12], Lorentz [29], DeVore and Lorentz [14], and Borwein and Erdélyi [7].

Markov Inequality. *The inequality*

$$\|p'\|_{[-1,1]} \leq n^2 \|p\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_n$.

Bernstein Inequality. *The inequality*

$$|p'(y)| \leq \frac{n}{\sqrt{1-y^2}}\, \|p\|_{[-1,1]}$$

holds for every $p \in \mathcal{P}_n$ *and* $y \in (-1, 1)$.

In the above two theorems and throughout the paper $\|\cdot\|_A$ denotes the supremum norm on $A \subset \mathbb{R}$. Markov- and Bernstein-type inequalities in L_p norms are discussed, for example, in Borwein and Erdélyi [7] and [8], DeVore and Lorentz [14], Lorentz, Golitschek, and Makovoz [30], Nevai [33], Máté and Nevai [31], Rahman and Schmeisser [38], Milovanović, Mitrinović, and Rassias [32]. Markov- and Bernstein-type inequalities have their own intrinsic interest. In addition, many of them play a key role in proving inverse theorems of approximation.

Markov- and Bernstein-type inequalities for classes of polynomials under various constraints have attracted a number of authors. For example, it has been observed by Bernstein [1] that Markov's inequality for monotone polynomials is not essentially better than for arbitrary polynomials. He proved that if n is odd, then

$$\sup_{0\neq p} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} = \left(\frac{n+1}{2}\right)^2 ,$$

where the supremum is taken for all $p \in \mathcal{P}_n$ that are monotone on $[-1,1]$. (For even n, the inequality

$$\sup_{0\neq p} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq \left(\frac{n+1}{2}\right)^2$$

still holds.) This may look quite surprising, since one would expect that if a polynomial is this far away from the "equioscillating" property of the Chebyshev polynomial, then there should be a more significant improvement in the Markov inequality. A Markov-Bernstein type inequality is proved by Borwein and Erdélyi [6] for quite general classes of polynomials with restricted zeros, namely

$$|p'(y)| \leq c \, \min\left\{ \sqrt{\frac{n(k+1)}{1-y^2}}\,, \;\; n(k+1) \right\} \|p\|_{[-1,1]} , \qquad y \in [-1,1] ,$$

for all $p \in \mathcal{P}_n$ having at most k zeros in the open unit disk, where c is an absolute constant. (Here and in what follows the expression "absolute constant" means a constant that is independent of all the variables in the inequality). For Markov- and Bernstein-type inequalities for classes of polynomials under various constraints, see Appendix 5 of our book [7].

A number of Markov- and Bernstein-type inequalities for polynomials with restricted coefficients may also be found in the literature. Most of these deal with polynomials with nonnegative coefficients in various bases. For example, Lorentz [28] proved that there is an absolute constant c such that

$$|p'(y)| \leq c \, \min\left\{ \sqrt{\frac{n}{1-y^2}}\,, \;\; n \right\} \|p\|_{[-1,1]} , \qquad y \in [-1,1] ,$$

for all polynomials p of the form

$$p(x) = \sum_{j=0}^{n} a_j (x+1)^j (x-1)^{n-j} , \qquad a_j \geq 0 .$$

Another attractive Markov-type inequality for polynomials with restricted coefficients is due to Newman [34]. It states that if $\Lambda := (\lambda_j)_{j=0}^{\infty}$ is a sequence of distinct nonnegative real numbers and $M_n(\Lambda) := \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots , x^{\lambda_n}\}$, then

$$\frac{2}{3} \sum_{j=0}^{n} \lambda_j \leq \sup_{0\neq p\in M_n(\Lambda)} \frac{|p'(1)|}{\|p\|_{[0,1]}} \leq \sup_{0\neq p\in M_n(\Lambda)} \frac{\|xp'(x)\|_{[0,1]}}{\|p\|_{[0,1]}} \leq 11 \sum_{j=0}^{n} \lambda_j .$$

It is our intention to establish Markov- and Bernstein-type inequalities for $\mathcal{F}_n$ on $[0,1]$ and on $[a,b] \subset [0,1)$. The class $\mathcal{F}_n$ and other classes of polynomials with restricted coefficients have been thoroughly studied in a number of (mainly number theoretic) papers. See, for example, Beck [2], Bloch and Pólya [3], Bombieri and Vaaler [4], Borwein, Erdélyi, and Kós [9], Borwein and Ingalls [10], Byrnes and Newman [11], Cohen [13], Erdős [17], Erdős and Turán [18], Ferguson [19], Hua [20], Kahane [21] and [22], Konjagin [23], Körner [24], Littlewood [25] and [26], Littlewood and Offord [27], Newman and Byrnes [35], Newman and Giroux [36], Odlyzko and Poonen [37], Salem and Zygmund [39], Schur [40], and Szegő [41].

2. Markov- and Bernstein-Type Inequalities for $\mathcal{F}_n$

Our first theorem shows that n^2 in the Markov inequality improves to at least $cn\log(n+1)$ for polynomials from $\mathcal{F}_n$.

Theorem 2.1 (Markov-Type Inequality for $\mathcal{F}_n$). *There is an absolute constant $c > 0$ such that*

$$\|p'\|_{[0,1]} \leq cn\log(n+1)\,\|p\|_{[0,1]}$$

for every $p \in \mathcal{F}_n$.[1]

A direct computation shows that $p(x) = x^n$ is not extremal for the inequality of Theorem 2.1. For example, the polynomial

$$p(x) = x^{10} - x^8 - x^6 + x^5$$

is the extremal polynomial for the inequality from $\mathcal{F}_{10}$ with

$$\frac{\|p'\|_{[0,1]}}{10\,\|p\|_{[0,1]}} = 3.701\ldots.$$

Our next theorem shows that $n(1-y^2)^{-1/2}$ in Bernstein's inequality improves to at least $c(1-y)^{-2}$ for polynomials from $\mathcal{F}_n$.

Theorem 2.2 (Bernstein-Type Inequality for $\mathcal{F}_n$). *There is an absolute constant $c > 0$ such that*

$$|p'(y)| \leq \frac{c}{(1-y)^2}\,\|p\|_{[0,1]}$$

for every $p \in \mathcal{F}_n$ and $y \in [0,1)$.

Theorem 2.2 follows immediately from the following more general result.

Theorem 2.3. *There is an absolute constant $c > 0$ such that*

$$|p'(y)| \leq \frac{c}{(1-y)^2}\,\|p\|_{[0,1]}$$

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[1] Up to the constant $c > 0$ this is the correct result as a construction suggested to us by F. Nazarov shows. This will be discussed in a later publication.

for every $p \in \mathcal{P}_n^c$ *of the form*

$$p(x) = \sum_{j=m}^{n} a_j x^j \,, \qquad |a_m| = 1 \,, \quad |a_j| \le 1 \,,$$

and for every $y \in [0,1)$.

It may be suspected that $(1-y)^{-2}$ can be replaced by some smaller factor in Theorems 2.2 and 2.3.[2]

Under slightly more restrictions we can prove the following better Bernstein-type inequality.

Theorem 2.4. *There is an absolute constant* $c > 0$ *such that*

$$|p'(y)| \le \frac{c}{(1-y)} \log\left(\frac{2}{1-y}\right) \|p\|_{[0,1]}$$

for every analytic function p *on the open unit disk* D *that satisfies the growth condition*

$$|p(0)| = 1 \,, \qquad |p(z)| \le \frac{1}{1-|z|} \,, \quad z \in D \,,$$

and for every $y \in [0,1)$.[3]

Our final result establishes an essentially sharp Markov-type inequality on an interval $[a,b] \subset [0,1)$ for the class in Theorem 2.3.

Theorem 2.5. *Suppose* $0 \le a < b < 1$. *There exists a constant* $c = c(a,b)$ *depending only on* a *and* b *such that*

$$\|p'\|_{[a,b]} \le cn \, \|p\|_{[a,b]}$$

for every $p \in \mathcal{P}_n^c$ *of the form*

$$p(x) = \sum_{j=m}^{n} a_j x^j \,, \qquad |a_m| = 1 \,, \quad |a_j| \le 1 \,.$$

3. LEMMAS FOR THEOREM 2.1

To prove Theorem 2.1 we need several lemmas.

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[2] We believe that we are now able to prove that $(1-y)^{-2}$ cannot be replaced by $(1-y)^{-1}$ in these results.

[3] We believe that we are now able to prove that Theorem 2.4 is, up to the constant $c > 0$, sharp.

Hadamard Three Circles Theorem. *Suppose f is regular in*

$$\{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}.$$

For $r \in [r_1, r_2]$, let

$$M(r) := \max_{|z|=r} |f(z)|.$$

Then

$$M(r)^{\log(r_2/r_1)} \leq M(r_1)^{\log(r_2/r)} M(r_2)^{\log(r/r_1)}.$$

Corollary 3.1. *Let $M \in \mathbb{R}$ and $n, m \in \mathbb{N}$. Suppose $m \leq M \leq 2n$. Suppose f is regular inside and on the ellipse $A_{n,M}$ with foci at 0 and 1 and with major axis*

$$\left[-\frac{M}{n}, 1 + \frac{M}{n}\right].$$

Let $B_{n,m,M}$ be the ellipse with foci at 0 and 1 and with major axis

$$\left[-\frac{m^2}{nM}, 1 + \frac{m^2}{nM}\right].$$

Then there is an absolute constant $c_1 > 0$ such that

$$\begin{aligned}\max_{z \in B_{n,m,M}} \log|f(z)| \leq \max_{z\in[0,1]} \log|f(z)| \\ + \frac{c_1 m}{M}\left(\max_{z \in A_{n,M}} \log|f(z)| - \max_{z \in [0,1]} \log|f(z)|\right).\end{aligned}$$

Proof. This follows from the Hadamard Three Circles Theorem with the substitution $w = \frac{1}{4}(z + z^{-1}) + \frac{1}{2}$. □

Lemma 3.2. *Let $p \in \mathcal{F}_n$ with $\|p\|_{[0,1]} =: \exp(-M)$, $M \geq \log(n+1)$. Suppose $m \in \mathbb{N}$ and $1 \leq m \leq M$. Then there is an absolute constant $c_2 > 0$ such that*

$$\max_{z \in B_{n,m,M}} |p(z)| \leq (c_2)^m \max_{z \in [0,1]} |p(z)|,$$

where $B_{n,m,M}$ is the same ellipse as in Corollary 3.1.

Proof. By Chebyshev's inequality, $\|p\|_{[0,1]} \geq 2 \cdot 4^{-n}$ for every $p \in \mathcal{P}_n$ with leading coefficient ± 1. Therefore $M \leq (\log 4)n$. Note that the assumption $p \in \mathcal{F}_n$ can be written as

$$\max_{z \in [0,1]} \log|p(z)| = -M.$$

Also, $p \in \mathcal{F}_n$ and $z \in A_{n,M}$ imply that

$$\begin{aligned}\log|p(z)| &\leq \log\left((n+1)\left(1 + \frac{M}{n}\right)^{n+1}\right) \\ &\leq \log(n+1) + (n+1)\frac{M}{n} \leq \log(n+1) + 2M \leq 3M.\end{aligned}$$

Now the lemma follows from Corollary 3.1. □

Lemma 3.3. *Let* $p \in \mathcal{F}_n$ *with* $\|p\|_{[0,1]} =: \exp(-M)$, $M \geq \log(n+1)$. *Suppose* $m \in \mathbb{N}$ *and* $1 \leq m \leq M$. *Then there is an absolute constant* $c_3 \geq 2$ *so that*

$$\|p^{(m)}\|_{[0,1]} \leq m! \left(\frac{c_3 n M}{m^2}\right)^m \|p\|_{[0,1]} \,.$$

Proof. This follows from Lemma 3.2 and the Cauchy Integral Formula □

Lemma 3.4. *Let* $p \in \mathcal{F}_n$ *with* $\|p\|_{[0,1]} =: \exp(-M)$, $M \geq 4\log(2n+2)$. *Suppose* $p \in \mathcal{F}_n$ *has exactly* k *zeros at* 1. *Let* $\mu := \min\{[M],\, k\}$. *Then*

$$|p'(y)| \leq 2c_3 n \log(2n+2) \|p\|_{[0,1]}$$

for every

$$y \in \left[1 - \frac{\mu^2}{2c_3 n M}, 1\right] \,,$$

where $c_3 \geq 2$ *is as in Lemma 3.3.*

In Lemma 3.4 and in what follows $[M]$ denotes the greatest integer not greater than M.

Proof. Let n be a positive integer. Suppose $p \in \mathcal{F}_n$ satisfies the assumptions of the lemma. First we note that $M \geq 4\log(2n+2)$ implies that $2 \leq \mu \leq k$. Indeed, since $|p^{(k)}(1)| \neq 0$ is an integer, Markov's inequality implies that

$$1 \leq |p^{(k)}(1)| \leq (2n)^{2k} \|p\|_{[0,1]} = (2n)^{2k} \exp(-M) \,.$$

Combining this with $M \geq 4\log(2n+2)$, we conclude

$$\mu := \min\{[M],\, k\} \geq \min\left\{M - 1\,,\, \frac{M}{2\log(2n)}\right\} \geq \frac{M}{2\log(2n+2)} \geq 2 \,. \tag{3.1}$$

Now using Taylor's Theorem and Lemma 3.3, we obtain

$$\begin{aligned}
|p'(y)| &\leq \frac{1}{(\mu-1)!} \,\|(p')^{(\mu-1)}\|_{[1-y,1]} (1-y)^{\mu-1} \\
&\leq \frac{\mu!}{(\mu-1)!} \left(\frac{c_3 n M}{\mu^2}\right)^{\mu} \|p\|_{[0,1]} (1-y)^{\mu-1} \\
&\leq \mu 2^{1-\mu} \frac{c_3 n M}{\mu^2} \,\|p\|_{[0,1]} \leq 2c_3 n \log(2n+2) \|p\|_{[0,1]}
\end{aligned}$$

whenever

$$y \in \left[1 - \frac{\mu^2}{2c_3 n M}, 1\right] \,.$$

Here we used again that $M \leq 2\mu \log(2n+2)$ by (3.1). This finishes the proof. □

Lemma 3.5. *Let $p \in \mathcal{F}_n$ with $\|p\|_{[0,1]} =: \exp(-M)$, $M \geq 4\log(2n+2)$. Suppose $p \in \mathcal{F}_n$ has exactly k zeros at 1. Let $\mu := \min\{[M], k\}$ as in Lemma 3.4. Then there is an absolute constant $c_4 > 0$ such that*

$$|p'(y)| \leq c_4 n \log(n+1) \|p\|_{[0,1]}$$

for every

$$y \in \left[\frac{1}{4}, 1 - \frac{\mu^2}{2c_3 nM}\right],$$

where $c_3 \geq 2$ is as in Lemma 3.3.

Proof. Using Lemma 3.2 with $m = 1$, the Cauchy Integral Formula, and the assumptions of the lemma, we obtain that there is an absolute constant $c_5 > 0$ such that

$$|p'(y)| \leq c_5 \left(\frac{1}{Mn}\right)^{-1/2} \left(\frac{\mu^2}{2c_3 nM}\right)^{-1/2} \|p\|_{[0,1]} \leq c_5 (2c_3)^{-1/2} \frac{M}{\mu} n \,\|p\|_{[0,1]}.$$

Note that $M \leq 2\mu \log(2n+2)$ as in the proof of Lemma 3.4. This, together with the previous line finishes the proof. $\square$

Lemma 3.6. *We have*

$$|p'(y)| \leq 2n \|p\|_{[0,1]}$$

for every $p \in \mathcal{F}_n$ and $y \in [0, \frac{1}{4}]$.

Proof. Suppose $0 \leq y \leq \frac{1}{4}$. If h denotes the smallest exponent occurring in p then

$$\begin{aligned} |p'(y)| &\leq ny^h(1 + y + y^2 + \cdots) \leq 2ny^h(1 - y - y^2 - \cdots) \\ &\leq 2n|p(y)| \leq 2n \max_{x \in [0,1]} |p(x)|. \quad \square \end{aligned}$$

Lemma 3.7. *There is an absolute constant $c_6 > 0$ such that*

$$\|p'\|_{[0,1]} \leq c_6 n \log(n+1) \|p\|_{[0,1]}$$

for every $p \in \mathcal{F}_n$ with $\|p\|_{[0,1]} \leq (2n+2)^{-4}$.

Proof. Combine Lemmas 3.4, 3.5, and 3.6. $\square$

Lemma 3.8. *There is an absolute constant $c_7 > 0$ such that*

$$\|p'\|_{[0,1]} \leq c_7 n \log(n+1) \|p\|_{[0,1]}$$

for every $p \in \mathcal{F}_n$ with $\|p\|_{[0,1]} \geq (2n+2)^{-4}$.

Proof. Applying Corollary 3.1 with $m = 1$ and $M = \log(n+2)$, we obtain that there is an absolute constant $c_8 > 0$ such that

$$\max_{z \in B_{n,1,\log(n+2)}} |p(z)| \leq c_8 \max_{z \in [0,1]} |p(z)|$$

for every $p \in \mathcal{F}_n$ with $\|p\|_{[0,1]} \geq (2n+2)^{-4}$. To see this note that

$$\max_{z \in [0,1]} \log |p(z)| \geq -4 \log(2n+2)$$

and

$$\max_{z \in A_{n,M}} \log |p(z)| \leq \log \left(n \left(1 + \frac{\log(n+2)}{n} \right)^n \right) \leq 2 \log(n+2)\,.$$

Now the Cauchy Integral Formula yields that

$$\|p'\|_{[0,1]} \leq c_7 n \log(n+1) \|p\|_{[0,1]}$$

with an absolute constant $c_7 > 0$. □

4. Lemmas for Theorems 2.3 and 2.4

Denote by $\mathcal{S}$ the collection of all analytic functions g on the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ that satisfy

$$|g(z)| \leq \frac{1}{1-|z|}\,, \qquad z \in D\,.$$

To prove Theorem 2.3, we need the following result. Its proof may be found in Borwein, Erdélyi, and Kós [9] where it also plays a key role.

Lemma 4.1. *There are absolute constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$|g(0)|^{c_1/a} \leq \exp\left(\frac{c_2}{a}\right) \|g\|_{[1-a,1]}$$

for every $g \in \mathcal{S}$ *and* $a \in (0,1]$.

Corollary 4.2. *There are absolute constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$|g(0)|^{c_1/a} \leq \exp\left(\frac{c_2}{a}\right) \|g\|_{[1-a,1-a/2]}$$

for every $g \in \mathcal{S}$ *and* $a \in (0,1]$.

Proof. This follows from Lemma 4.1 by a linear scaling. □

Lemma 4.3. *Let* $y \in [1/2, 1)$ *and* $\tilde{y} := y + (1-y)/2$. *Suppose* f *is regular inside and on the ellipse* A_y *with foci at* 0 *and* $\tilde{y}$ *and with major axis*

$$\left[-\frac{1-y}{4},\ \tilde{y} + \frac{1-y}{4} \right].$$

Let B_y *be the ellipse with foci at* 0 *and* $\tilde{y}$ *and with major axis*

$$\left[-\frac{(1-y)^3}{4},\ \tilde{y} + \frac{(1-y)^3}{4} \right].$$

Then there is an absolute constant $c_3 > 0$ *such that*

$$\max_{z \in B_y} \log |f(z)| \leq \max_{z \in [0,\tilde{y}]} \log |f(z)| + c_3 (1-y) \left(\max_{z \in A_y} \log |f(z)| - \max_{z \in [0,\tilde{y}]} \log |f(z)| \right).$$

Proof. This follows from the Hadamard Three Circles Theorem with the substitution $w = (\tilde{y}/4)(z + z^{-1}) + (\tilde{y}/2)$. □

Lemma 4.4. *Let $y \in [1/2, 1)$ and $\tilde{y} := y + (1-y)/2$. Suppose f is regular inside and on the ellipse A_y with foci at 0 and $\tilde{y}$ and with major axis*

$$\left[-\frac{1-y}{4},\ \tilde{y} + \frac{1-y}{4} \right].$$

Let C_y be the ellipse with foci at 0 and $\tilde{y}$ and with major axis

$$\left[-\frac{1-y}{4\log^2\left(\frac{2}{1-y}\right)},\ \tilde{y} + \frac{1-y}{4\log^2\left(\frac{2}{1-y}\right)} \right].$$

Then there is an absolute constant $c_4 > 0$ such that

$$\begin{aligned}\max_{z \in C_y} \log|f(z)| \leq \max_{z \in [0,\tilde{y}]} \log|f(z)| \\ + c_4 \left(\log\left(\frac{2}{1-y} \right) \right)^{-1} \left(\max_{z \in A_y} \log|f(z)| - \max_{z \in [0,\tilde{y}]} \log|f(z)| \right).\end{aligned}$$

Proof. This follows from the Hadamard Three Circles Theorem with the substitution $w = (\tilde{y}/4)(z + z^{-1}) + (\tilde{y}/2)$. $\square$

Lemma 4.5. *Let $y \in [1/2, 1)$ and $\tilde{y} := y + (1-y)/2$. Let k be a nonnegative integer not greater than $c(1-y)^{-2}$. Suppose f is of the form*

$$f(z) = z^k g(z), \qquad g \in \mathcal{S}, \quad |g(0)| = 1.$$

Then there is an absolute constant $c_5 > 0$ such that

$$\max_{z \in B_y} |f(z)| \leq c_5 e^c \max_{z \in [0,\tilde{y}]} |f(z)|,$$

where B_y is as in Lemma 4.3.

Proof. Lemma 4.2, $k \leq c(1-y)^{-2}$, $f(z) = z^k g(z)$, $g \in \mathcal{S}$, and $|g(0)| = 1$ imply that

$$\begin{aligned}\max_{z \in [0,\tilde{y}]} \log|f(z)| \geq \max_{z \in [y,\tilde{y}]} \log|f(z)| \\ \geq \log(y^k) + \max_{z \in [y,\tilde{y}]} \log|g(z)| \geq -\frac{c}{1-y} - \frac{c_2}{1-y}.\end{aligned}$$

Also $z \in A_y$ (A_y is defined in Lemma 4.3), $f(z) = z^k g(z)$, and $g \in \mathcal{S}$ imply that

$$\log|f(z)| \leq \log\left(\frac{4}{1-y} \right) \leq \frac{4}{1-y}.$$

Now the lemma follows from Lemma 4.3. $\square$

Lemma 4.6. *Let $y \in [1/2, 1)$ and $\tilde{y} := y + (1-y)/2$. Suppose*

$$f \in \mathcal{S}\,, \qquad |f(0)| = 1\,.$$

Then there is an absolute constant $c_6 > 0$ such that

$$\max_{z \in C_y} |f(z)| \leq c_6 \max_{z \in [0,\tilde{y}]} |f(z)|\,,$$

where C_y is as in Lemma 4.4.

Proof. The assumption $|f(0)| = 1$ implies that

$$\max_{z \in [0,\tilde{y}]} \log |f(z)| \geq 0\,.$$

Also $z \in A_y$ (A_y is defined in Lemma 4.4) and $f \in \mathcal{S}$ imply that

$$\log |f(z)| \leq \log \left(\frac{4}{1-y} \right)\,.$$

Now the lemma follows from Lemma 4.4. $\square$

Lemma 4.7. *Let $y \in [1/2, 1)$ and $\tilde{y} := y + (1-y)/2$. Let $c := 8c_2 + 1$, where c_2 is as in Lemma 4.1 Let k be a nonnegative integer greater than $c(1-y)^{-2}$. Suppose f is of the form*

$$f(z) = z^k g(z)\,, \qquad g \in \mathcal{S}\,, \quad |g(0)| = 1\,.$$

Then there exists an absolute constant $c_7 > 0$ such that

$$|f'(y)| \leq c_7 \|f\|_{[\tilde{y},1]}\,.$$

Proof. Lemma 4.1, $k > c(1-y)^{-2}$, $f(z) = z^k g(z)$, and $|g(0)| = 1$ imply that

$$\begin{aligned}
|f'(y)| &\leq \frac{ky^{k-1}}{(1-y)^2} \leq \frac{2ky^k}{(1-y)^2} \exp\left(\frac{2c_2}{1-y}\right) \|g\|_{[\tilde{y},1]} \\
&\leq \frac{2ky^k}{(1-y)^2} \exp\left(\frac{2c_2}{1-y}\right) \tilde{y}^{-k} \|f\|_{[\tilde{y},1]} \\
&= \frac{1}{(1-y)^2} 2k \left(\frac{y}{\tilde{y}}\right)^k \exp\left(\frac{2c_2}{1-y}\right) \|f\|_{[\tilde{y},1]} \\
&\leq \frac{1}{(1-y)^2} c_8 \exp\left(-\frac{c}{4(1-y)}\right) \exp\left(\frac{2c_2}{1-y}\right) \|f\|_{[\tilde{y},1]} \\
&\leq \frac{1}{(1-y)^2} \exp\left(-\frac{1}{1-y}\right) \|f\|_{[\tilde{y},1]} \leq c_7 \|f\|_{[\tilde{y},1]}\,,
\end{aligned}$$

where $c_7 > 0$ and $c_8 > 0$ are absolute constants. $\square$

Lemma 4.8. *There is an absolute constant* $c_9 > 0$ *such that*

$$|f(z)| \le \exp\left(\frac{c_9}{1-a}\right) \|f\|_{[a,1]}$$

holds for every polynomial $f \in \mathcal{P}_n^c$ *of the form*

$$f(x) = \sum_{j=m}^{n} a_j x^j\,, \qquad |a_m| = 1\,, \quad |a_j| \le 1\,, \quad a_j \in \mathbb{C}\,,$$

for every $a \in [0,1)$, *and for every* $z \in \mathbb{C}$ *with* $|z| \le a$.

Proof. This follows from Lemma 4.1. □

5. LEMMAS FOR THEOREM 2.5

Lemma 5.1. *Suppose* $r \in (0,1)$. *Every polynomial* $p \in \mathcal{P}_n^c$ *of the form*

$$p(x) = \sum_{j=m}^{n} a_j x^j\,, \qquad |a_m| = 1\,, \quad |a_j| \le 1\,.$$

has at most $(4/r)\log(2/r)$ *zeros different from* 0 *in the open disk*

centered at 0 *with radius* $1-r$.

Proof. The proof is a simple application of the Jensen formula. We omit the details.

Lemma 5.2. *Suppose* $0 \le k \le n$. *The inequality*

$$\|p'\|_{[a,b]} \le \frac{18n(k+1)}{b-a} \|p\|_{[a,b]}$$

holds for every $p \in \mathcal{P}_n$ *that has at most* k *zeros in the open disk with diameter* $[a,b]$.

Proof. See Borwein [5], Erdélyi [16], or Borwein and Erdélyi [7].

We remark that the following complex analogue of Lemma 5.2 also holds, but its proof has not been published yet.

Lemma 5.3. *Suppose* $0 \le k \le n$. *There exists an absolute constant* $c > 0$ *such that*

$$\|p'\|_{[a,b]} \le \frac{cn\,\max\{k+1, \log n\}}{b-a} \|p\|_{[a,b]}$$

for every $p \in \mathcal{P}_n^c$ *that has at most* k *zeros in the open disk with diameter* $[a,b]$.

6. Proofs of the Main Results

Proof of Theorem 2.1. Combine Lemmas 3.7 and 3.8. □

Proof of Theorem 2.3. When $y \in [0, 1/2)$ the statement follows from Lemma 4.8 and the Cauchy Integral formula. If $y \in [1/2, 1)$, the theorem follows from Lemmas 4.5 and 4.7 (Lemma 4.5 has to be combined with the Cauchy integral formula). □

Proof of Theorem 2.4. When $y \in [0, 1/2)$ the statement follows from Lemma 4.8 and the Cauchy Integral formula. If $y \in [1/2, 1)$, the theorem follows from Lemma 4.6 and the Cauchy integral formula. □

Proof of Theorem 2.5. This is a straightforward corollary of Lemmas 5.1 and 5.3. In the case when the coefficients are real we need Lemma 5.2 (the proof of which is published) rather than Lemma 5.3 (the proof of which has not been published yet). □

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